

Prove that each of the following problems is NP-hard.

1. Given an undirected graph  $G$ , does  $G$  contain a simple path that visits all but 374 vertices?

**Solution:** We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph  $G$ , let  $H$  be the graph obtained from  $G$  by adding 374 isolated vertices. Call a path in  $H$  **almost-Hamiltonian** if it visits all but 374 vertices. I claim that  $G$  contains a Hamiltonian path if and only if  $H$  contains an almost-Hamiltonian path.

$\Rightarrow$  Suppose  $G$  has a Hamiltonian path  $P$ . Then  $P$  is an almost-Hamiltonian path in  $H$ , because it misses only the 374 isolated vertices.

$\Leftarrow$  Suppose  $H$  has an almost-Hamiltonian path  $P$ . This path must miss all 374 isolated vertices in  $H$ , and therefore must visit every vertex in  $G$ . Every edge in  $H$ , and therefore every edge in  $P$ , is also an edge in  $G$ . We conclude that  $P$  is a Hamiltonian path in  $G$ .

Given  $G$ , we can easily build  $H$  in polynomial time by brute force. ■

2. Given an undirected graph  $G$ , does  $G$  have a spanning tree in which every node has degree at most 374?

**Solution:** We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph  $G$ , let  $H$  be the graph obtained by attaching a fan of 372 edges to every vertex of  $G$ . Call a spanning tree of  $H$  **almost-Hamiltonian** if it has maximum degree 374. I claim that  $G$  contains a Hamiltonian path if and only if  $H$  contains an almost-Hamiltonian spanning tree.

$\Rightarrow$  Suppose  $G$  has a Hamiltonian path  $P$ . Let  $T$  be the spanning tree of  $H$  obtained by adding every fan edge in  $H$  to  $P$ . Every vertex  $v$  of  $H$  is either a leaf of  $T$  or a vertex of  $P$ . If  $v \in P$ , then  $\deg_P(v) \leq 2$ , and therefore  $\deg_H(v) = \deg_P(v) + 372 \leq 374$ . We conclude that  $H$  is an almost-Hamiltonian spanning tree.

$\Leftarrow$  Suppose  $H$  has an almost-Hamiltonian spanning tree  $T$ . The leaves of  $T$  are precisely the vertices of  $H$  with degree 1; these are also precisely the vertices of  $H$  that are not vertices of  $G$ . Let  $P$  be the subtree of  $T$  obtained by deleting every leaf of  $T$ . Observe that  $P$  is a spanning tree of  $G$ , and for every vertex  $v \in P$ , we have  $\deg_P(v) = \deg_T(v) - 372 \leq 2$ . We conclude that  $P$  is a Hamiltonian path in  $G$ .

Given  $G$ , we can easily build  $H$  in polynomial time by brute force. ■

3. Given an undirected graph  $G$ , does  $G$  have a spanning tree with at most 374 leaves?

**Solution:** We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem.<sup>a</sup> Given an arbitrary graph  $G$ , let  $H$  be the graph obtained from  $G$  by adding the following vertices and edges:

- First we add a vertex  $z$  with edges to every other vertex in  $z$ .
- Then we add 373 vertices  $\ell_1, \dots, \ell_{373}$ , each with edges to  $t$  and nothing else.

Call a spanning tree of  $H$  **almost-Hamiltonian** if it has at most 374 leaves. I claim that  $G$  contains a Hamiltonian path if and only if  $H$  contains an almost-Hamiltonian spanning tree.

$\Rightarrow$  Suppose  $G$  has a Hamiltonian path  $P$ . Suppose  $P$  starts at vertex  $s$  and ends at vertex  $t$ . Let  $T$  be subgraph of  $H$  obtained by adding the edge  $tz$  and all possible edges  $z\ell_i$ . Then  $T$  is a spanning tree of  $H$  with exactly 374 leaves, namely  $s$  and all 373 new vertices  $\ell_i$ .

$\Leftarrow$  Suppose  $H$  has an almost-Hamiltonian spanning tree  $T$ . Every node  $\ell_i$  is a leaf of  $T$ , so  $T$  must consist of the 373 edges  $z\ell_i$  and a simple path from  $z$  to some vertex  $s$  of  $G$ . Let  $t$  be the only neighbor of  $z$  in  $T$  that is not a leaf  $\ell_i$ , and let  $P$  be the unique path in  $T$  from  $s$  to  $t$ . This path visits every vertex of  $G$ ; in other words,  $P$  is a Hamiltonian path in  $G$ .

Given  $G$ , we can easily build  $H$  in polynomial time by brute force. ■

<sup>a</sup>Are you noticing a pattern here?

4. Recall that a 5-coloring of a graph  $G$  is a function that assigns each vertex of  $G$  a “color” from the set  $\{0, 1, 2, 3, 4\}$ , such that for any edge  $uv$ , vertices  $u$  and  $v$  are assigned different “colors”. A 5-coloring is *careful* if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

**Solution:** We prove that careful 5-coloring is NP-hard by reduction from the standard 5COLOR problem.

Given a graph  $G$ , we construct a new graph  $H$  by replacing each edge in  $G$  with a path of length three. I claim that  $H$  has a careful 5-coloring if and only if  $G$  has a (not necessarily careful) 5-coloring.

$\Leftarrow$  Suppose  $G$  has a 5-coloring. Consider a single edge  $uv$  in  $G$ , and suppose  $color(u) = a$  and  $color(v) = b$ . We color the path from  $u$  to  $v$  in  $H$  as follows:

- If  $b = (a + 1) \pmod{5}$ , use colors  $(a, (a + 2) \pmod{5}, (a - 1) \pmod{5}, b)$ .
- If  $b = (a - 1) \pmod{5}$ , use colors  $(a, (a - 2) \pmod{5}, (a + 1) \pmod{5}, b)$ .
- Otherwise, use colors  $(a, b, a, b)$ .

In particular, every vertex in  $G$  retains its color in  $H$ . The resulting 5-coloring of  $H$  is careful.

$\Rightarrow$  On the other hand, suppose  $H$  has a careful 5-coloring. Consider a path  $(u, x, y, v)$  in  $H$  corresponding to an arbitrary edge  $uv$  in  $G$ . There are exactly eight careful colorings of this path with  $color(u) = 0$ , namely:  $(0, 2, 0, 2)$ ,  $(0, 2, 0, 3)$ ,  $(0, 2, 4, 1)$ ,  $(0, 2, 4, 2)$ ,  $(0, 3, 0, 3)$ ,  $(0, 3, 0, 2)$ ,  $(0, 3, 1, 3)$ ,  $(0, 3, 1, 4)$ . It follows immediately that  $color(u) \neq color(v)$ . Thus, if we color each vertex of  $G$  with its color in  $H$ , we obtain a valid 5-coloring of  $G$ .

Given  $G$ , we can clearly construct  $H$  in polynomial time. ■

5. Prove that the following problem is NP-hard: Given an undirected graph  $G$ , find *any* integer  $k > 374$  such that  $G$  has a proper coloring with  $k$  colors but  $G$  does not have a proper coloring with  $k - 374$  colors.

**Solution:** Let  $G'$  be the union of 374 copies of  $G$ , with additional edges between *every* vertex of each copy and *every* vertex in *every* other copy. Given  $G$ , we can easily build  $G'$  in polynomial time by brute force. Let  $\chi(G)$  and  $\chi(G')$  denote the minimum number of colors in any proper coloring of  $G$ , and define  $\chi(G')$  similarly.

$\implies$  Fix any coloring of  $G$  with  $\chi(G)$  colors. We can obtain a proper coloring of  $G'$  with  $374 \cdot \chi(G)$  colors, by using a distinct set of  $\chi(G)$  colors in each copy of  $G$ . Thus,  $\chi(G') \leq 374 \cdot \chi(G)$ .

$\impliedby$  Now fix any coloring of  $G'$  with  $\chi(G')$  colors. Each copy of  $G$  in  $G'$  must use its own distinct set of colors, so at least one copy of  $G$  uses at most  $\lfloor \chi(G')/374 \rfloor$  colors. Thus,  $\chi(G) \leq \lfloor \chi(G')/374 \rfloor$ .

These two observations immediately imply that  $\chi(G') = 374 \cdot \chi(G)$ . It follows that if  $k$  is an integer such that  $k - 374 < \chi(G') \leq k$ , then  $\chi(G) = \chi(G')/374 = \lceil k/374 \rceil$ . Thus, if we could compute such an integer  $k$  in polynomial time, we could compute  $\chi(G)$  in polynomial time. But computing  $\chi(G)$  is NP-hard! ■

6. A **bicoloring** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.
- (a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

**Solution:** It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3COLOR problem.

Let  $G$  be an arbitrary undirected graph. I claim that  $G$  has a proper 3-coloring if and only if  $G$  has a weak bicoloring with 3 colors.

- Suppose  $G$  has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of  $G$  using only the colors cyan, magenta, and yellow by recoloring each red vertex with {magenta, yellow}, recoloring each blue vertex with {magenta, cyan}, and recoloring each green vertex with {yellow, cyan}.
- Suppose  $G$  has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of  $G$  by defining red = {magenta, yellow}, defining blue = {magenta, cyan}, and defining green = {yellow, cyan}.

More generally, for any integer  $k$  and any graph  $G$ , every weak  $k$ -bicoloring of  $G$  is also a proper  $\binom{k}{2}$ -coloring of  $G$ , and vice versa. ■

- (b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

**Solution:** It suffices to prove that deciding whether a graph has a strong bicoloring with six colors is NP-hard, using the following reduction from the standard 3COLOR problem.

Let  $G$  be an arbitrary undirected graph. We build a new graph  $H$  from  $G$  as follows:

- For every vertex  $v$  in  $G$ , the graph  $H$  contains three vertices  $v_1, v_2$ , and  $v_3$  and three edges  $v_1v_2, v_2v_3$ , and  $v_3v_1$ .
- For every edge  $uv$  in  $G$ , the graph  $H$  contains three edges  $u_1v_1, u_2v_2$ , and  $u_3v_3$ .

I claim that  $G$  has a proper 3-coloring if and only if  $H$  has a strong bicoloring with six colors. Without loss of generality, we can assume that  $G$  (and therefore  $H$ ) is connected; otherwise, consider each component independently.

⇒ Suppose  $G$  has a proper 3-coloring with colors red, green, and blue. Then we define a strong bicoloring of  $H$  with colors 1, 2, 3, 4, 5, 6 as follows:

- For every red vertex  $v$  in  $G$ , let  $color(v_1) = \{1, 2\}$  and  $color(v_2) = \{3, 4\}$  and  $color(v_3) = \{5, 6\}$ .
- For every blue vertex  $v$  in  $G$ , let  $color(v_1) = \{3, 4\}$  and  $color(v_2) = \{5, 6\}$  and  $color(v_3) = \{1, 2\}$ .
- For every green vertex  $v$  in  $G$ , let  $color(v_1) = \{5, 6\}$  and  $color(v_2) = \{1, 2\}$  and  $color(v_3) = \{3, 4\}$ .

Exhaustive case analysis confirms that every pair of adjacent vertices of  $H$  has disjoint color sets.

- Suppose  $H$  has a strong bicoloring with six colors. Fix an arbitrary vertex  $v$  in  $G$ , and without loss of generality, suppose  $color(v_1) = \{1, 2\}$  and  $color(v_2) = \{3, 4\}$  and  $color(v_3) = \{5, 6\}$ . Exhaustive case analysis implies that for any edge  $uv$ , each vertex  $u_i$  must be colored either  $\{1, 2\}$  or  $\{3, 4\}$  or  $\{5, 6\}$ . It follows by induction that every vertex in  $H$  must be colored either  $\{1, 2\}$  or  $\{3, 4\}$  or  $\{5, 6\}$ .

Now for each vertex  $w$  in  $G$ , color  $w$  red if  $color(w_1) = \{1, 2\}$ , blue if  $color(w_2) = \{3, 4\}$ , and green if  $color(w_3) = \{5, 6\}$ . This assignment of colors is a proper 3-coloring of  $G$ .

Given  $G$ , we can build  $H$  in polynomial time by brute force. ■

*I believe that deciding whether a graph has a strong bicoloring with five colors is also NP-hard, but I don't have a proof yet. A graph has a strong bicoloring with four colors if and only if it is bipartite, and a strong bicoloring with two or three colors if and only if it has no edges.*