1. This is to help you recall Boolean formulae. A Boolean function f over r variables $a_1, a_2, \ldots, a_r$ is a function $f : \{0,1\}^r \to \{0,1\}$ which assigns 0 or 1 to each possible assignment of values to the variables. One can specify a Boolean function in several ways including a truth table. Here is a truth table for a function on 3 variables a_1, a_2, a_3.

a_1	a_2	a_3	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Suppose we are given a Boolean function on r variables $a_1, a_2, \ldots, a_r$ via a truth table. We wish to express f as a CNF formula using variables $a_1, a_2, \ldots, a_r$.

It may be easier to first think about expressing using a DNF formula (a disjunction of one more conjunctions of a set of literals). For instance the function above can be expressed as

$$(\bar{a}_1 \wedge \bar{a}_2 \wedge a_3) \vee (\bar{a}_1 \wedge a_2 \wedge \bar{a}_3) \vee (a_1 \wedge \bar{a}_2 \wedge a_3) \vee (a_1 \wedge a_2 \wedge \bar{a}_3) \vee (a_1 \wedge a_2 \wedge a_3).$$

- What is a CNF formula for the function? *Hint:* Think of the complement function and complement the DNF formula.
- Describe how one can express an arbitrary Boolean function f over r variables as a CNF formula over the variables using at most 2^r clauses.

Solution: We consider the Boolean function $\bar{f}$ which is the complement of f. We can express $\bar{f}$ in DNF form using at most 2^r terms. We then complement the resulting DNF formula to obtain our desired CNF formula which has at most 2^r clauses.

For the example function we obtain a DNF formula for $\bar{f}$ as

$$(\bar{a}_1 \wedge \bar{a}_2 \wedge \bar{a}_3) \vee (\bar{a}_1 \wedge a_2 \wedge a_3) \vee (a_1 \wedge \bar{a}_2 \wedge \bar{a}_3).$$

Thus the CNF formula for f is obtained by complementing this DNF formula and we obtain:

$$(a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee \bar{a}_2 \vee \bar{a}_3) \wedge (\bar{a}_1 \vee a_2 \vee a_3).$$

■

2. A *Hamiltonian cycle* in a graph G is a cycle that goes through every vertex of G exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

   A ***tonian cycle*** in a graph G is a cycle that goes through at least *half* of the vertices of G. Prove that deciding whether a graph contains a tonian cycle is NP-hard.

Solution (duplicate the graph): I'll describe a polynomial-time reduction from HamiltonianCycle. Let G be an arbitrary graph. Let H be a graph consisting of two disjoint copies of G, with no edges between them; call these copies G_1 and G_2. I claim that G has a Hamiltonian cycle if and only if H has a tonian cycle.

$\Longrightarrow$ Suppose G has a Hamiltonian cycle C. Let C_1 be the corresponding cycle in G_1. C_1 contains exactly half of the vertices of H, and thus is a tonian cycle in H.

$\Longleftarrow$ On the other hand, suppose H has a tonian cycle C. Because there are no edges between the subgraphs G_1 and G_2, this cycle must lie entirely within one of these two subgraphs. G_1 and G_2 each contain exactly half the vertices of H, so C must also contain exactly half the vertices of H, and thus is a *Hamiltonian* cycle in either G_1 or G_2. But G_1 and G_2 are just copies of G. We conclude that G has a Hamiltonian cycle.

Given G, we can construct H in polynomial time easily. ■

Solution (add n new vertices): I'll describe a polynomial-time reduction from HamiltonianCycle. Let G be an arbitrary graph, and suppose G has n vertices. Let H be a graph obtained by adding n new vertices to G, but no additional edges. I claim that G has a Hamiltonian cycle if and only if H has a tonian cycle.

$\Longrightarrow$ Suppose G has a Hamiltonian cycle C. Then C visits exactly half the vertices of H, and thus is a tonian cycle in H.

$\Longleftarrow$ On the other hand, suppose H has a tonian cycle C. This cycle cannot visit any of the new vertices, so it must lie entirely within the subgraph G. Since G contains exactly half the vertices of H, the cycle C must visit every vertex of G, and thus is a Hamiltonian cycle in G.

Given G, we can construct H in polynomial time easily. ■

3. *Big Clique* is the following decision problem: given a graph $G = (V, E)$, does G have a clique of size at least $n/2$ where $n = |V|$ is the number of nodes? Prove that *Big Clique* is NP-hard.

Solution: Recall that an instance of CLIQUE consists of a graph $G = (V, E)$ and integer k. (G, k) is a YES instance if G has a clique of size at least k, otherwise it is a NO instance. For simplicity we will assume n is an even number.

We describe a polynomial-time reduction from CLIQUE to BIG CLIQUE. We consider two cases depending on whether $k \leq n/2$ or not. If $k \leq n/2$ we obtain a graph $G' = (V', E')$ as follows. We add a set of X new vertices where $|X| = n - 2k$; thus $V' = V \uplus X$. We make X a clique by adding all possible edges between vertices of X. In addition we connect each vertex $v \in X$ to each vertex $u \in V$. In other words $E' = E \cup \{(u, v) \mid u \in V, v \in X\} \cup \{(a, b) \mid a, b \in X\}$. If $k > n/2$ we let $G' = (V', E')$ where $V' = V \uplus X$ and $E' = E$, where $|X| = 2k - n$. In other words we add $2k - n$ new vertices which are isolated and have no edges incident on them.

We make the following relatively easy claims that we leave as exercises.

Claim 1. *Suppose $k \leq n/2$. Then for any clique S in G, $S \cup X$ is a clique in G'. For any clique $S' \in G'$ the set $S' \setminus X$ is a clique in G.*

Claim 2. *Suppose $k > n/2$. Then S is a clique in G' iff $S \cap X = \emptyset$ and S is a clique in G.*

Now we prove the correctness of the reduction. We need to show that G has a clique of size k if and only if G' has a clique of size $n'/2$ where n' is the number of nodes in G'.

$\Longrightarrow$ Suppose G has a clique S of size k. We consider two cases. If $k > n/2$ then $n' = n + 2k - n = 2k$; note that S is a clique in G' as well and hence S is a big clique in G' since $|S| = k \geq n'/2$. If $k \leq n/2$, by the first claim, $S \cup X$ is a clique in G' of size $k + |X| = k + n - 2k = n - k$. Moreover, $n' = n + n - 2k = 2n - 2k$ and hence $S \cup X$ is a big clique in G'. Thus, in both cases G' has a big clique.

$\Longleftarrow$ Suppose G' has a clique of size at least $n'/2$ in G'. Let it be S'; $|S'| \geq n'/2$. We consider two cases again. If $k \leq n/2$, we have $n' = 2n - 2k$ and $|S'| \geq n - k$. By the first claim, $S = S' \setminus X$ is a clique in G. $|S| \geq |S'| - |X| \geq n - k - (n - 2k) \geq k$. Hence G has a clique of size k. If $k > n/2$, by the second claim S' is a clique in G and $|S'| \geq n'/2 = (n + 2k - n)/2 = k$. Therefore, in this case as well G has a clique of size k.

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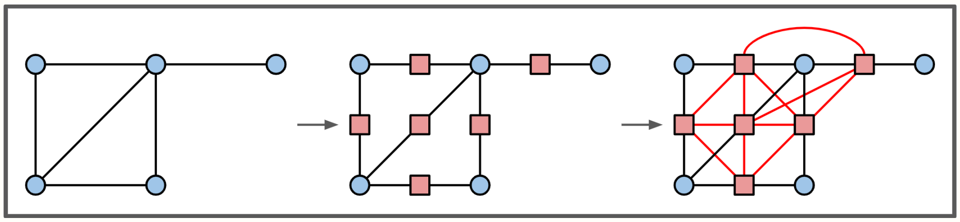
4. A *strongly independent* set is a subset of vertices S in a graph G such that for any two vertices in S, there is no path of length two in G. Prove that *Strongly Independent Set* is NP-hard.

Solution: To show that strongly independent set is NP-hard we do a reduction from independent set.

For any graph G we construct a new graph H where we add vertices that correspond to each edge, these edge vertices are connected to the vertices that the original edges were connected to and to any edge vertex where the original edges shared a vertex.



$\rightarrow$ Let G have an independent set X, then X is a strongly independent set in H. If u, v in G are 2 length away, then the path in H is u connects to the edge vertex of u, which connects to the edge vertex of v which connects to v, so u, v are 3 length away in H. Therefore if G has an independent set of size k, then H has a strongly independent set of size k.

$\leftarrow$ Let H have a strongly independent set Y. Then let w be an edge vertex in Y where the original edge in G connects to vertices a, b. The path from any vertex z (not a) to w is less than or equal to the path from z to a (or b). This is because a connects only to the edge vertices that represent the edges a connects to in G, but these edge vertices that connect to a are all connected to one another. So any path to a has to go through one of these edge vertices. If this edge vertex is w then the path to w is 1 shorter then the path to a. If this edge vertex is not w then this path could go to w instead of a next so the path to w is the same as the path to a. This mean for every edge vertex in Y we can swap it out for a vertex that the edge connected to in G and preserve strong independence. Let Y' be the strongly independent set obtained from swapping out all of the edge vertices. Y' is an independent set in G because if all of the paths from u to v in Y' are greater than 2 in H then u and v cannot be adjacent in G. Therefore if H has a strongly independent set of size k, then G has an independent set of size k.

We have proved both directions in the reduction and constructing the new graph is polynomial time. Therefore because independent set is NP-hard, strongly independent set must also be NP-hard.

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5. Recall the following kColor problem: Given an undirected graph G, can its vertices be colored with k colors, so that every edge touches vertices with two different colors?

   (a) Describe a direct polynomial-time reduction from 3Color to 4Color.

Solution: Suppose we are given an arbitrary graph G. Let H be the graph obtained from G by adding a new vertex a (called an *apex*) with edges to every vertex of G. I claim that G is 3-colorable if and only if H is 4-colorable.

$\Longrightarrow$ Suppose G is 3-colorable. Fix an arbitrary 3-coloring of G, and call the colors "red", "green", and "blue". Assign the new apex a the color "plaid". Let uv be an arbitrary edge in H.

- If both u and v are vertices in G, they have different colors.
- Otherwise, one endpoint of uv is plaid and the other is not, so u and v have different colors.

We conclude that we have a valid 4-coloring of H, so H is 4-colorable.

$\Longleftarrow$ Suppose H is 4-colorable. Fix an arbitrary 4-coloring; call the apex's color "plaid" and the other three colors "red", "green", and "blue". Each edge uv in G is also an edge of H and therefore has endpoints of two different colors. Each vertex v in G is adjacent to the apex and therefore cannot be plaid. We conclude that by deleting the apex, we obtain a valid 3-coloring of G, so G is 3-colorable.

We can easily transform G into H in polynomial time by brute force. ■

(b) Prove that kColor problem is NP-hard for any $k \geq 3$.

Solution (direct): The lecture notes include a proof that 3Color is NP-hard. For any integer $k > 3$, I'll describe a direct polynomial-time reduction from 3Color to kColor.

Let G be an arbitrary graph. Let H be the graph obtain from G by adding $k-3$ new vertices $a_1, a_2, \ldots, a_{k-3}$, each with edges to every other vertex in H (including the other a_i's). I claim that G is 3-colorable if and only if H is k-colorable.

$\Longrightarrow$ Suppose G is 3-colorable. Fix an arbitrary 3-coloring of G. Color the new vertices $a_1, a_2, \ldots, a_{k-3}$ with $k-3$ new distinct colors. Every edge in H is either an edge in G or uses at least one new vertex a_i; in either case, the endpoints of the edge have different colors. We conclude that H is k-colorable.

$\Longleftarrow$ Suppose H is k-colorable. Each vertex a_i is adjacent to every other vertex in H, and therefore is the only vertex of its color. Thus, the vertices of G use only three distinct colors. Every edge of G is also an edge of H, so its endpoints have different colors. We conclude that the induced coloring of G is a proper 3-coloring, so G is 3-colorable.

Given G, we can construct H in polynomial time by brute force. ■

Solution (induction): Let k be an arbitrary integer with $k \geq 3$. Assume that jColor is NP-hard for any integer $3 \leq j < k$. There are two cases to consider.

- If $k = 3$, then kColor is NP-hard by the reduction from 3Sat in the lecture notes.
- Suppose $k = 3$. The reduction in part (a) directly generalizes to a polynomial-time reduction from $(k-1)$Color to kColor: To decide whether an arbitrary graph G is $(k-1)$-colorable, add an apex and ask whether the resulting graph is k-colorable. The induction hypothesis implies that $(k-1)$Color is NP-hard, so the reduction implies that kColor is NP-hard.

In both cases, we conclude that kColor is NP-hard. ■

To think about later:

6. Let G be an undirected graph with weighted edges. A Hamiltonian cycle in G is ***heavy*** if the total weight of edges in the cycle is at least half of the total weight of all edges in G. Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.

Solution (two new vertices): I'll describe a polynomial-time a reduction from the Hamiltonian ***path*** problem. Let G be an arbitrary undirected graph (without edge weights). Let H be the edge-weighted graph obtained from G as follows:

- Add two new vertices s and t.
- Add edges from s and t to all the other vertices (including each other).
- Assign weight 1 to the edge st and weight 0 to every other edge.

The total weight of all edges in H is 1. Thus, a Hamiltonian cycle in H is heavy if and only if it contains the edge st. I claim that H contains a heavy Hamiltonian cycle if and only if G contains a Hamiltonian path.

$\Longrightarrow$ First, suppose G has a Hamiltonian path from vertex u to vertex v. By adding the edges vs, st, and tu to this path, we obtain a Hamiltonian cycle in H. Moreover, this Hamiltonian cycle is heavy, because it contains the edge st.

$\Longleftarrow$ On the other hand, suppose H has a heavy Hamiltonian cycle. This cycle must contain the edge st, and therefore must visit all the other vertices in H contiguously. Thus, deleting vertices s and t and their incident edges from the cycle leaves a Hamiltonian path in G.

Given G, we can easily construct H in polynomial time by brute force. ■

Solution (smartass): I'll describe a polynomial-time a reduction from the standard Hamiltonian cycle problem. Let G be an arbitrary graph (without edge weights). Let H be the edge-weighted graph obtained from G by assigning each edge weight 0. I claim that H contains a heavy Hamiltonian cycle if and only if G contains a Hamiltonian path.

$\Longrightarrow$ Suppose G has a Hamiltonian cycle C. The total weight of C is at least half the total weight of all edges in H, because $0 \geq 0/2$. So C is a heavy Hamiltonian cycle in H.

$\Longleftarrow$ Suppose H has a heavy Hamiltonian cycle C. By definition, C is also a Hamiltonian cycle in G.

Given G, we can easily construct H in polynomial time by brute force. ■