

The algorithms portion of the course will require you to evaluate mathematical expressions. Recursive algorithms have a natural synergy with recurrence relations and hence for this course, you will be expected to solve simple recurrence relations.

1.  $T(n) = 2T(n - 1) + cn$
2.  $T(n) = 2T\left(\frac{n}{2}\right) + cn$
3.  $T(n) = 2T\left(\frac{n}{4}\right) + cn$

Recurrence relations are a fascinating and area of discrete mathematics. Feel free to explore more advanced equations and solving techniques!<sup>1</sup>

Here are several problems that are easy to solve in  $O(n)$  time, essentially by brute force. Your task is to design algorithms for these problems that are significantly faster using binary search related ideas.

1. Suppose we are given an array  $A[1..n]$  of  $n$  distinct integers, which could be positive, negative, or zero, sorted in increasing order so that  $A[1] < A[2] < \dots < A[n]$ .
  - (a) Describe a fast algorithm that either computes an index  $i$  such that  $A[i] = i$  or correctly reports that no such index exists.
  - (b) Formulate a recurrence relation that describes your algorithm.
  - (c) Suppose we know in advance that  $A[1] > 0$ . Describe an even faster algorithm that either computes an index  $i$  such that  $A[i] = i$  or correctly reports that no such index exists. [*Hint: This is really easy.*]
2. Suppose we are given an array  $A[1..n]$  of  $n$  integers, which could be positive, negative, or zero, sorted in increasing order so that  $A[1] \leq A[2] \leq \dots \leq A[n]$ . Suppose we wanted to count the number of times some integer value  $x$  occurs in  $A$ . Describe an algorithm (as fast as possible) which returns the number of elements containing value  $x$ .
3. Suppose we are given an array  $A[1..n]$  such that  $A[1] \geq A[2]$  and  $A[n-1] \leq A[n]$ . We say that an element  $A[x]$  is a **local minimum** if both  $A[x-1] \geq A[x]$  and  $A[x] \leq A[x+1]$ . For example, there are exactly six local minima in the following array:



Describe and analyze a fast algorithm that returns the index of one local minimum. For example, given the array above, your algorithm could return the integer 9, because  $A[9]$  is a local minimum. [*Hint: With the given boundary conditions, any array must contain at least one local minimum. Why?*]

4. Suppose you are given two sorted arrays  $A[1..n]$  and  $B[1..n]$  containing distinct integers. Describe a fast algorithm to find the median (meaning the  $n$ th smallest element) of the union  $A \cup B$ . For example, given the input

$$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \quad B[1..8] = [2, 4, 5, 8, 17, 19, 21, 23]$$

<sup>1</sup>[http://discrete.openmathbooks.org/dmoi2/sec\\_recurrence.html](http://discrete.openmathbooks.org/dmoi2/sec_recurrence.html)

your algorithm should return the integer 9. [Hint: What can you learn by comparing one element of  $A$  with one element of  $B$ ?]

**To think about later:**

4. Now suppose you are given two sorted arrays  $A[1..m]$  and  $B[1..n]$  and an integer  $k$ . Describe a fast algorithm to find the  $k$ th smallest element in the union  $A \cup B$ . For example, given the input

$$A[1..8] = [0, 1, 6, 9, 12, 13, 18, 20] \quad B[1..5] = [2, 5, 7, 17, 19] \quad k = 6$$

your algorithm should return the integer 7.

5. Suppose you have an algorithm that given as input a directed graph  $G = (V, E)$ , nodes  $s, t \in V$ , and an integer  $k$ , outputs whether the *number* of distinct shortest paths from  $s$  to  $t$  is at least  $k$ . Describe an algorithm that counts the number of distinct shortest  $s$ - $t$  paths in  $G$ . Does your algorithm run in polynomial time?