ECE 374 B: Algorithms and Models of Computation, Fall 2023

Midterm 2 – October 31, 2023

- **You will have 75 minutes (1.25 hours) to solve 6 problems. Most have multiple parts.** Don't spend too much time on questions you don't understand and focus on answering as much as you can!
- *No* resources are allowed for use during the exam except a multi-page cheatsheet and scratch paper on the back of the exam. ***Do not tear out the cheatsheet or the scratch paper!*** It messes with the auto-scanner.
- You should write your answers *completely* in the space given for the question. We will not grade parts of any answer written outside of the designated space.
- Please *use a dark-colored pen* unless you are *absolutely* sure your pencil writing is forceful enough to be legible when scanned. We will take off points if we have difficulty reading the uploaded document.
- Incorrect algorithms will receive a score of 0, but slower than necessary but correct algorithms will *always* receive some points, even brute force ones. Thus, *you should prioritize the correctness of your submitted algorithms over speed*; you will receive more points that way. On the other hand, submit the fastest algorithms that you know are correct; faster algorithms will receive more points.
- Any recursive backtracking algorithm or dynamic programming algorithm given without an *English* description of the recursive function (i.e., a description of the output of the function *in terms of their inputs*) will receive a score of 0.
- Any greedy algorithm or a modification of a standard graph algorithm given without a proof of correctness will receive a score of 0.
- Any algorithms written in actual code instead of pseudocode will receive a score of 0.
- For problems with a graph given as input, you may assume the graph is simple (i.e., it has no self-loops or parallel edges).
- Only algorithms referenced in the cheat sheet may be referred to as a "black box". You may not simply refer to a prior lab/homework for the solution and must give the full answer.
- Unless explicitly mentioned, **a runtime analysis is required for each given algorithm**.
- ***Don't cheat.*** If we catch you, you will get an F in the course.
- ***Good luck!***

Name: ______________________________

NetID: ______________________________

Date: ______________________________

1 Short answer (2 questions) - 20 points

Answer the following questions. You may **briefly** (no more than 2 sentences) justify your answers, but a complete proof is not required.

(a) Give a *tight* asymptotic upper-bound for the following recurrences :

(i)

$$A(n) = 2A(n-2) + n \qquad A(0) = A(1) = A(2) = 1$$

(ii)

$$B(n) = 3B(n/2) + n^2 \qquad B(0) = B(1) = 1$$

(b) Give the recurrence that describes the following program. What is the asymptotic, upper-bound, of the running time?

```
foolishness(n)
    sum=0
    if n = 1,
        return sum = 1
    for i from 1 to n^3:
        sum += 4*i
    return foolishness (n-1)
```

Recurrence:

Running time:

2 Short answer II (5 questions) - 30 points

Answer the following questions. You *may* **briefly** (no more than 2 sentences) justify your answers, but a complete proof is not required. For the following graph problems, use the notation $G = (V, E)$, $n = |V|$ and $m = |E|$

(a) Suppose we implemented a vanilla version of QuickSort where the pivot was always chosen from the first element of the array. Under what input array will the QuickSort algorithm result in a running time of $O\left(n^2\right)$

(b) In the longest increasing subsequence algorithm (found in the cheat sheet), we defined a recurrence $LIS(i, j)$. **Give an English description (no more than 2 sentences) of what $LIS(i, j)$ represents.** Note: what's in the cheat sheet does not constitute a english description for the recurrence.

(c) Given a graph with n vertices and no edges, how many topological sorts will this graph have?

(d) You are given a graph ($G = (V, E)$) and two vertices a and b. Write a algorithm that determines if both a and b are part of the same strongly connected component. What is the run-time of this algorithm?

(e) In the Bellman-Ford algorithm (found in the cheat sheet), we defined a recurrence $d(v, k)$. **Give an English description (no more than 2 sentences) of what $d(v, k)$ represents.** Note: what's in the cheat sheet does not constitute a english description for the recurrence.

3 Recursion - 10 points

Suppose we are given an array $A[1..n]$ of n *distinct integers*, which could be positive, negative, or zero, sorted in increasing order so that $A[1] < A[2] < \cdots < A[n]$. Describe a fast algorithm that either computes an index i such that $A[i] = n - i + 1$ or correctly reports that no such index exists.

(You *cannot* simply reference a prior lab/homework. You must give the actual solution.)

4 Finding outliers - 10 points

The ECE374 staff has just completed grading midterm 2. They have a list of S[1...n] of unsorted exam scores and are searching for outliers on either end. The interquartile range is defined by Q3 – Q1, that is the 75th percentile score – 25th percentile score. Specifically, an outlier is someone who scores $> Q3 + 1.5 * IQR$ (interquartile range) or someone who scores $< Q1 - 1.5 * IQR$. Give an efficient O(n) program to (1) compute the interquartile range, and (2) output all the outlier scores. A solution that uses hashing or sorting will be awarded 0 points.

5 Dynamic programming (1 questions) - 15

You are given an integer value x and an array A where each element of the array represents a coin denomination. Describe a dynamic programming algorithm that returns the number of ways to make change for x.

Example: $A = [1,2,3]$ and $x = 5$. Output is 5 ($\{1,1,1,1,1\}, \{1,1,1,2\}, \{1,1,3\}, \{1,2,2\}, \{2,3\}$).

(You *cannot* simply reference a prior lab/homework. You must give the actual solution.)

Recurrence and short English description(in terms of the parameters):

Memoization data structure and evaluation order:

Return value:

Time Complexity:

6 Graph algorithm - 15 points

You are given two weighted, directed acyclic graphs ($G = (V, E_G)$ and $H = (V, E_H)$) that have the same vertices but different edges. Every edge is assigned an integer value that could be positive or negative.

You are given two nodes s and t. The length of a path in each graph is defined as $\ell_G(s,t)$ for G and $\ell_H(s,t)$ for H. You need to find the path that exists in both G and H with the minimum combined length (in other words, find minimum value of $\ell_G(s,t) + \ell_H(s,t)$). (Hint: topological sort is your friend)

(You *cannot* simply reference a prior lab/homework. You must give the actual solution.)

Note: These details don't necessarily matter but might be useful to some. Both graphs are given as adjacency matrices where 0 indicates no edge and 1 indicates an edge. The weights are given by a weight matrix $w_G[i,j]$ or $w_H[i,j]$ for G and H respectively. Remember pseudo-code is one of the worst ways to describe your algorithm(s); be clear and succinct.

This page is for additional scratch work!

ECE 374 B Algorithms: Cheatsheet

1 Recursion

Simple recursion

Definitions

- **Reduction:** solve one problem using the solution to another.
- **Recursion:** a special case of reduction - reduce problem to a *smaller* instance of *itself* (self-reduction).
  - Problem instance of size n is reduced to *one or more* instances of size $n-1$ or less.
  - For termination, problem instances of small size are solved by some other method as *base cases*

Tower of Hanoi

Arguably the most famous example of recursion. The goal is to move n disks one at a time from the first peg to the last peg.

Pseudocode: Tower of Hanoi

```
Hanoi (n, src, dest, tmp):
    if (n > 0) then
        Hanoi (n − 1, src, tmp, dest)
        Move disk n from src to dest
        Hanoi (n − 1, tmp, dest, src)
```

Recurrences

Suppose you have a recurrence of the form $T(n) = rT(n/c) + f(n)$.

The *master theorem* gives a good asymptotic estimate of the recurrence. If the work at each level is:

Decreasing:	$rf(n/c) = \kappa f(n)$ where $\kappa < 1$	$T(n) = O(f(n))$
Equal:	$rf(n/c) = f(n)$	$T(n) = O(f(n) \cdot \log_c n)$
Increasing:	$rf(n/c) = Kf(n)$ where $K > 1$	$T(n) = O(n^{\log_c r})$

Some useful identities:

- Sum of integers: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- Geometric series closed-form formula: $\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$
- Logarithmic identities: $\log(ab) = \log a + \log b$, $\log(a/b) = \log a - \log b$, $a^{\log_c b} = b^{\log_c a}$ $(a, b, c > 1)$, $\log_a b = \log_c b / \log_c a$.

Backtracking

Backtracking is the algorithm paradigm involving guessing the solution to a single step in some multi-step process and recursing backwards if it doesn't lead to a solution. For instance, consider the longest increasing subsequence (LIS) problem. You can either check all possible subsequences:

Pseudocode: LIS - Naive enumeration

```
algLISNaive(A[1..n]):
    maxmax = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|
    return max
```

On the other hand, we don't need to generate every subsequence; we only need to generate the subsequences that are increasing:

Pseudocode: LIS - Backtracking

```
LIS_smaller(A[1..n], x):
    if n = 0 then return 0
    max = LIS_smaller(A[1..n − 1], x)
    if A[n] < x then
        max = max {max, 1 + LIS_smaller(A[1..(n − 1)], A[n])}
    return max
```

Divide and conquer

Divide and conquer is an algorithm paradigm involving the decomposition of a problem into the same subproblem, solving them separately and combining their results to get a solution for the original problem.

Sorting algorithms

Algorithm	Runtime	Space
Mergesort	$O(n \log n)$	$O(n \log n)$ $O(n)$ (if optimized)
Quicksort	$O(n^2)$ $O(n \log n)$ if using MoM	$O(n)$

Karatsuba's algorithm

We can divide and conquer multiplication like so:

$$bc = 10^n b_L c_L + 10^{n/2}(b_L c_R + b_R c_L) + b_R c_R.$$

We can rewrite the equation as:

$$\begin{aligned} bc = b(x)c(x) = (b_L x + b_R)(c_L x + c_R) = (b_L c_L)x^2 \\ + ((b_L + b_R)(c_L + c_R) - b_L c_L - b_R c_R)\,x \\ + b_R c_R, \end{aligned}$$

Its running time is $O(n^{\log_2 3}) = O(n^{1.585})$.

Linear time selection

The *median of medians* (MoM) algorithms give a element that is larger than $\frac{3}{10}$'s and smaller than $\frac{7}{10}$'s of the array elements. This is used in the linear time selection algorithm to find element of rank k.

Pseudocode: Quickselect with median of medians

```
Median-of-medians (A, i):
    sublists = [A[j:j+5] for j ← 0, 5, . . . , len(A)]
    medians = [sorted (sublist)[len (sublist)/2]
              for sublist ∈ sublists]

    // Base case
    if len (A) ≤ 5 return sorted (a)[i]

    // Find median of medians
    if len (medians) ≤ 5
        pivot = sorted (medians)[len (medians)/2]
    else
        pivot = Median-of-medians (medians, len/2)

    // Partitioning step
    low = [j for j ∈ A if j < pivot]
    high = [j for j ∈ A if j > pivot]

    k = len (low)
    if i < k
        return Median-of-medians (low, i)
    else if i > k
        return Median-of-medians (low, i-k-1)
    else
    return pivot
```

Dynamic programming

Dynamic programming (DP) is the algorithm paradigm involving the computation of a recursive backtracking algorithm iteratively to avoid the recomputation of any particular subproblem.

Longest increasing subsequence

The longest increasing subsequence problem asks for the length of a longest increasing subsequence in a unordered sequence, where the sequence is assumed to be given as an array. The recurrence can be written as:

$$\mathit{LIS}(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ \mathit{LIS}(i-1,j) & \text{if } A[i] \geq A[j] \\ \max \begin{cases} \mathit{LIS}(i-1,j) \\ 1 + \mathit{LIS}(i-1,i) \end{cases} & \text{else} \end{cases}$$

Pseudocode: LIS - DP

LIS-Iterative$(A[1..n])$:
$A[n+1] = \infty$
for $j \leftarrow 0$ to n
if A[i] $\leq$ A[j] **then** $LIS[0][j] = 1$

for $i \leftarrow 1$ to $n-1$ **do**
for $j \leftarrow i$ to $n-1$ **do**
if $A[i] \geq A[j]$
$LIS[i,j] = LIS[i-1,j]$
else
$LIS[i,j] = \max\{LIS[i-1,j], 1 + LIS[i-1,i]\}$
return $LIS[n, n+1]$

Edit distance

The edit distance problem asks how many edits we need to make to a sequence for it to become another one. The recurrence is given as:

$$\mathrm{Opt}(i,j) = \min \begin{cases} \alpha_{x_i y_j} + \mathrm{Opt}(i-1,j-1), \\ \delta + \mathrm{Opt}(i-1,j), \\ \delta + \mathrm{Opt}(i,j-1) \end{cases}$$

Base cases: $\mathrm{Opt}(i,0) = \delta \cdot i$ and $\mathrm{Opt}(0,j) = \delta \cdot j$

Pseudocode: Edit distance - DP

$EDIST(A[1..m], B[1..n])$
for $i \leftarrow 1$ to m **do** $M[i,0] = i\delta$
for $j \leftarrow 1$ to n **do** $M[0,j] = j\delta$

for $i = 1$ to m **do**
for $j = 1$ to n **do**

$$M[i][j] = \min \begin{cases} COST\left[A[i]\right]\left[B[j]\right] \\ \quad + M[i-1][j-1], \\ \delta + M[i-1][j], \\ \delta + M[i][j-1] \end{cases}$$

2 Graph algorithms

Graph basics

A graph is defined by a tuple $G = (V, E)$ and we typically define $n = |V|$ and $m = |E|$. We define (u, v) as the edge from u to v. Graphs can be represented as **adjacency lists**, or **adjacency matrices** though the former is more commonly used.

- *path*: sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ (the number of edges in the path). *Note:* a single vertex u is a path of length 0.
- *cycle*: sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k-1$ and $(v_k, v_1) \in E$. A single vertex is not a cycle according to this definition.
Caveat: Sometimes people use the term cycle to also allow vertices to be repeated; we will use the term *tour*.
- A vertex u is *connected* to v if there is a path from u to v.
- The *connected component* of u, $\mathrm{con}(u)$, is the set of all vertices connected to u.
- A vertex u can *reach* v if there is a path from u to v. Alternatively v can be reached from u. Let **rch**(u) be the set of all vertices reachable from u.

Directed acyclic graphs

Directed acyclic graphs (dags) have an intrinsic ordering of the vertices that enables dynamic programming algorithms to be used on them.
A *topological ordering* of a dag $G = (V, E)$ is an ordering $\prec$ on V such that if $(u, v) \in E$ then $u \prec v$.

Pseudocode: Kahn's algorithm

Kahn$(G(V, E), u)$:
toposort $\leftarrow$ empty list
for $v \in V$:
  $\text{in}(v) \leftarrow |\{u \mid u \to v \in E\}|$
while $v \in V$ that has $\text{in}(v) = 0$:
  Add v to end of toposort
  Remove v from V
  for v in $u \to v \in E$:
    $\text{in}(v) \leftarrow \text{in}(v) - 1$
return toposort

Running time: $O(n + m)$

- A dag may have multiple topological sorts.
- A topological sort can be computed by DFS, in particular by listing the vertices in decreasing post-visit order.

DFS and BFS

Pseudocode: Explore (DFS/BFS)

Explore(G, u):
  for $i \leftarrow 1$ to n:
    Visited$[i] \leftarrow$ False
  Add u to ToExplore and to S
  Visited$[u] \leftarrow$ True
  Make tree T with root as u
  while B is non-empty **do**
    Remove node x from B
    for each edge (x, y) in $Adj(x)$ **do**
      if Visited$[y]$ = False
        Visited$[y] \leftarrow$ True
        Add y to B, S, T (with x as parent)

Note:

- If B is a queue, *Explore* becomes BFS.
- If B is a stack, *Explore* becomes DFS.

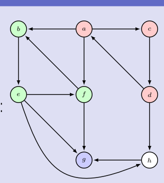
Pre/post numbering

Pre and post numbering aids in analyzing the graph structure. By looking at the numbering we can tell if a edge (u, v) is a:

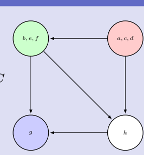
- *Forward edge*: $\text{pre}(u) < \text{pre}(v) < \text{post}(v) < \text{post}(u)$
- *Backward edge*: $\text{pre}(v) < \text{pre}(u) < \text{post}(u) < \text{post}(v)$
- *Cross edge*: $\text{pre}(u) < \text{post}(u) < \text{pre}(v) < \text{post}(v)$

Strongly connected components

- Given G, u is *strongly connected* to v if $v \in \text{rch}(u)$ *and* $u \in \text{rch}(v)$.
- A *maximal* group of G: vertices that are all strongly connected to one nother is called a strong component.



G^{SCC}



Pseudocode: Metagraph - linear time

Metagraph$(G(V, E))$:
  Compute rev(G) by brute force
  ordering $\leftarrow$ reverse postordering of V in rev(G)
    by **DFS**(rev(G), s) for any vertex s
  Mark all nodes as unvisited
  for each u in ordering **do**
    if u is not visited and $u \in V$ **then**
      $S_u \leftarrow$ nodes reachable by u by **DFS**(G, u)
      Output S_u as a strong connected component
      $G(V, E) \leftarrow G - S_u$

Shortest paths

Dijkstra's algorithm:
Find minimum distance from vertex s to **all** other vertices in graphs *without* negative weight edges.

Pseudocode: Dijkstra

for $v \in V$ **do**
  $d(v) \leftarrow \infty$
$X \leftarrow \varnothing$
$d(s, s) \leftarrow 0$
for $i \leftarrow 1$ to n **do**
  $v \leftarrow \arg\min_{u \in V - X} d(u)$
  $X = X \cup \{v\}$
  for u in Adj(v) **do**
    $d(u) \leftarrow \min\{(d(u),\ d(v) + \ell(v, u))\}$
return d

Running time: $O(m + n\log n)$ (if using a Fibonacci heap as the priority queue)

Bellman-Ford algorithm:
Find minimum distance from vertex s to **all** other vertices in graphs *without* negative cycles. It is a DP algorithm with the following recurrence:

$$d(v, k) = \begin{cases} 0 & \text{if } v = s \text{ and } k = 0 \\ \infty & \text{if } v \neq s \text{ and } k = 0 \\ \min \begin{cases} \min_{uv \in E} \{d(u, k-1) + \ell(u, v)\} \\ d(v, k-1) \end{cases} & \text{else} \end{cases}$$

Base cases: $d(s, 0) = 0$ and $d(v, 0) = \infty$ for all $v \neq s$.

Pseudocode: Bellman-Ford

for each $v \in V$ **do**
  $d(v) \leftarrow \infty$
$d(s) \leftarrow 0$

for $k \leftarrow 1$ to $n - 1$ **do**
  for each $v \in V$ **do**
    for each edge $(u, v) \in \text{in}(v)$ **do**
      $d(v) \leftarrow \min\{d(v), d(u) + \ell(u, v)\}$

return d

Running time: $O(nm)$

Floyd-Warshall algorithm:
Find minimum distance from *every* vertex to *every* vertex in a graph *without* negative cycles. It is a DP algorithm with the following recurrence:

$$d(i, j, k) = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } (i, j) \notin E \text{ and } k = 0 \\ \min \begin{cases} d(i, j, k-1) \\ d(i, k, k-1) + d(k, j, k-1) \end{cases} & \text{else} \end{cases}$$

Then $d(i, j, n-1)$ will give the shortest-path distance *from* i *to* j.

Pseudocode: Floyd-Warshall

Metagraph$(G(V, E))$:
  for $i \in V$ **do**
    for $j \in V$ **do**
      $d(i, j, 0) \leftarrow \ell(i, j)$
      (* $\ell(i, j) \leftarrow \infty$ if $(i, j) \notin E$, 0 if $i = j$ *)

  for $k \leftarrow 0$ to $n - 1$ **do**
    for $i \in V$ **do**
      for $j \in V$ **do**
        $d(i, j, k) \leftarrow \min \begin{cases} d(i, j, k-1), \\ d(i, k, k-1) + d(k, j, k-1) \end{cases}$
  for $v \in V$ **do**
    if $d(i, i, n-1) < 0$ **then**
      return "$\exists$ negative cycle in G"

  return $d(\cdot, \cdot, n-1)$

Running time: $\Theta(n^3)$