Bellman-Ford and Dynamic Programming on Graphs

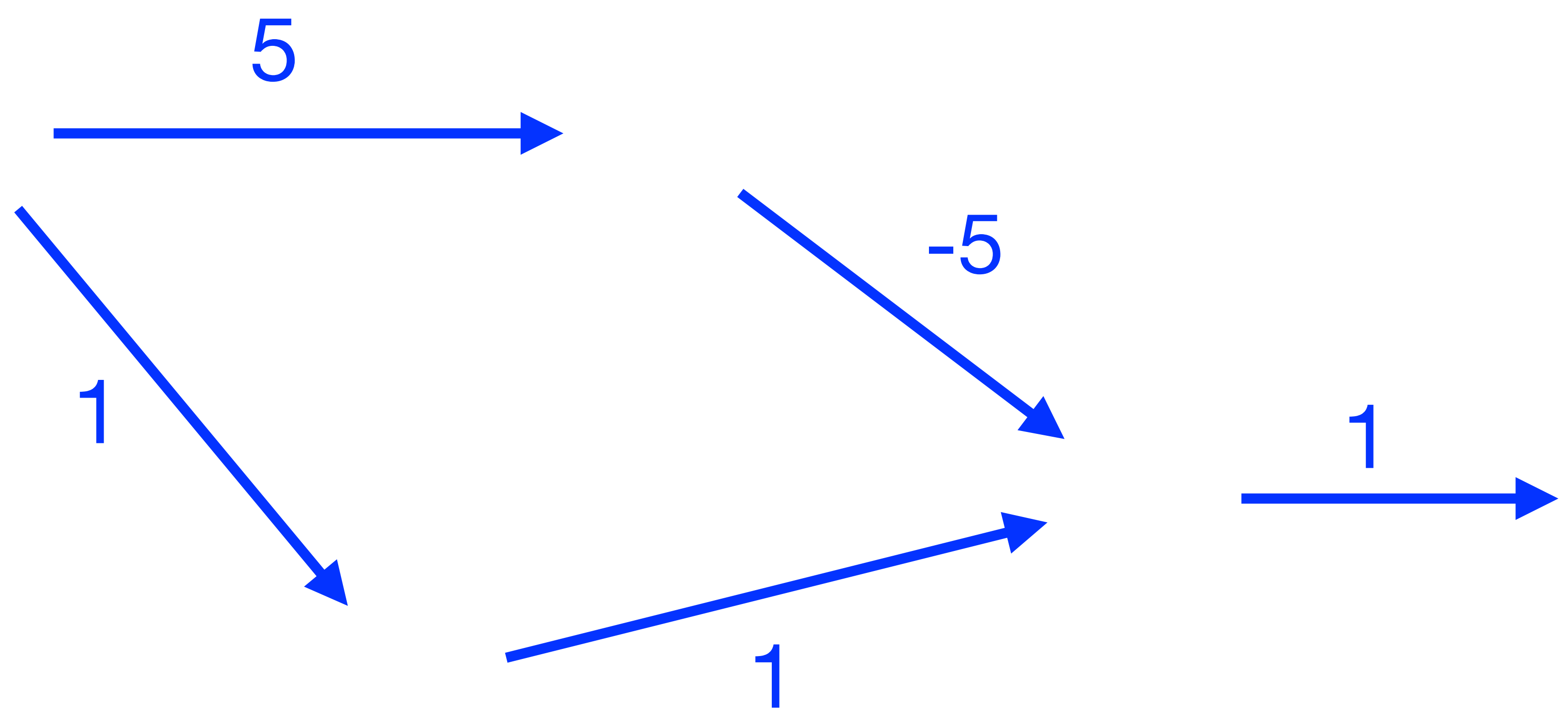
Sides based on material by Kani, Erickson, Chekuri, et. al.

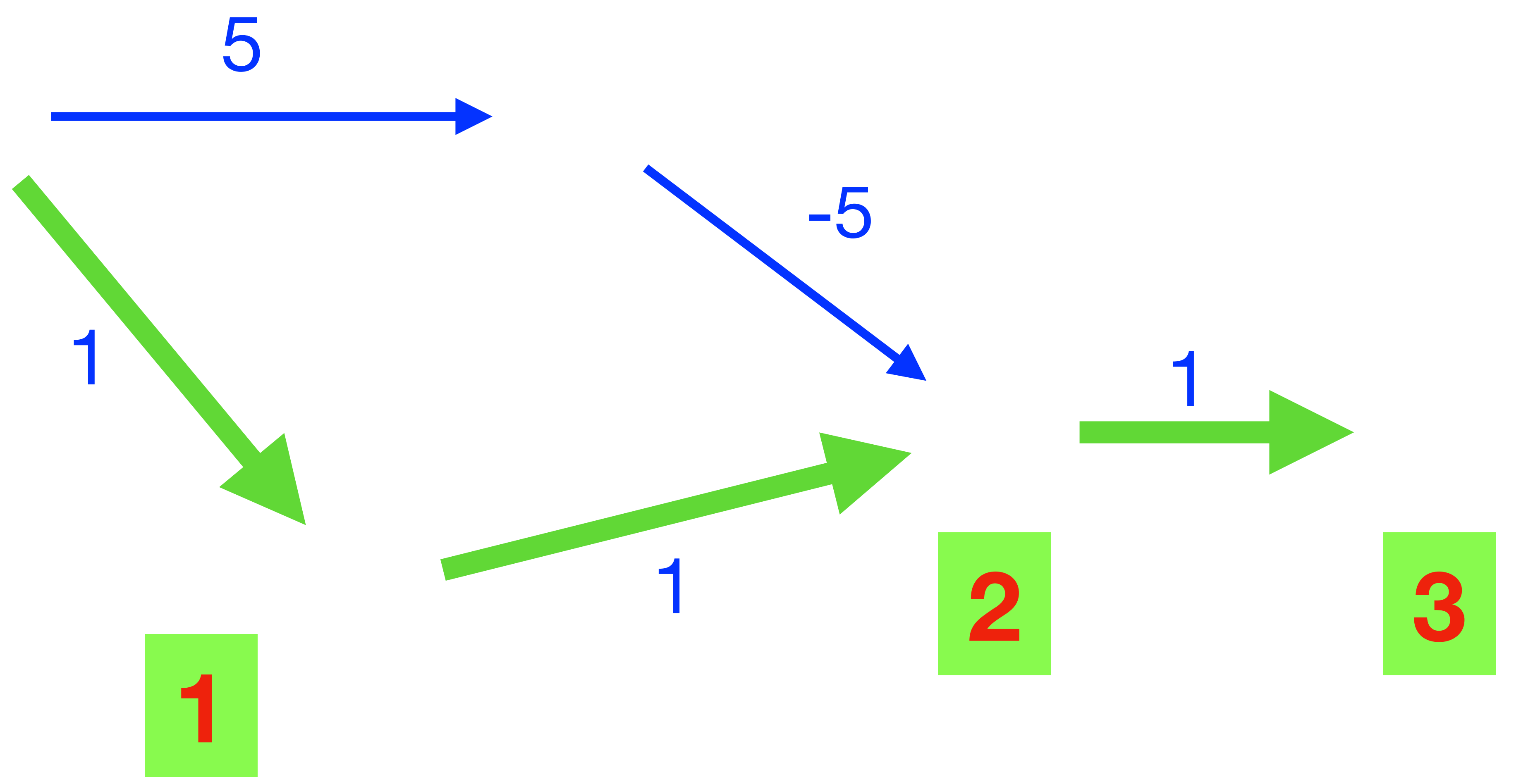
All mistakes are my own! - Ivan Abraham (Fall 2024)

Image by ChatGPT (probably collaborated with DALL-E)

Why Dijkstra’s algorithm fails with negative edges

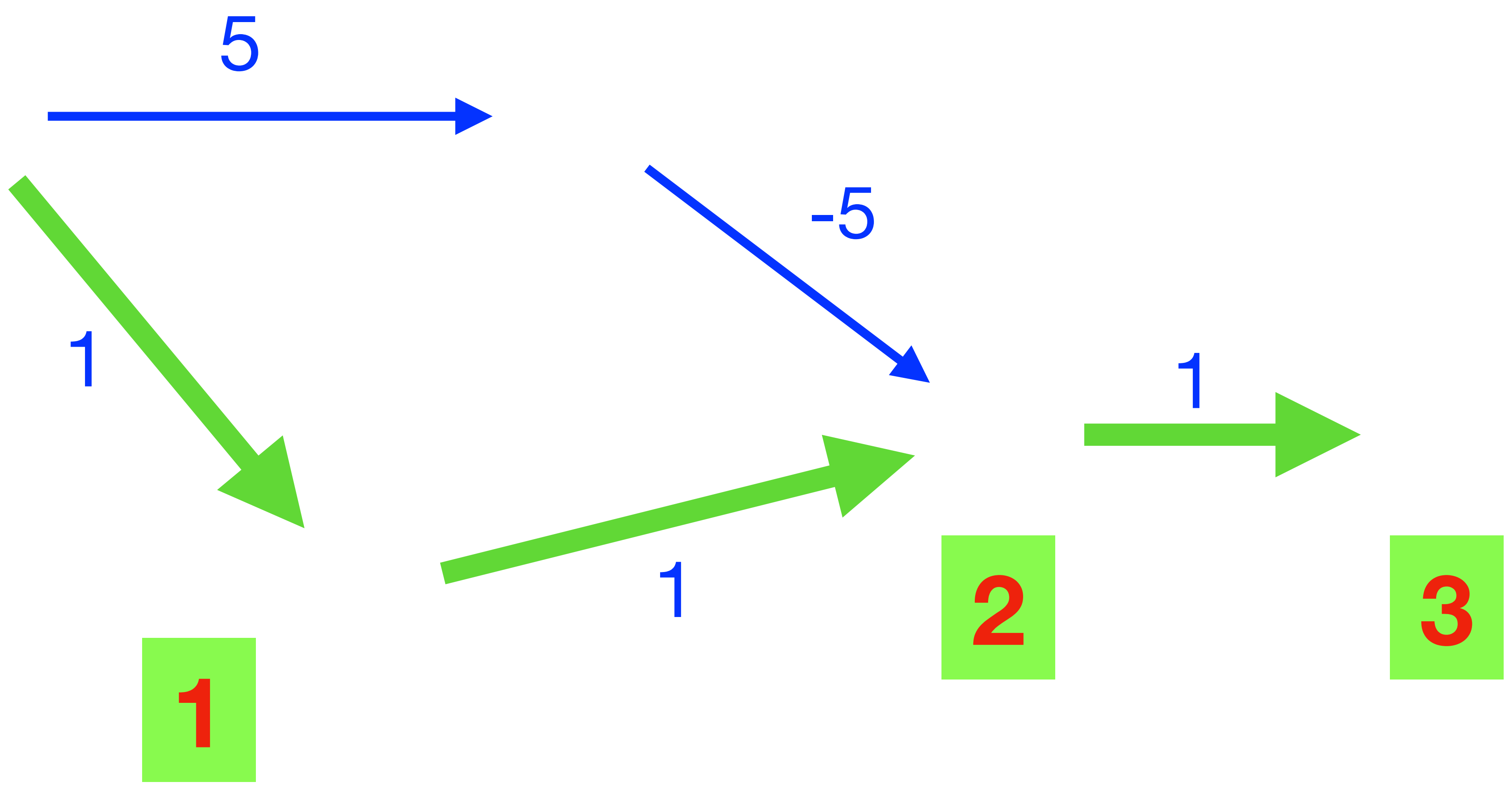
What are the distances computed by Dijkstra's algorithm?

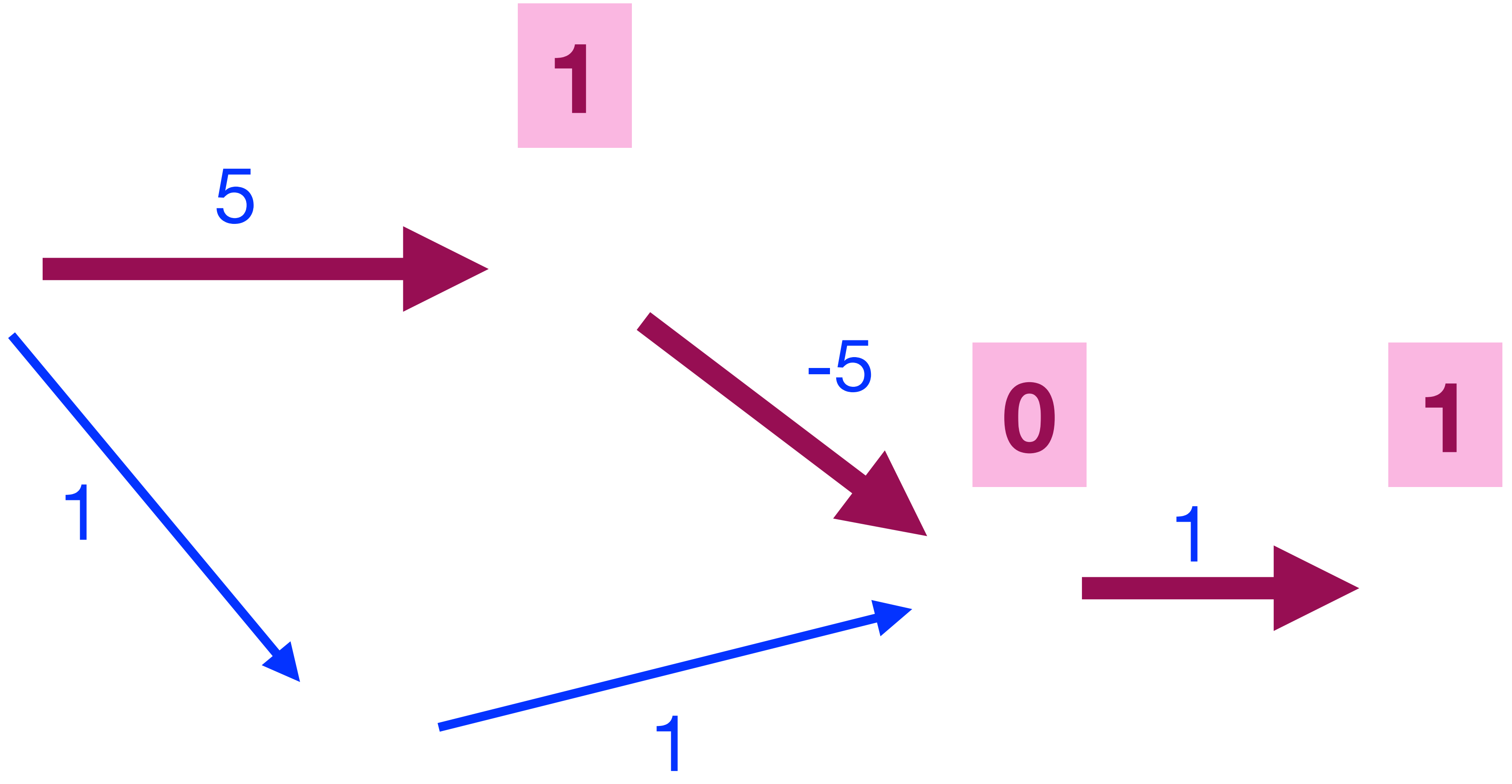




What are the distances computed by Dijkstra's algorithm?

With negative length edges, Dijkstra's algorithm can fail!





False assumption: If $s \to v_0 \to v_1 \to v_2 \ldots \to v_k$ is a shortest path from s to v_k then $\text{dist}(s, v_i) \leq \text{dist}(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.

Shortest paths with negative lengths

Lemma: Let G be a directed graph with *arbitrary* edge lengths and let

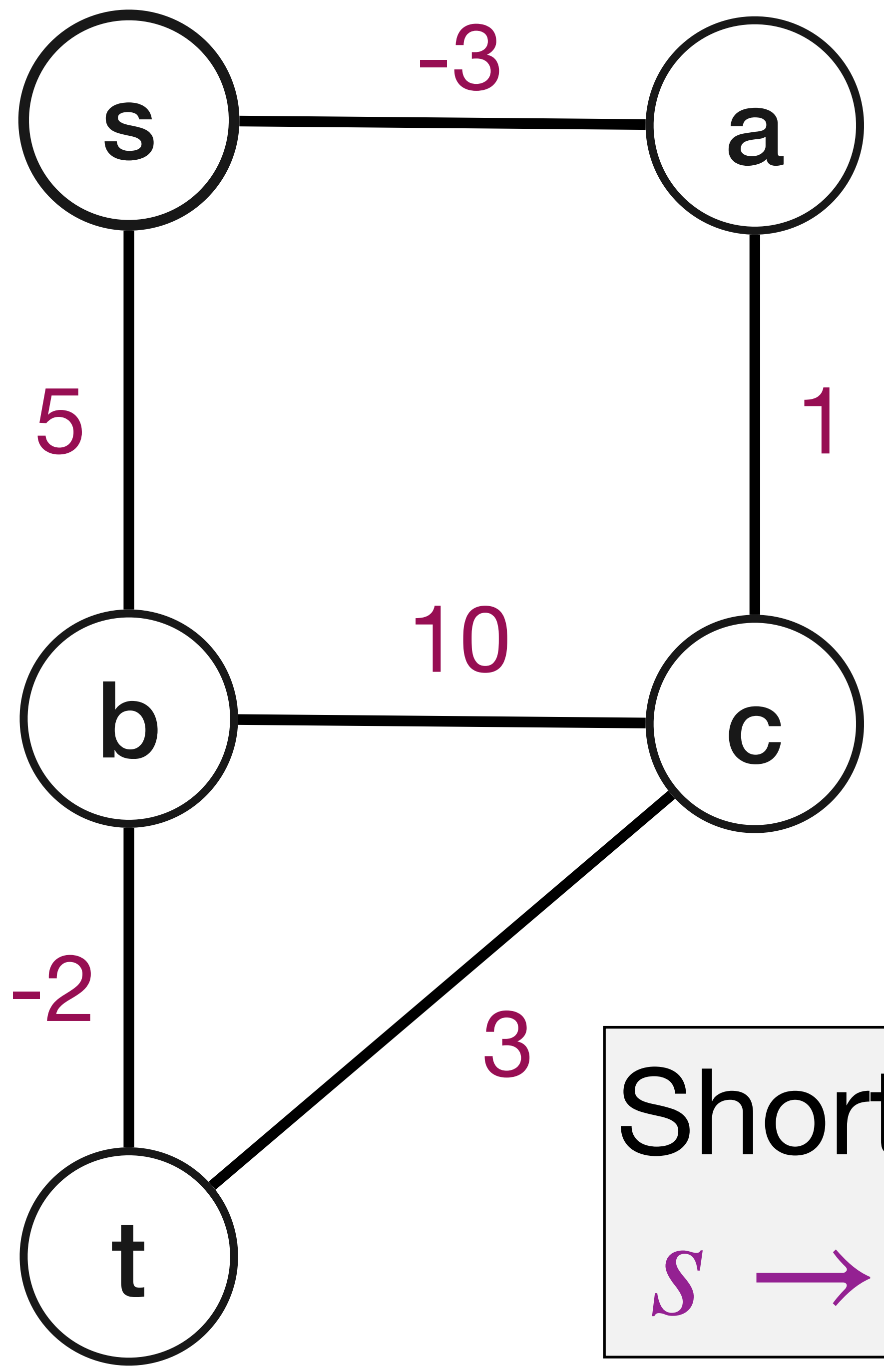
$$s = v_0 \to v_1 \to v_2 \ldots \to v_k = t$$

be a shortest path from s to t then for $1 \leq i < k$:

- $s = v_0 \to v_1 \to v_2 \to \ldots v_i$ is a shortest path from s to v_i
- False: $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$ for $1 \leq i < k$.

Why can't we just re-normalize the edge lengths?

Instinctual thought



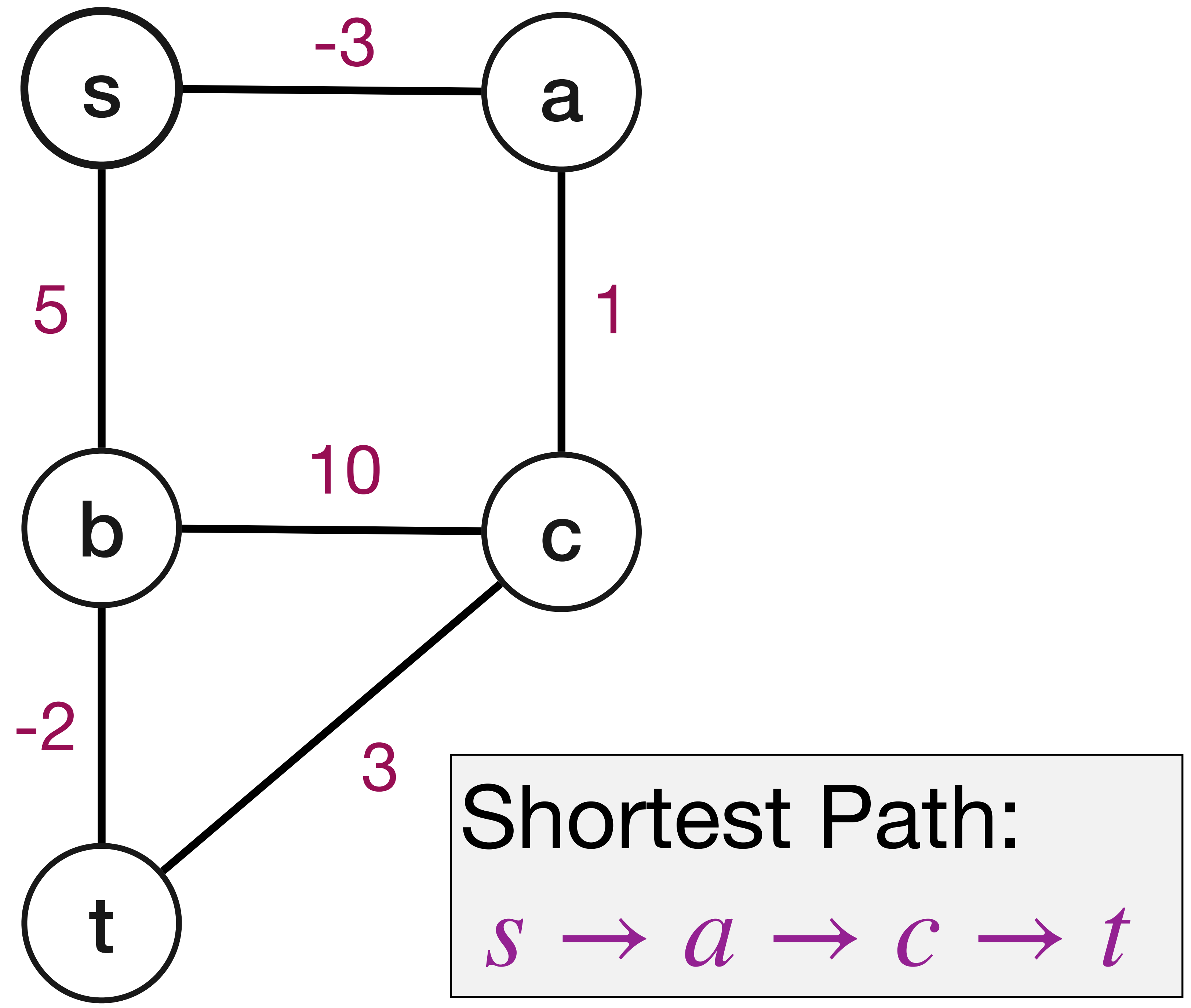
Shortest Path:
$s \to a \to c \to t$

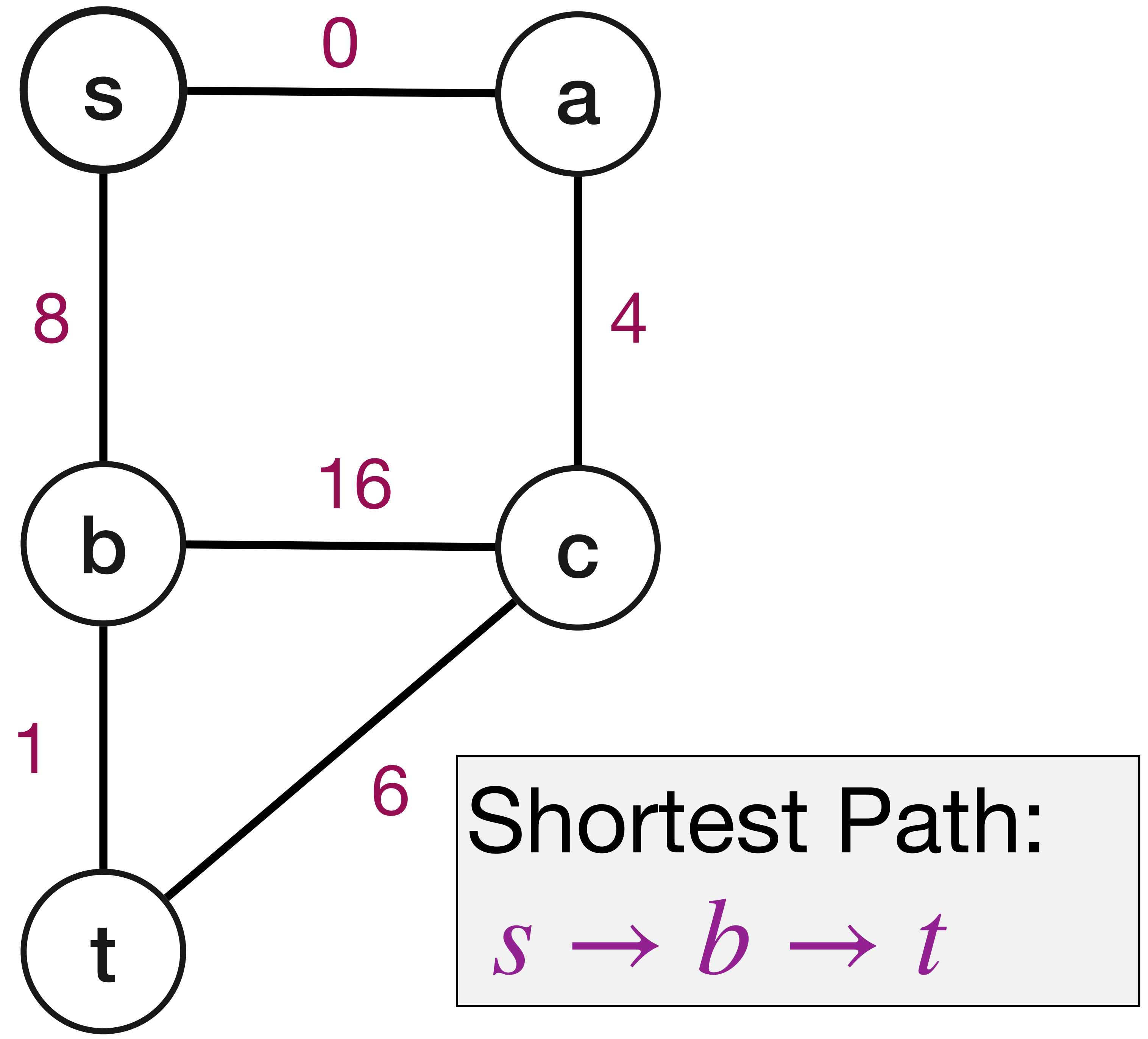
Why can't we simply add a weight to each edge so that the shortest length is 0 (or positive)?

Why can't we just re-normalize the edge lengths?

Instinctual thought

Adding weights to edges penalizes paths with more edges, gives wrong path on original graph.

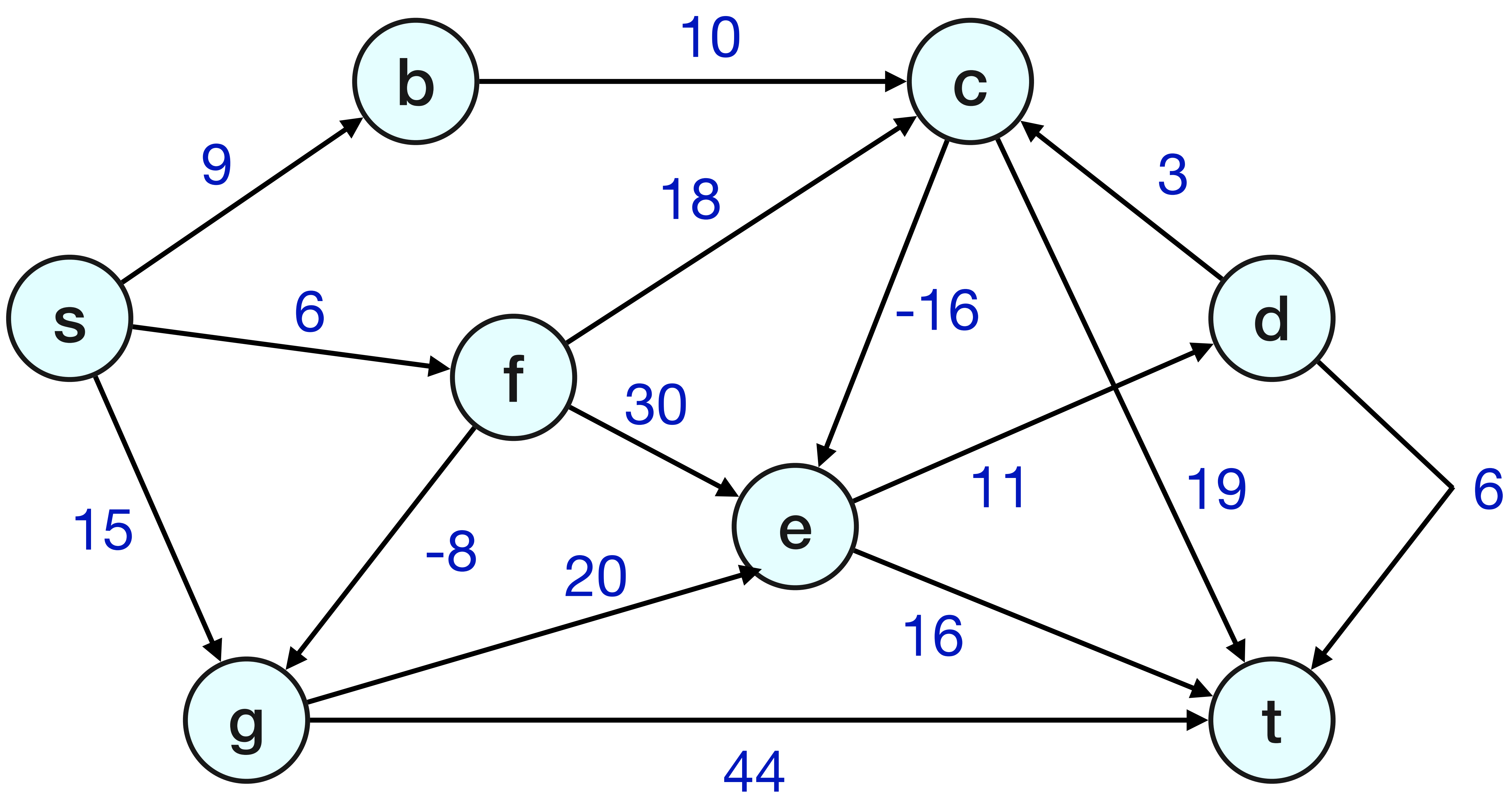




Negative length cycles

Definition

A cycle C is a negative length cycle if the sum of the edge lengths of C is negative



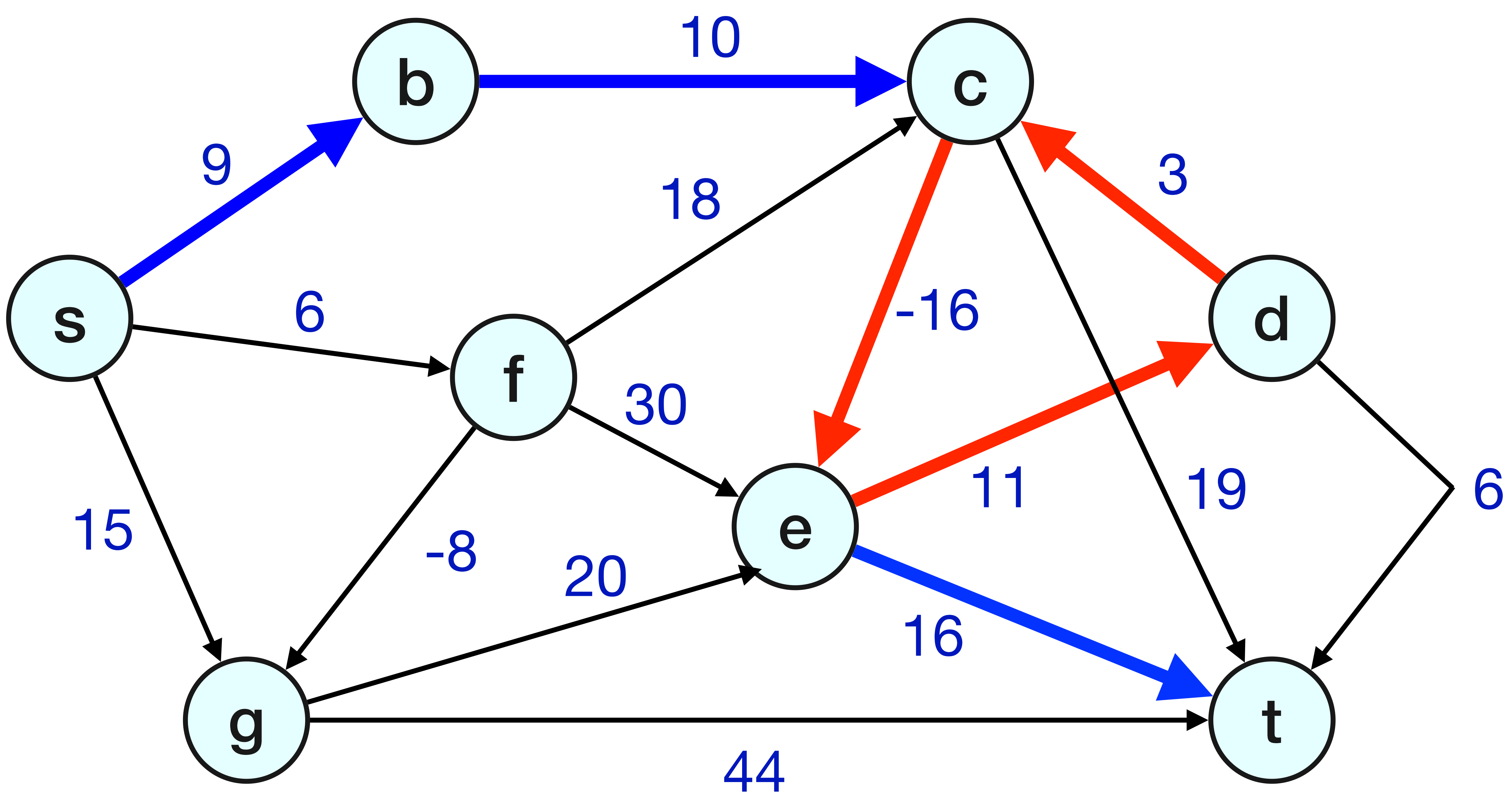
Negative length cycles

Definition

A cycle C is a negative length cycle if the sum of the edge lengths of C is negative

What is the shortest path distance between s and t ?

Reminder: Paths have to be simple...



Shortest paths and negative cycles

Given $G = (V, E)$ with edge lengths and s, t. Suppose

- G has a negative length cycle C, and s can reach C and C can reach t.

Question: What is the shortest distance from s to t ?

Possible answers:

- undefined, that is $-\infty$, OR
- the length of a shortest ***simple*** path from s to t.

Restating problem of shortest path with negative edges

Alternatively: Finding shortest walks

Recall that given a graph $G = (V, E)$:

- A path is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$.
- A walk is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$.

Define $\text{dist}(u, v)$ to be the length of a shortest walk from u to v

- If there is a walk from u to v that contains negative length cycle then $\text{dist}(u, v) = -\infty$
- Else, there is a path with at most $n - 1$ edges whose length is equal to the length of a shortest walk and $\text{dist}(u, v)$ is finite

Shortest paths with negative edges

Algorithmic problems

Input: A directed graph $G = (V, E)$ with edge lengths (could be negative). For edge $e = (u, v)$, $l(e) = l(u, v)$ is its length.

Questions:

- Given nodes s, t either find a negative length cycle C that s can reach or find a shortest path from s to t.
- Given node s, either find a negative length cycle C that s can reach or find shortest path distances from s to all reachable nodes.
- Check if G has a negative length cycle or not.

Bellman Ford Algorithm

Shortest paths and recursion

- Is it possible to compute the shortest path distance from s to t ***recursively***?
- What are the ***smaller*** sub-problems?

Lemma: Let G be a directed graph with arbitrary edge lengths. If

$$s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$$

is a shortest path from s to v_k then for $1 \leq i < k$:

$s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ is a shortest path from s to v_i

Sub-problem idea: paths of *fewer* hops/edges

Hop-based recursion

Bellman-Ford Algorithm

Single-source problem: Fix source s.

Assumptions: All nodes can be reached from s in G. Assume G has no negative-length cycle (for now).

Define, $d(v, k)$ as the shortest *walk* length from s to v using at most k edges. Then note, $\mathrm{dist}(s, v) = d(v, n-1)$. Recursion for $d(v, k)$:

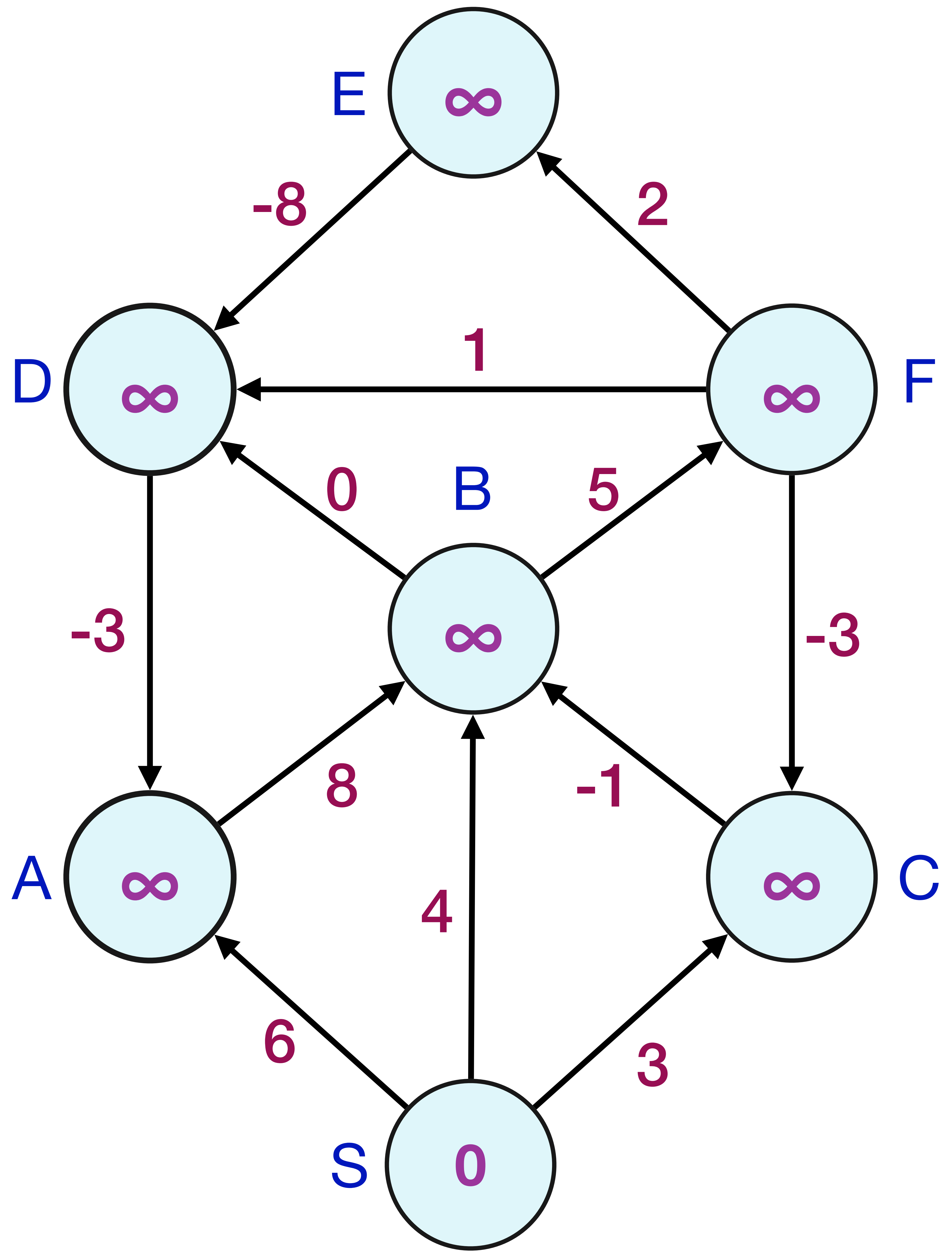
$$d(v,k) = \min \begin{cases} \min\limits_{u \in V} \left(d(u, k-1) + l(u, v)\right) \\ d(v, k-1) \end{cases}$$

Base case: $d(s,0) = 0$ and $d(v,0) = \infty$ for all $v \neq s$

Bellman-Ford Algorithm

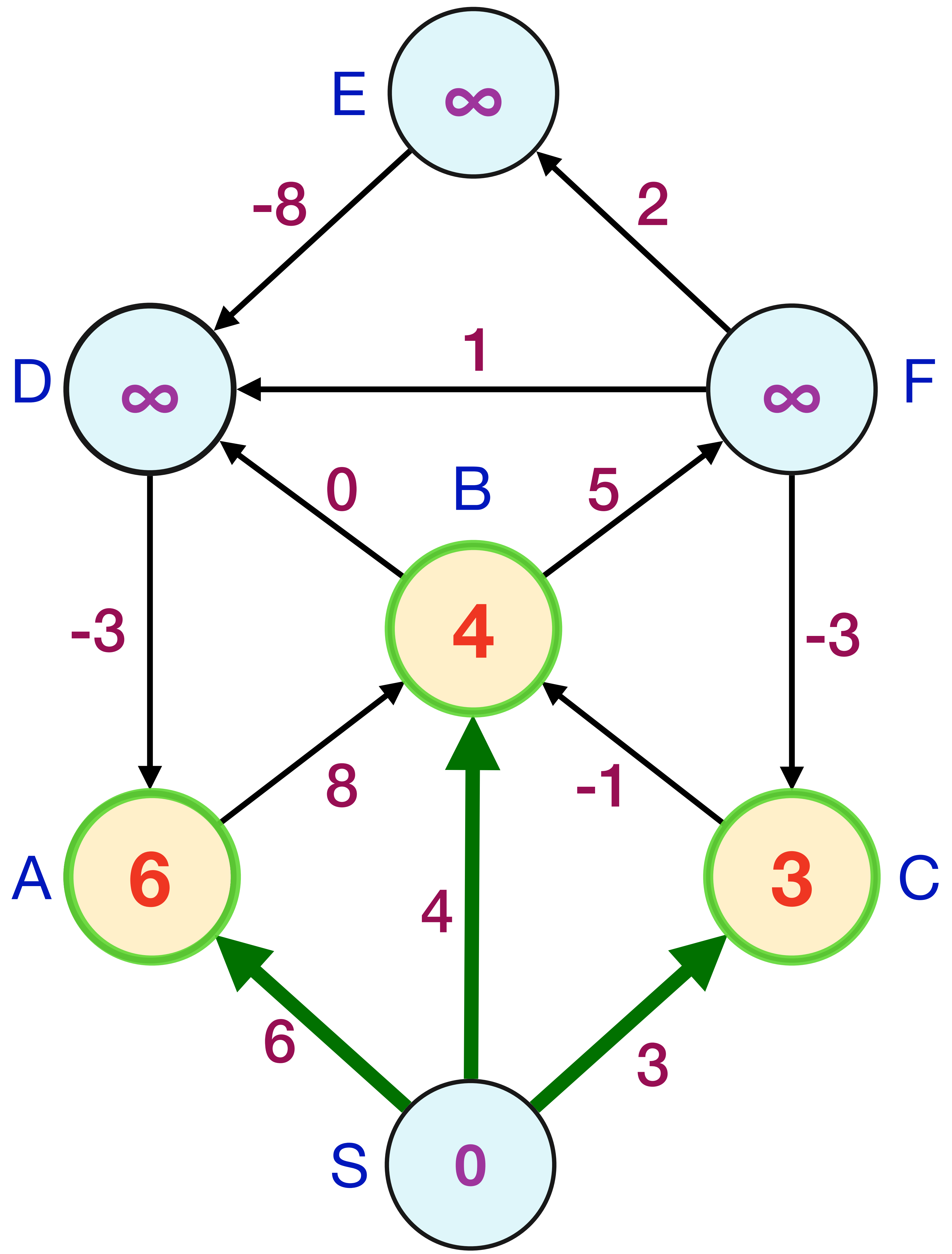
Example

Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1							
2							
3							
4							
5							
6							



Bellman-Ford Algorithm
Example

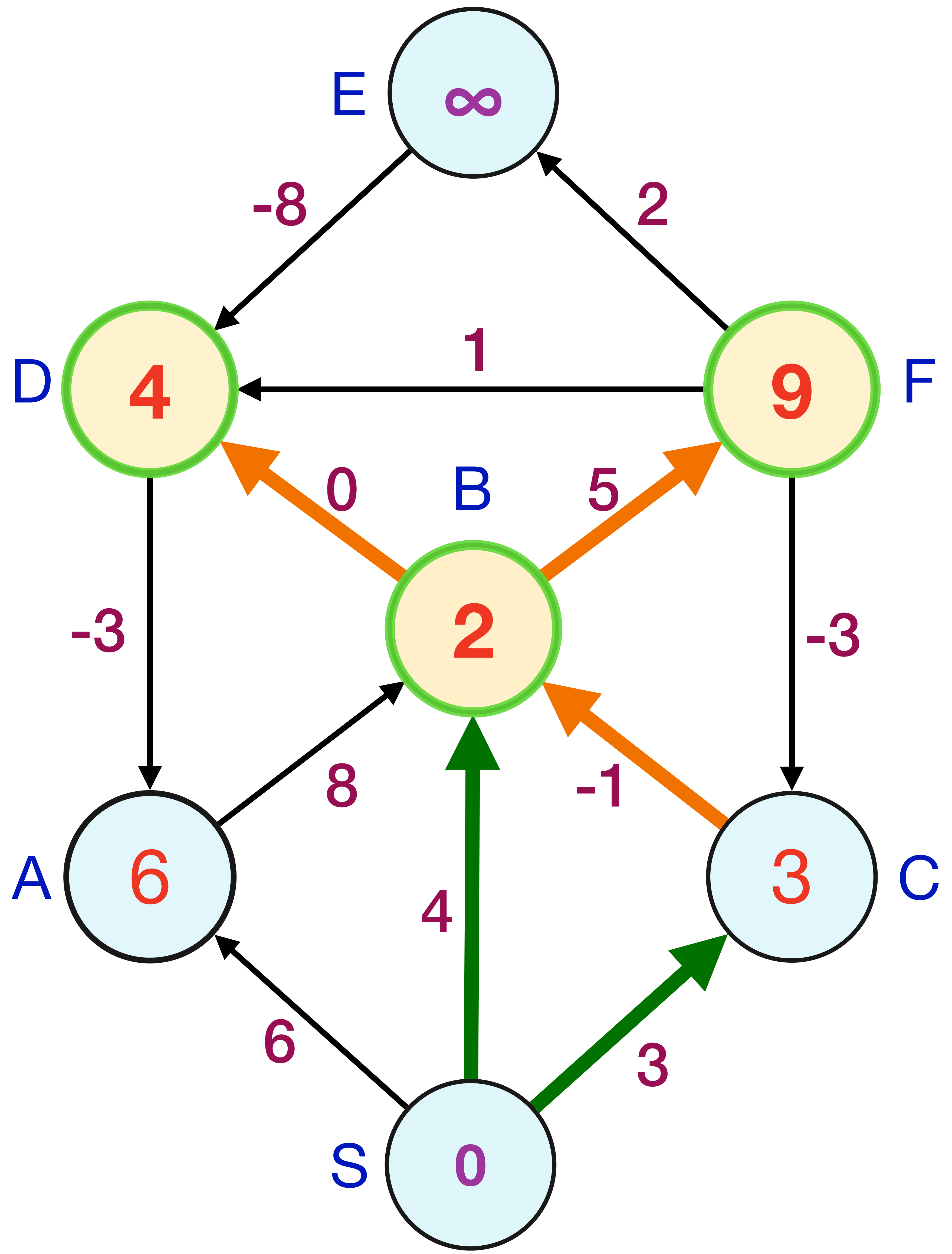
Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1	0	**6**	**4**	**3**	∞	∞	∞
2							
3							
4							
5							
6							



Bellman-Ford Algorithm

Example

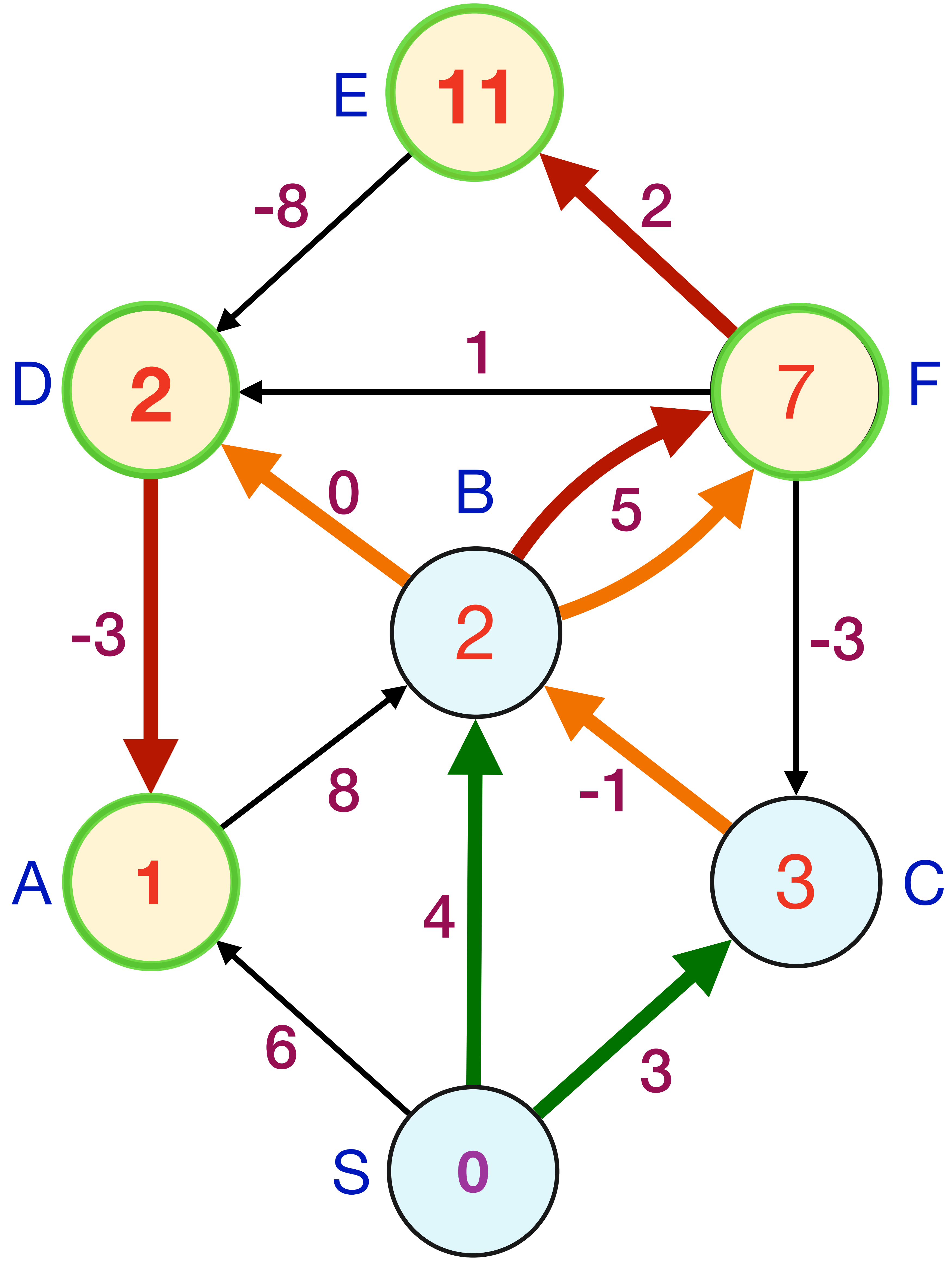
Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3							
4							
5							
6							



Bellman-Ford Algorithm

Example

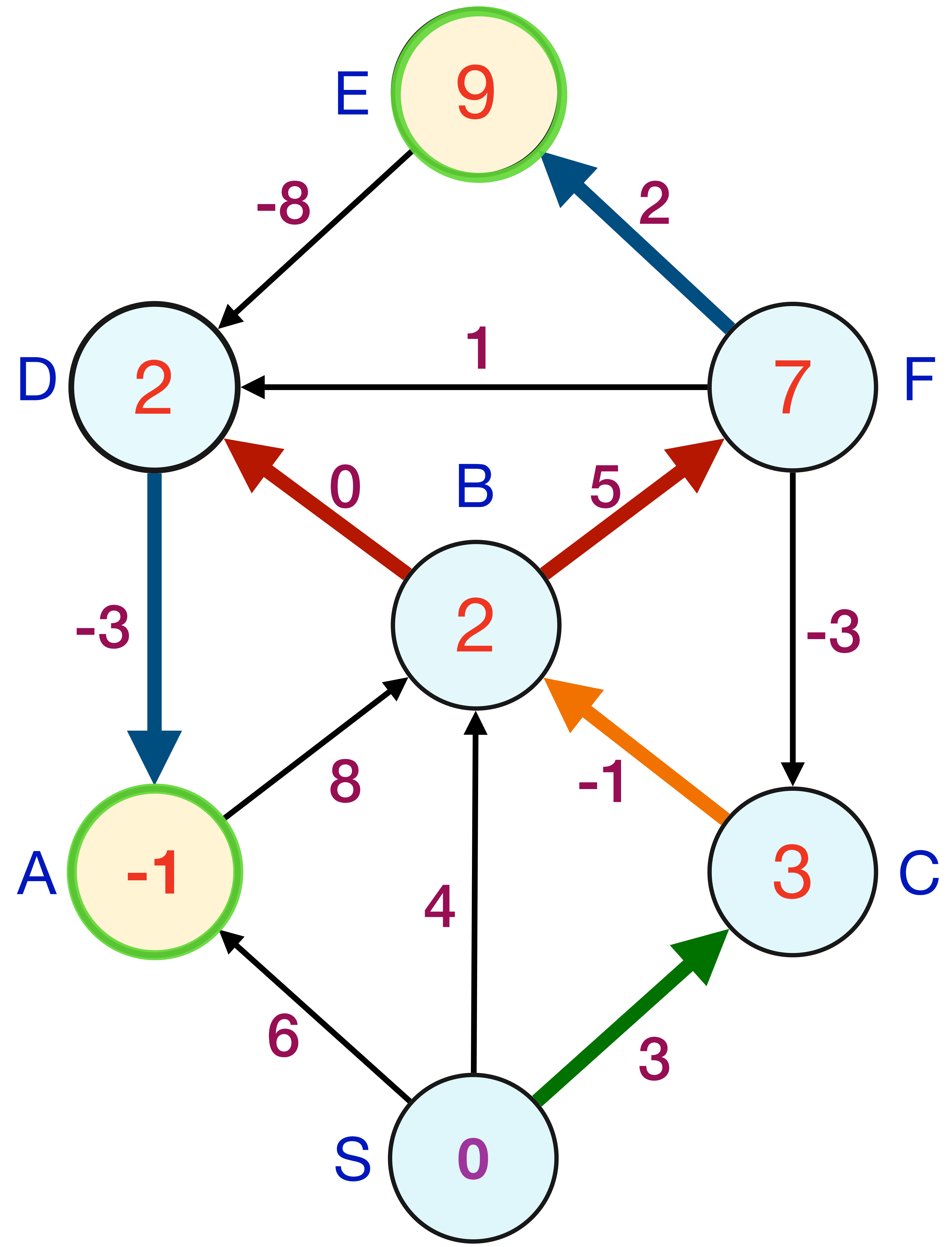
Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	**1**	2	3	**2**	**11**	**7**
4							
5							
6							



Bellman-Ford Algorithm

Example

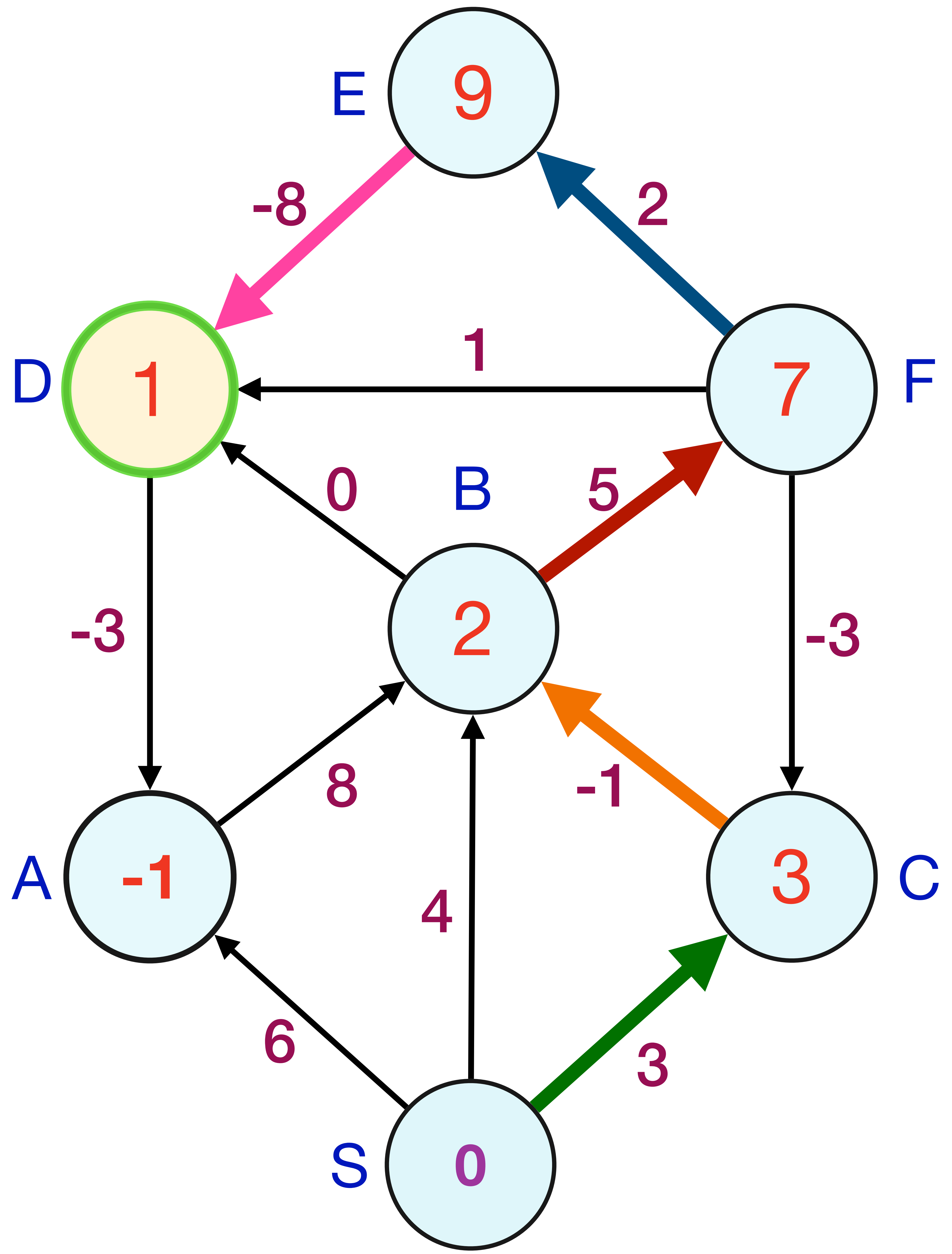
Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	**-1**	2	3	2	**9**	7
5							
6							



Bellman-Ford Algorithm

Example

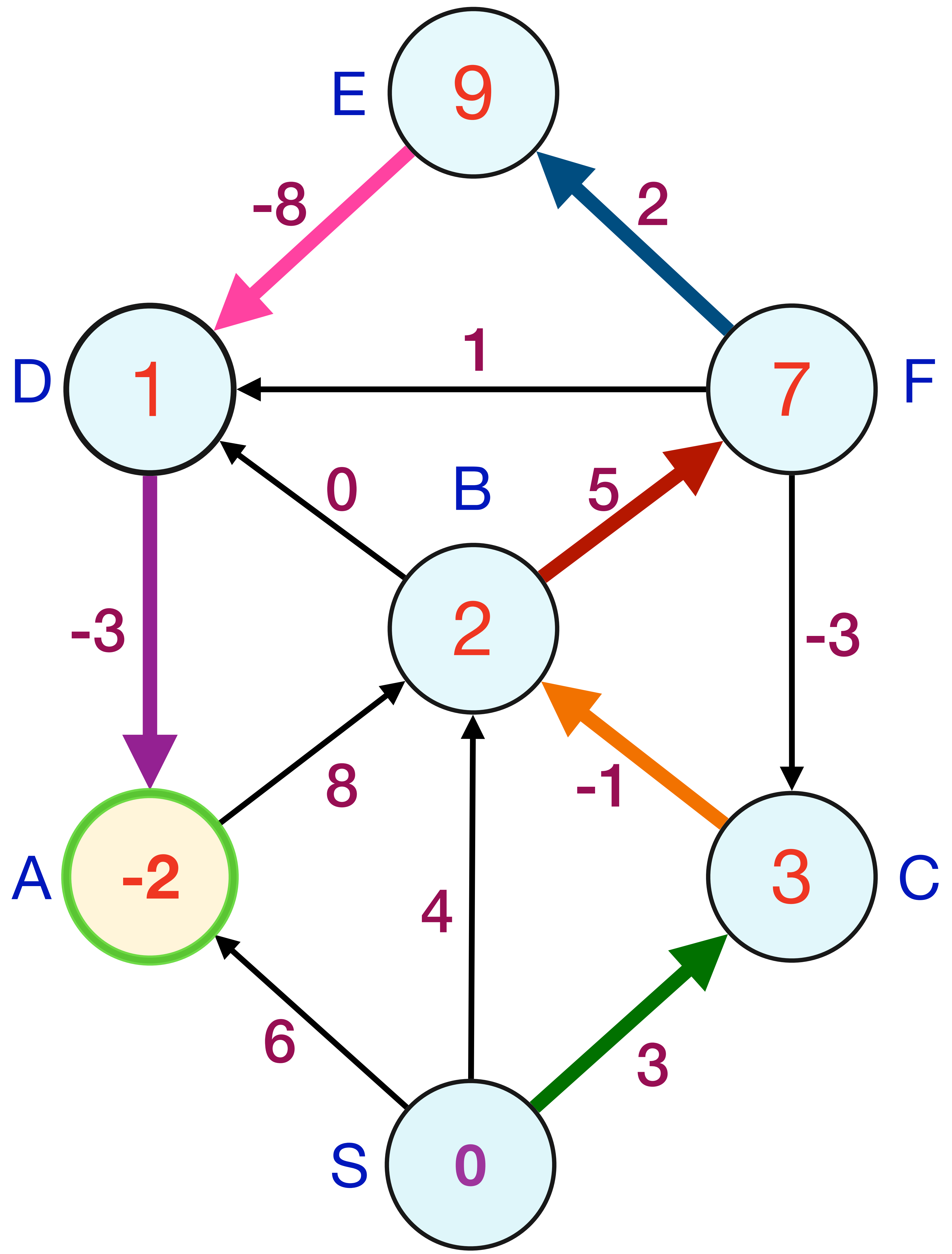
Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	-1	2	3	2	9	7
5	0	-1	2	3	1	9	7
6							



Bellman-Ford Algorithm

Example

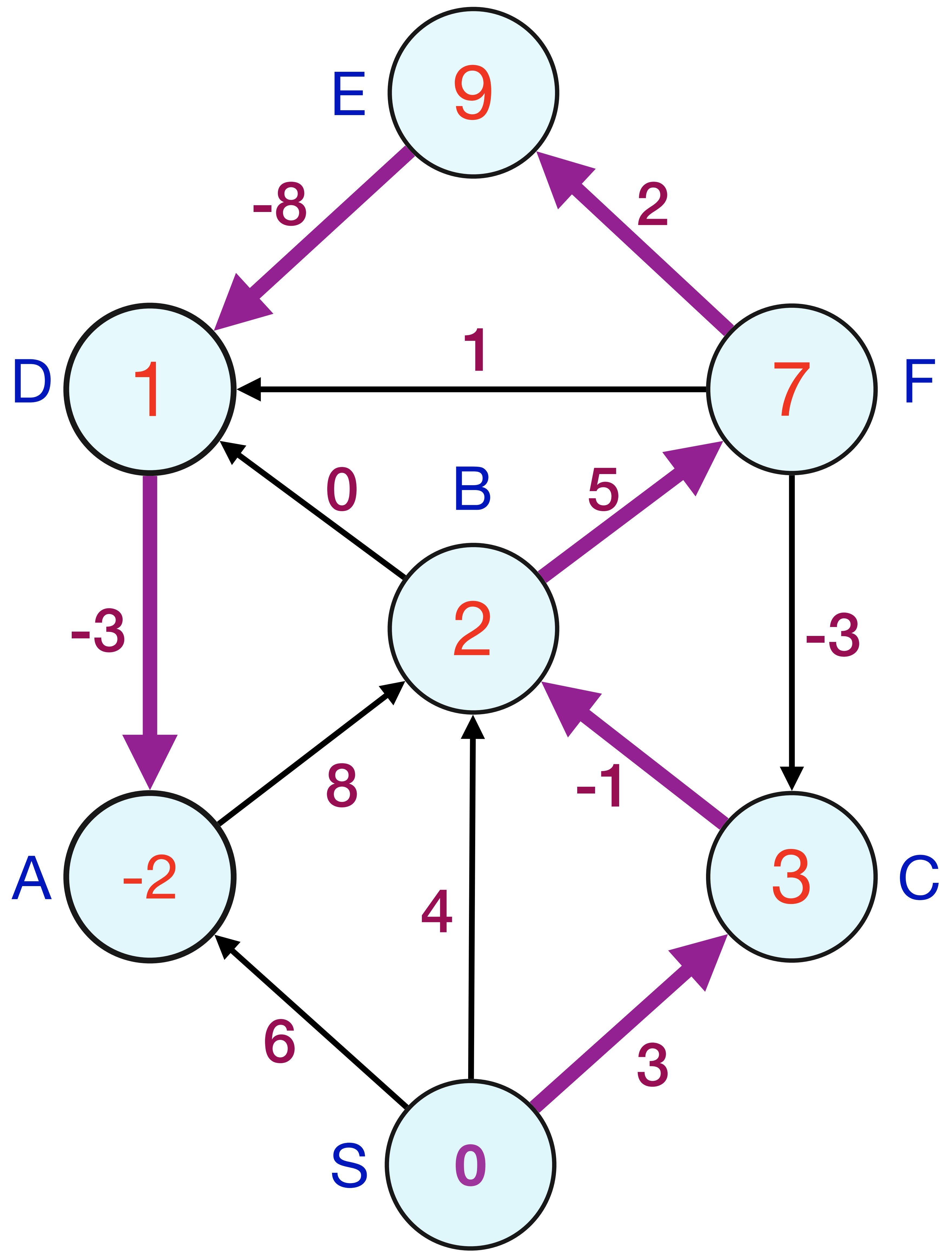
Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	-1	2	3	2	9	7
5	0	-1	2	3	1	9	7
6	0	**-2**	2	3	1	9	7



Bellman-Ford Algorithm

Example

Round	S	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞	∞
1	0	6	4	3	∞	∞	∞
2	0	6	2	3	4	∞	9
3	0	1	2	3	2	11	7
4	0	-1	2	3	2	9	7
5	0	-1	2	3	1	9	7
6	0	-2	2	3	1	9	7



Bellman-Ford Algorithm

Algorithm

Create $\text{In}(G)$ list from $\text{adj}(G)$

for each $u \in V$ **do**
    $d(u,0) \leftarrow \infty$
$d(s,0) \leftarrow 0$

for k = 1 to n − 1 **do**
    for each $v \in V$ **do**
        d(v, k) ← d(v, k − 1)
            for each edge $(u, v) \in \text{In}(v)$ **do**
                d(v, k) = min{d(v, k), d(u, k − 1) + l(u, v)}

for each $v \in V$ **do**
    dist(s, v) ← d(v, n − 1)

Running time: $O(n(n+m))$

Space: $O(m+n^2)$

Space can be reduced to $O(m+n)$

Bellman-Ford Algorithm

Algorithm - cleaner version

for each $u \in V$ **do**
    $d(u,0) \leftarrow \infty$
$d(s,0) \leftarrow 0$

for k = 1 to n - 1 **do**
    for each $v \in V$ **do**
        for each edge $(u, v) \in \text{In}(v)$ **do**
            d(v) = min{d(v), d(u) + l(u, v)}

for each $v \in V$ **do**
    dist(s, v) ← d(v, n - 1)

Running time: $O(mn)$

Space: $O(m + n)$

Do we need the $\text{In}(V)$ list?

Bellman-Ford Algorithm

Algorithm - cleaner version

for each $u \in V$ **do**
    $d(u,0) \leftarrow \infty$
$d(s,0) \leftarrow 0$

for k = 1 to n − 1 **do**
        for each edge $(u, v) \in G$ **do**
            d(v) = min{d(v), d(u) + l(u, v)}

for each $v \in V$ **do**
    dist(s, v) ← d(v, n − 1)

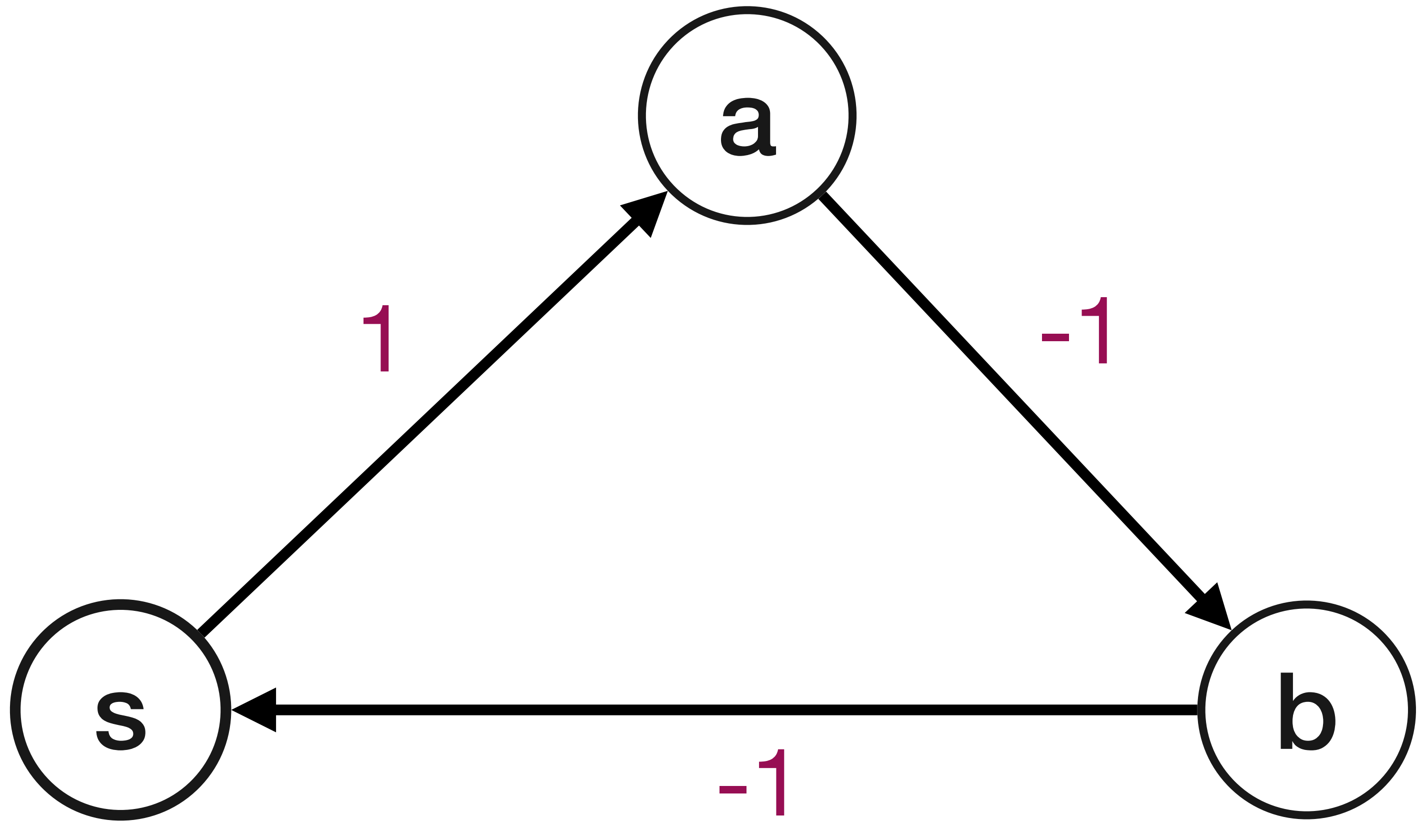
Running time: $O(mn)$

Space: $O(n)$

Bellman-Ford Algorithm

Negative cycles

What happens if we run this on a graph with negative cycles?



Round	s	a	b
0	0	∞	∞
1	0	1	∞
2	0	1	0
3	-1	1	0
4	-1	0	0
5	-1	0	-1

Correctness: detecting negative length cycle

Lemma: Suppose G has a negative cycle C reachable from s. Then there is some node $v \in C$ such that $d(v, n) < d(v, n-1)$.

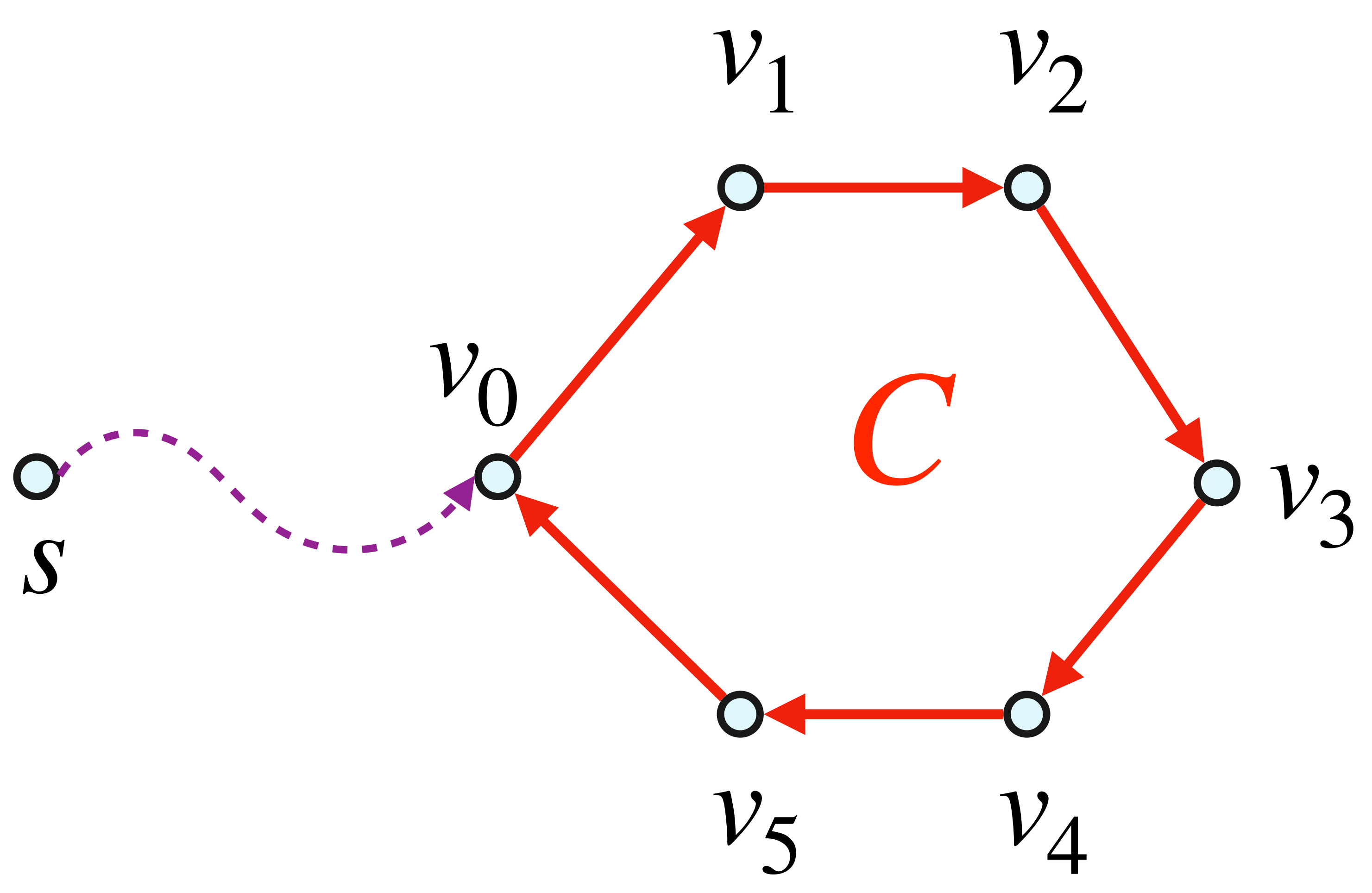
Proof: Suppose not. Let $C = v_1 \to v_2 \to \ldots \to v_h \to v_1$ be negative length cycle reachable from s. Then $d(v_i, n-1)$ is finite for $1 \leq i \leq h$ since C is reachable from s.

By assumption $d(v, n) \geq d(v, n-1)$ for all $v \in C$; implies no change in n^{th} iteration; $d(v_i, n-1) = d(v_i, n)$ for $1 \leq i \leq h$.

This means $d(v_i, n-1) \leq d(v_{i-1}, n-1) + l(v_{i-1}, v_i)$ for $2 \leq i \leq h$ and $d(v_1, n-1) \leq d(v_n, n-1) + l(v_n, v_1)$.

Summing/telescoping these inequalities results in $0 \leq l(C)$ which contradicts the assumption that $l(C) < 0$!

Proof of lemma …



$$d(v_1, n) \leq d(v_0, n-1) + l(v_0, v_1)$$

$$d(v_2, n) \leq d(v_1, n-1) + l(v_1, v_2)$$

$$\ldots$$

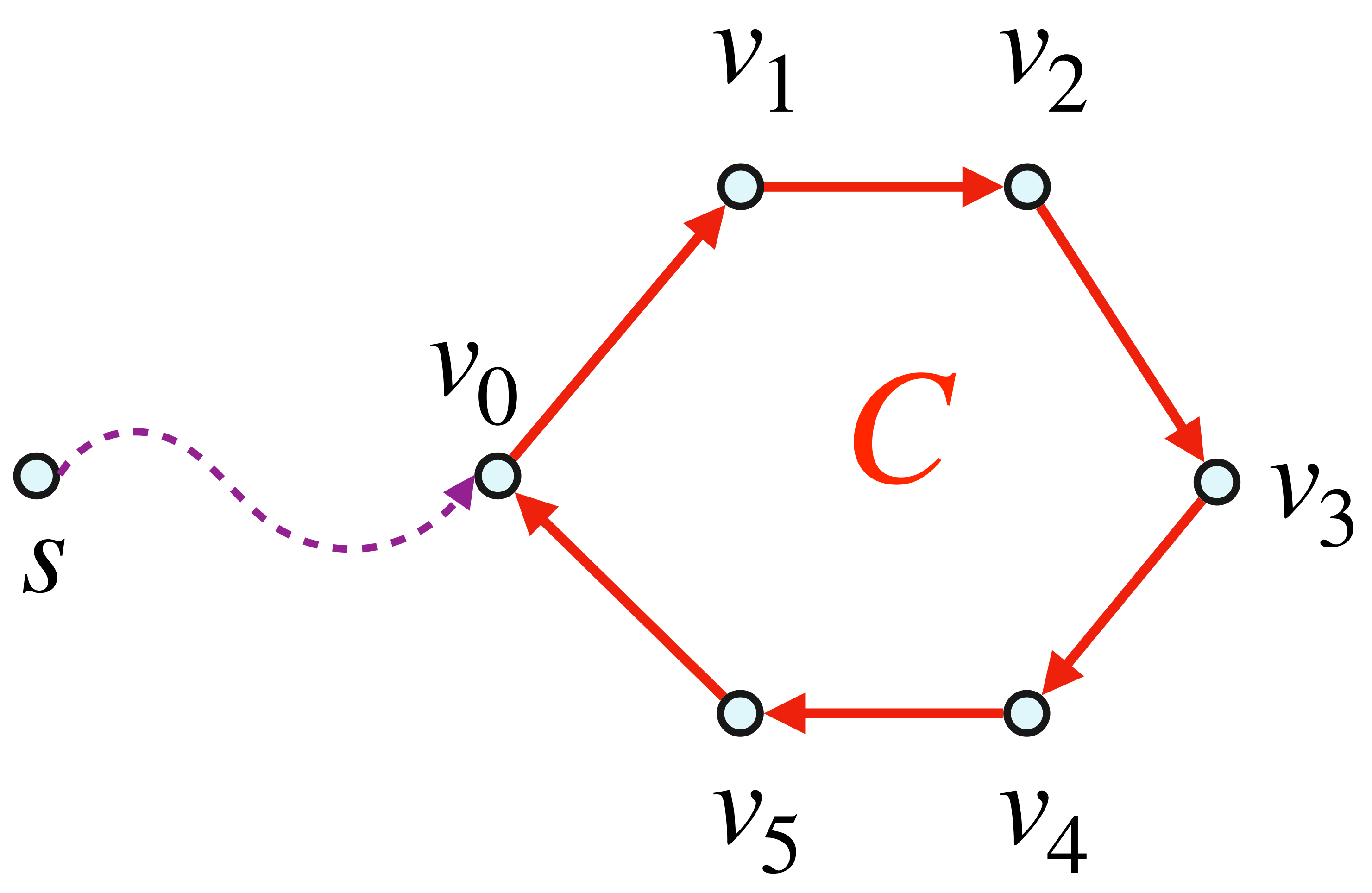
$$d(v_i, n) \leq d(v_{i-1}, n-1) + l(v_{i-1}, v_i)$$

$$\ldots$$

$$d(v_k, n) \leq d(v_{k-1}, n-1) + l(v_{k-1}, v_k)$$

$$d(v_0, n) \leq d(v_k, n-1) + l(v_k, v_0)$$

Proof of lemma …



$$\sum_{i=0}^{k} d(v_i, n) \leq \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} l(v_{i+1}, v_i) + l(v_k, v_0)$$

$$0 \leq \sum_{i=1}^{k} l(v_{i-1}, v_i) + l(v_k, v_0) = \mathrm{len}(C)$$

C is a not a negative cycle. Contradiction!

Negative cycles can not hide

Lemma restated: If G does ***not*** has a negative length cycle reachable from $s \Rightarrow \forall v : d(v, n) = d(v, n-1)$.

Also, $d(v, n-1)$ is the length of the shortest path between s and v.

Put together are the following:

Lemma: If G has a negative length cycle reachable from $s \iff$ there is some node v such that $d(v, n) < d(v, n-1)$.

Bellman-Ford: negative cycle detection

Final version

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0
for k = 1 to n − 1 do
    for each v ∈ V do
        for each edge (u, v) ∈ In(v) do
          d(v) = min{d(v), d(u) + l(u, v)}

(* One more iteration to check if distances change *)

for each v ∈ V do
    for each edge (u, v) ∈ In(v) do
         if (d(v) > d(u) + l(u, v))
            Output “Negative Cycle”
for each v ∈ V do
    dist(s, v) ← d(v)
```

Variants on Bellman-Ford

Finding the Paths and a Shortest Path Tree

How do we find a shortest path tree in addition to distances?

- For each v the $d(v)$ can only get smaller as the algorithm proceeds.
- If $d(v)$ becomes smaller it is because we found a vertex u such that $d(v) > d(u) + l(u, v)$ and we update $d(v) = d(u) + l(u, v)$. That is, we found a shorter path to v through u.
- For each v have a $\mathrm{prev}(v)$ pointer and update it to point to u if v finds a shorter path via u.
- At the end of the algorithm $\mathrm{prev}(v)$ pointers give a shortest path tree oriented towards the source s.

Negative Cycle Detection

Negative Cycle Detection

Given directed graph G with arbitrary edge lengths, does it have a negative length cycle?

- Bellman-Ford checks whether there is a negative cycle C that is reachable from a specific vertex s. There may negative cycles not reachable from s.

- Run Bellman-Ford $|V|$ times, once from each node u?

- Add a new node s' and connect it to all nodes of G with zero length edges. Bellman-Ford from s' will fill find a negative length cycle if there is one. Exercise: why does this work?

- Negative cycle detection can be done with one Bellman-Ford invocation.

Shortest paths in a DAG

Single-Source Shortest Path Problems

Input: A directed acyclic graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v), l(e) = l(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.

Simplification of algorithms for DAGs

- No cycles and hence no negative length cycles! Hence can find shortest paths even for negative length edges
- Can order nodes using topological sort

Algorithm for DAGs

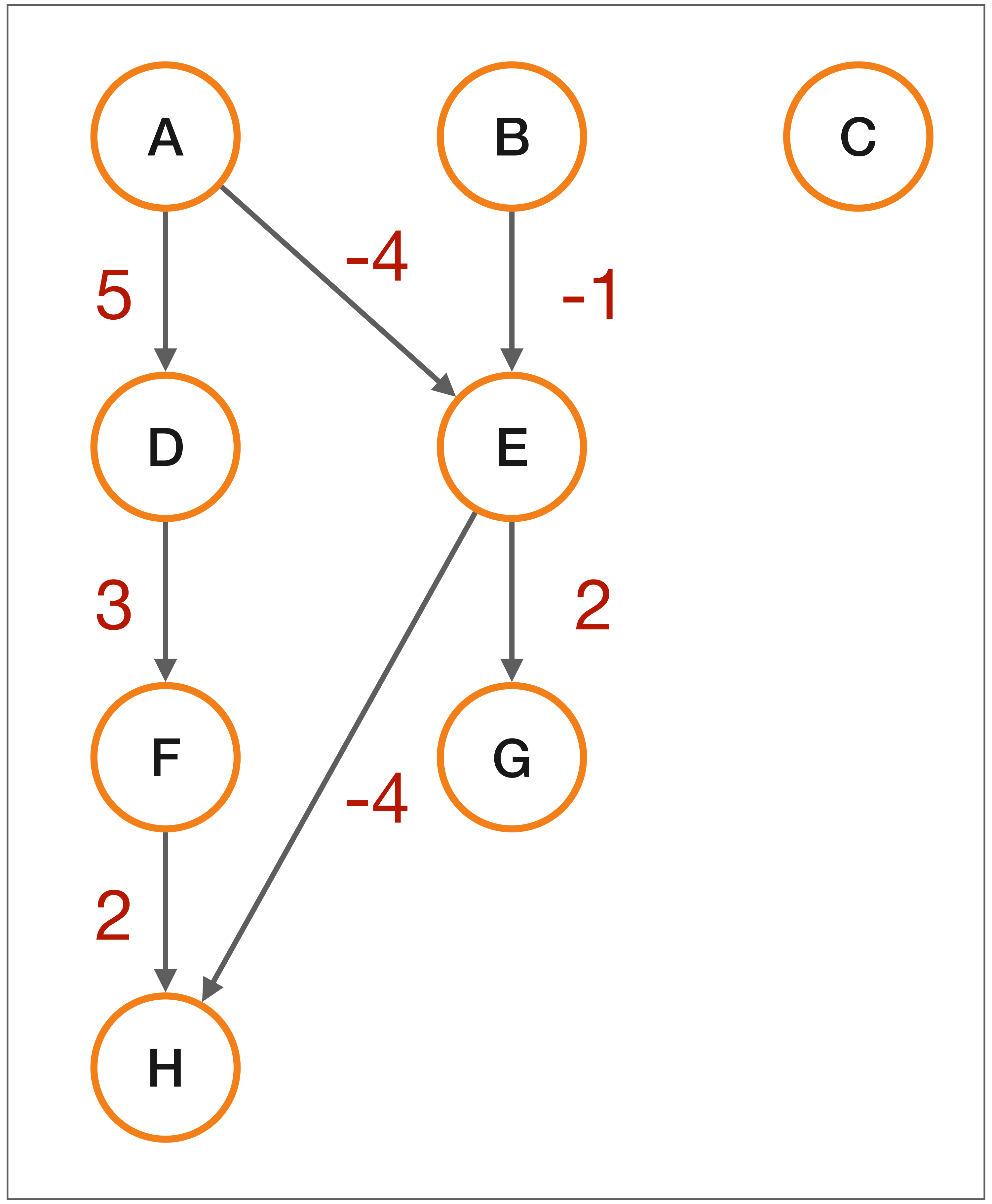
- Want to find shortest paths from s. Ignore nodes not reachable from s.
- Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of G.

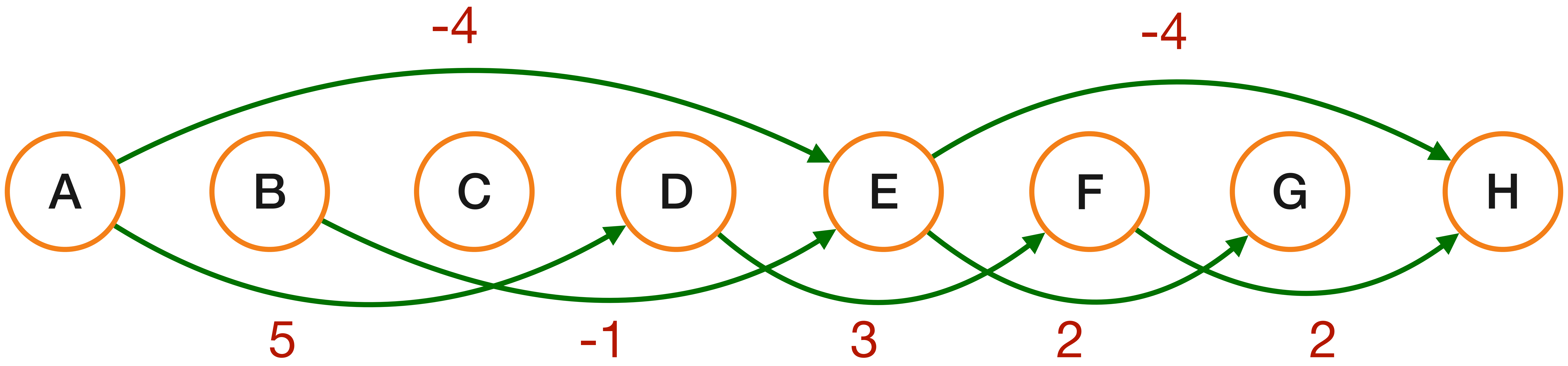
Observation:

- shortest path from s to v_i cannot use any node from $v_{i+1}, \ldots, v_n$
- can find shortest paths in topological sort order

Shortest Paths for DAGs

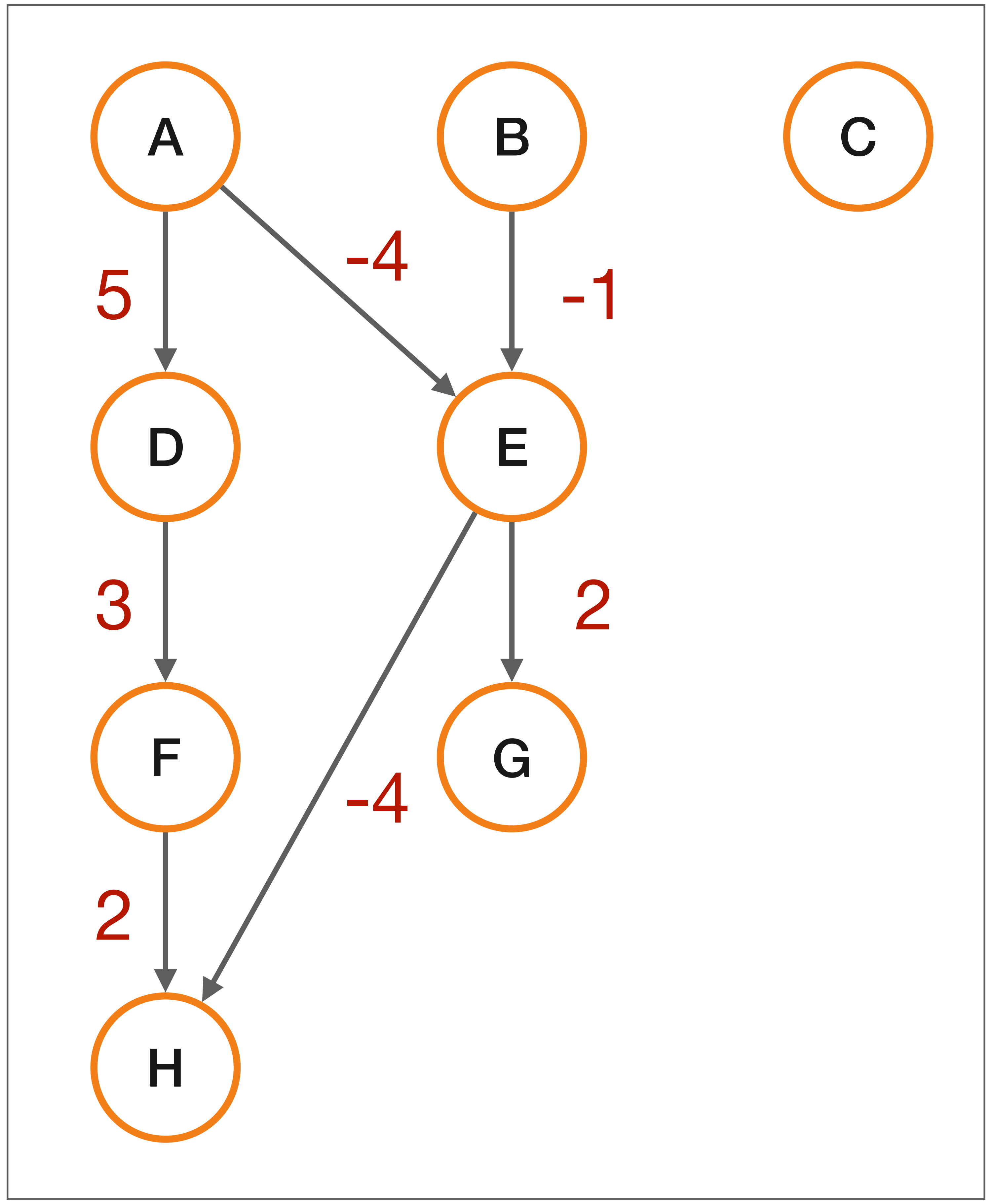
Example

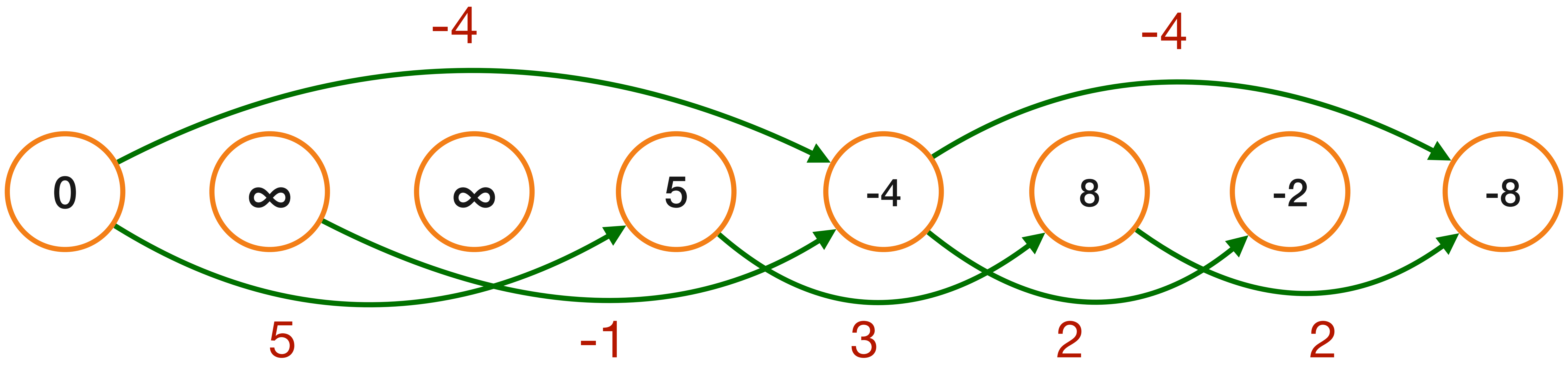




Shortest Paths for DAGs

Example





Algorithm for DAGs

```
for i = 1 to n do
    d(s, vi) = ∞

d(s, s) = 0
for i = 1 to n − 1 do
   for each edge (vi , vj) in Adj(vi) do
        d(s, vj) = min{d(s, vj), d(s, vi) + l(vi, vj)}

return d(s, ·) values computed
```

Correctness: Induction on i and observations in previous slide.

Running time: $O(m+n)$ time algorithm! Works for negative edge lengths and hence can find longest paths in a DAG.

All pairs shortest paths

Shortest Path Problems

Input: A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $l(e) = l(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.
- Find shortest paths for all pairs of nodes.

SSSP: Single-Source Shortest Paths

Input: A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $l(e) = l(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes

Dijkstra's algorithm for non-negative edge lengths.
Running time: $O((m + n)\ \log n)$ with heaps and $O(m + n \log n)$ with advanced priority queues.

Bellman-Ford algorithm for arbitrary edge lengths. Running time: $O(nm)$.

All-Pairs Shortest Paths - Using known algorithms...

All-Pairs Shortest Path Problem

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$,$l(e) = l(u, v)$ is its length.

Find shortest paths for all pairs of nodes.

Apply single-source algorithms n times, once for each vertex.

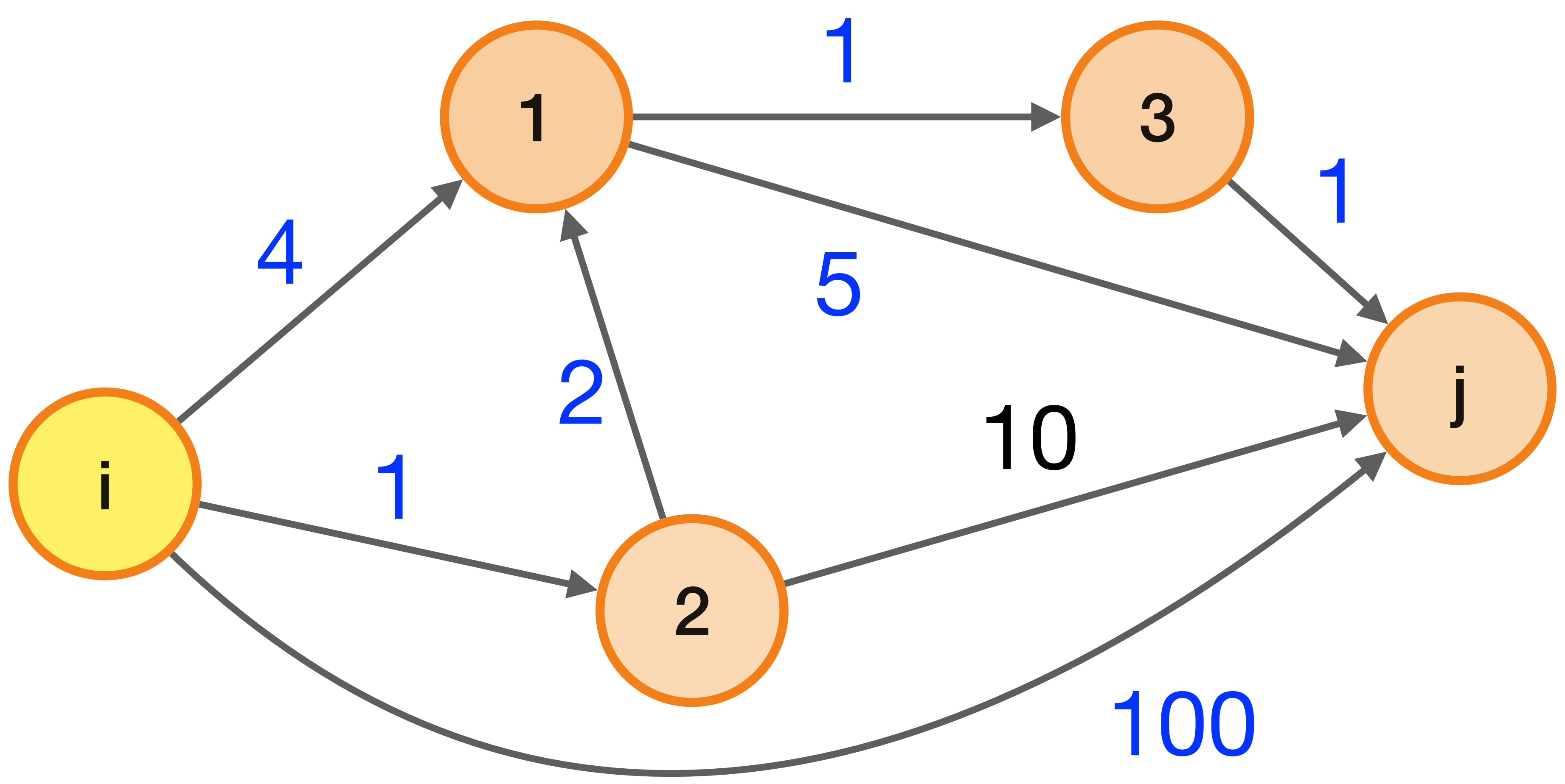
- Non-negative lengths. $O(nm \log n)$ with heaps and $O(nm + n^2 \log n)$ using advanced priority queues.
- Arbitrary edge lengths: $O(n^2 m)$.
  $\Theta(n^4)$ if $m = \Omega(n^2)$

Can we do better?

All Pairs Shortest Paths: A recursive solution

All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- dist(i, j, k): length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



dist$(i, j, 0) =$

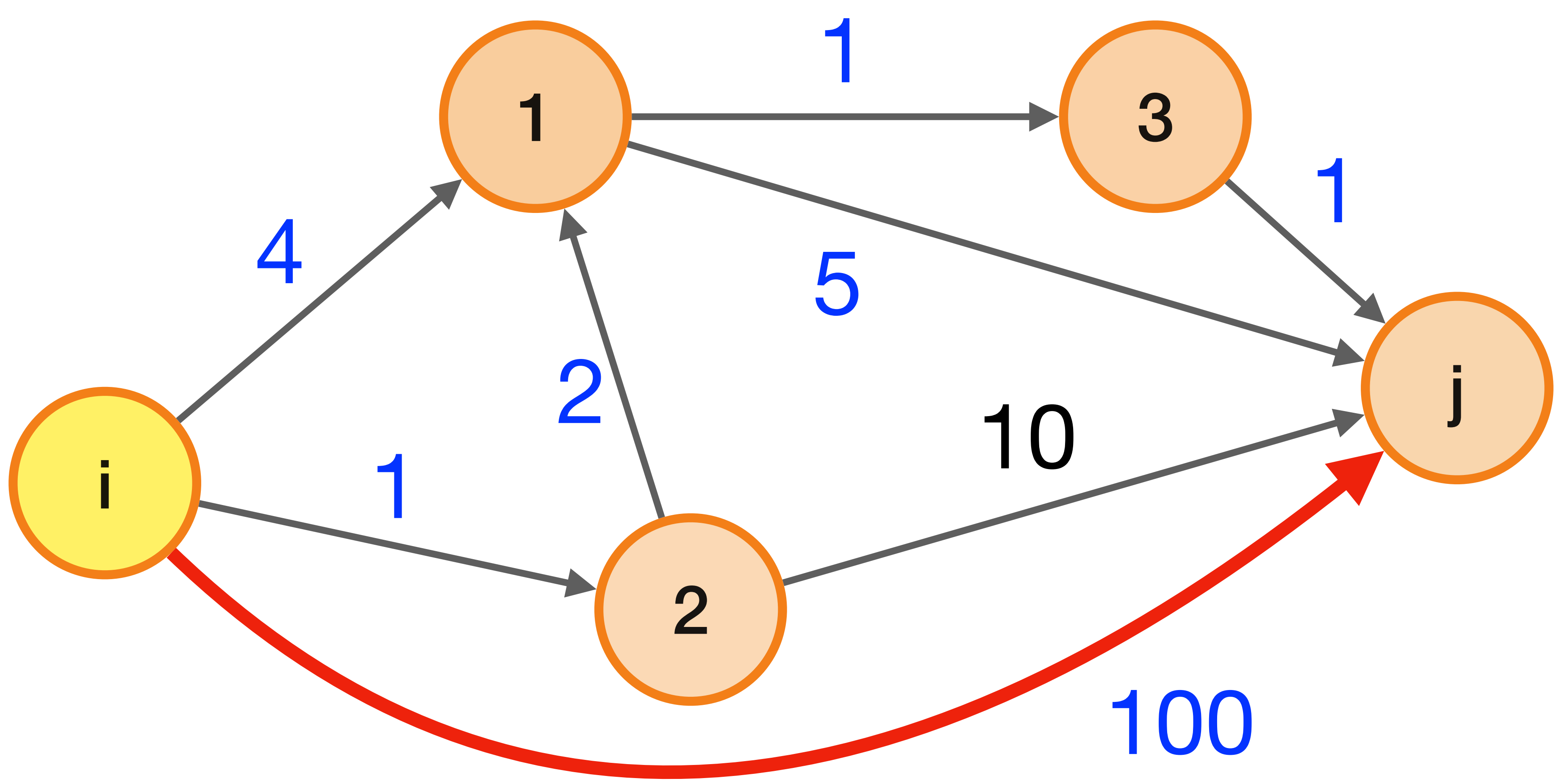
dist$(i, j, 1) =$

dist$(i, j, 2) =$

dist$(i, j, 3) =$

All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- dist(i, j, k): length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



dist$(i, j, 0) = 100$

dist$(i, j, 1) =$

dist$(i, j, 2) =$

dist$(i, j, 3) =$

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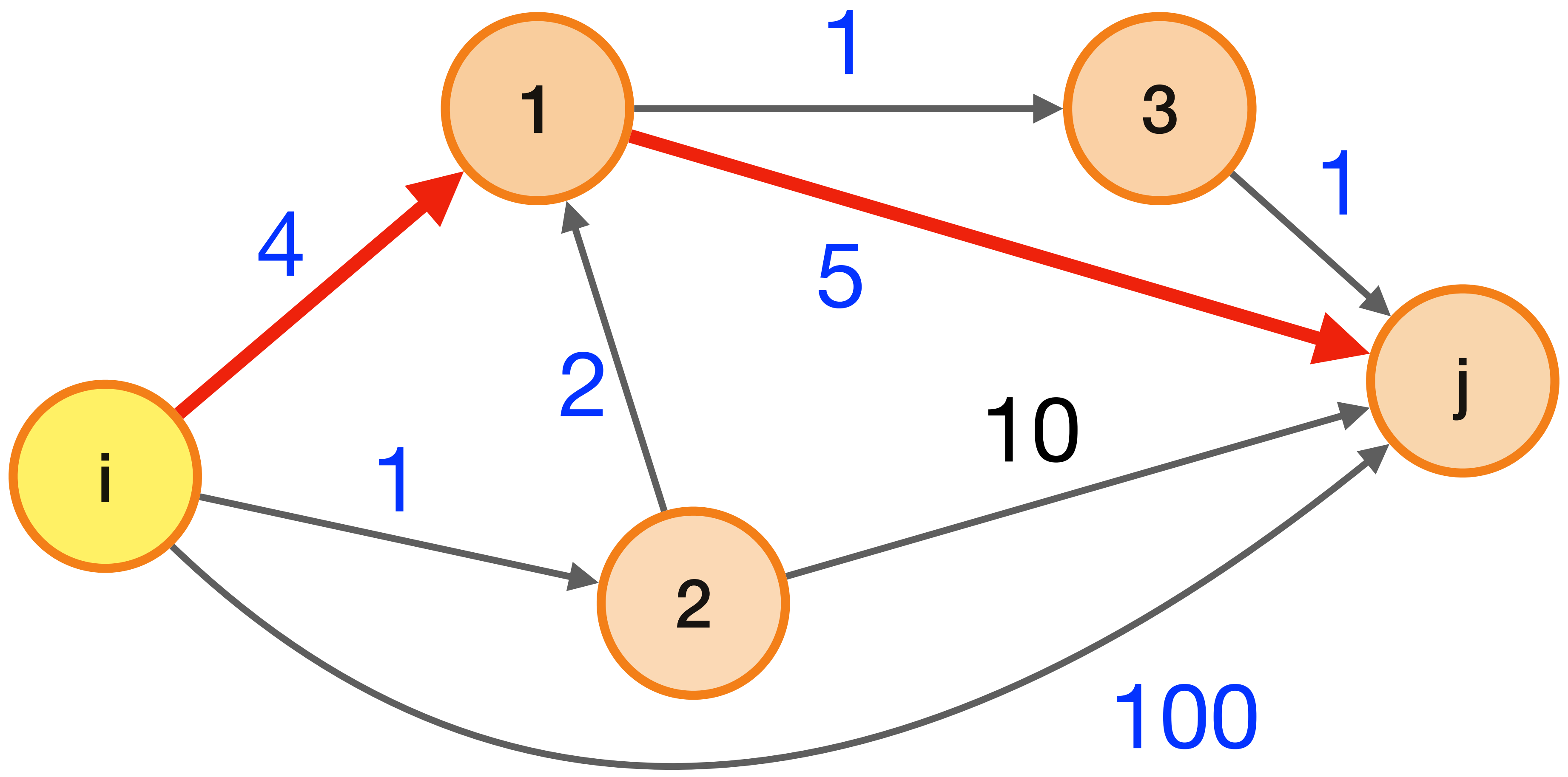


dist$(i, j, 0) = 100$

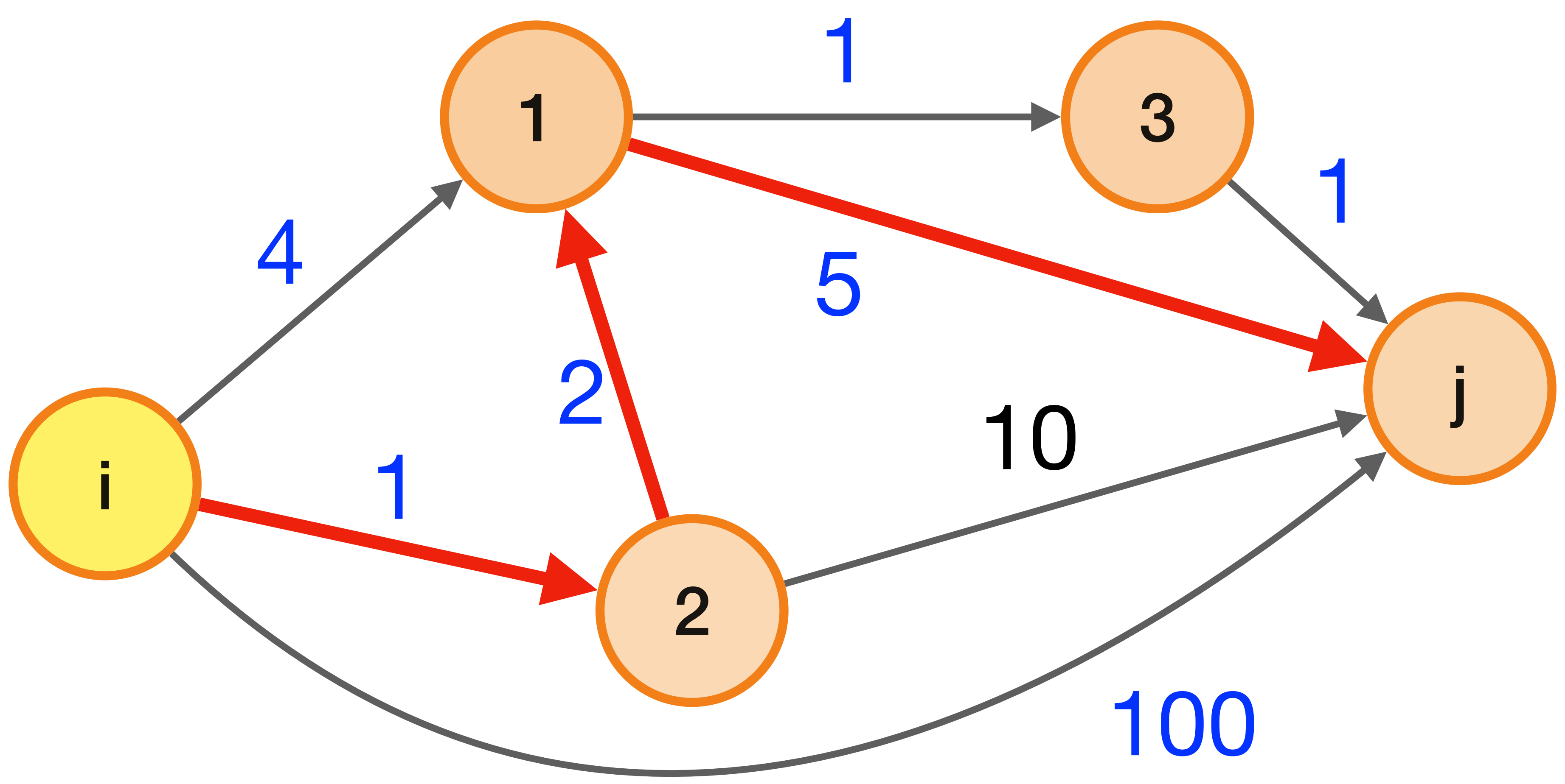
dist$(i, j, 1) = 9$

dist$(i, j, 2) =$

dist$(i, j, 3) =$

All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- dist(i, j, k): length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



dist$(i, j, 0) = 100$

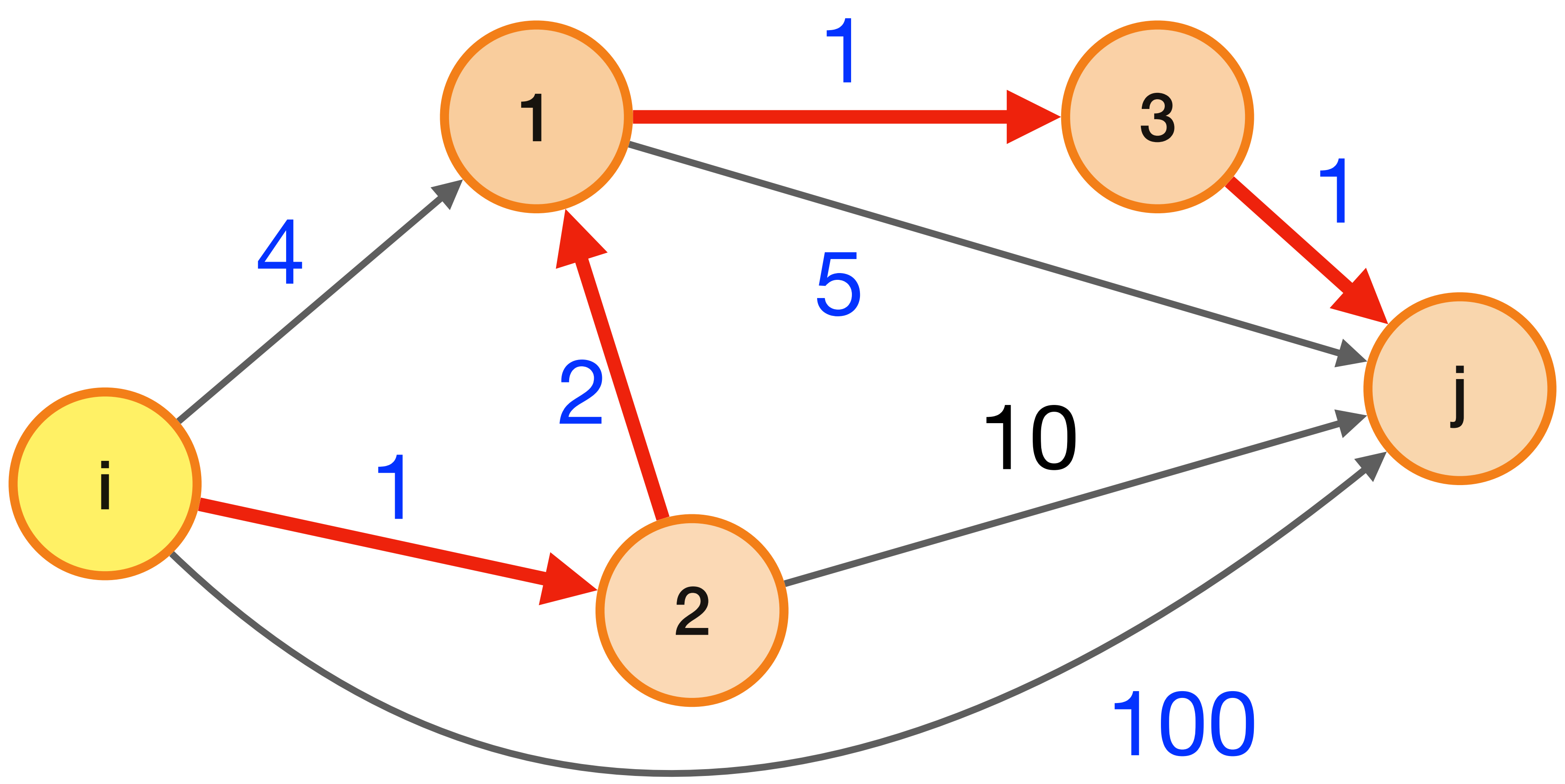
dist$(i, j, 1) = 9$

dist$(i, j, 2) = 8$

dist$(i, j, 3) =$

All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $\text{dist}(i, j, k)$: length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



$\text{dist}(i, j, 0) = 100$

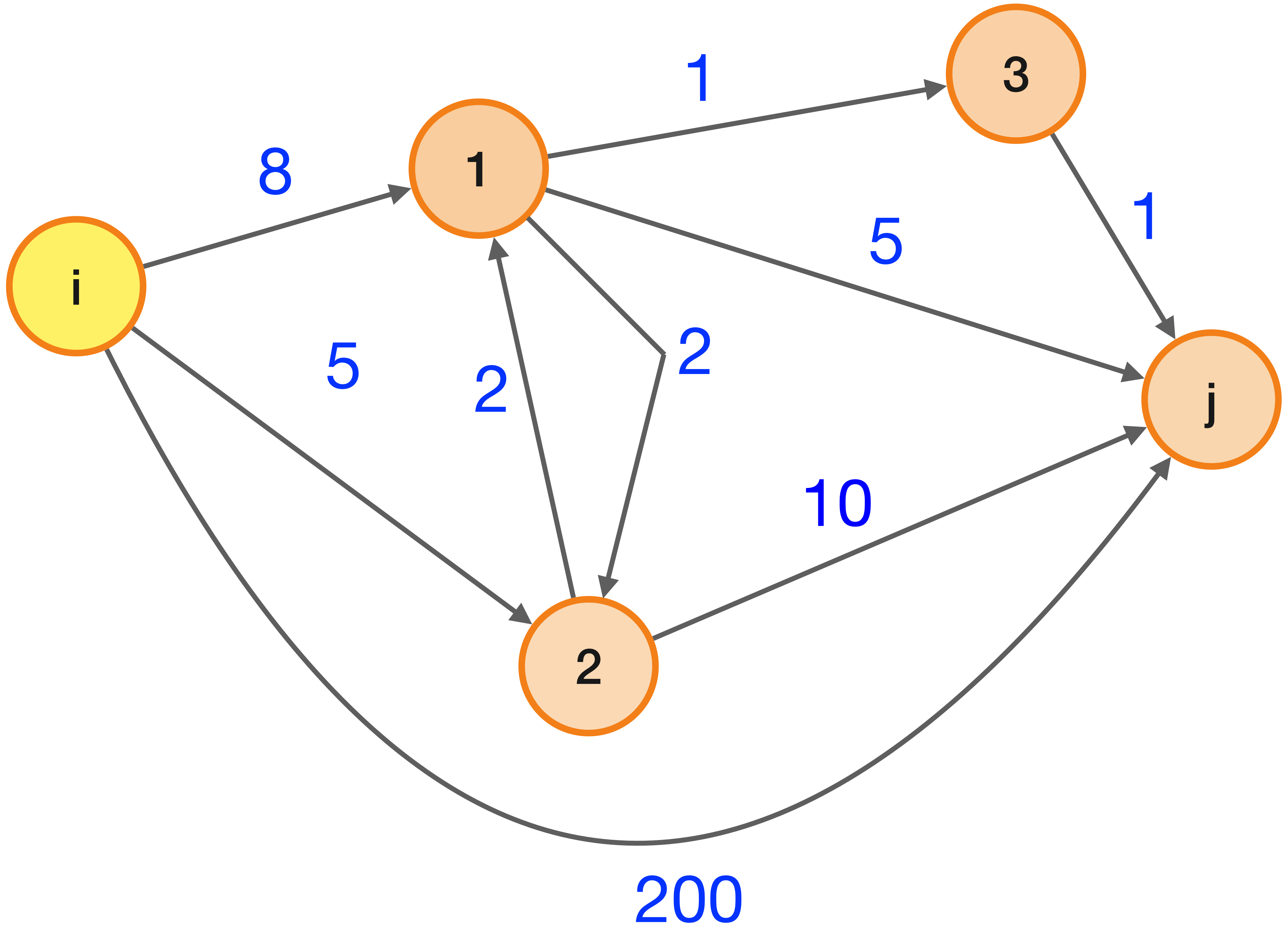
$\text{dist}(i, j, 1) = 9$

$\text{dist}(i, j, 2) = 8$

$\text{dist}(i, j, 3) = 5$

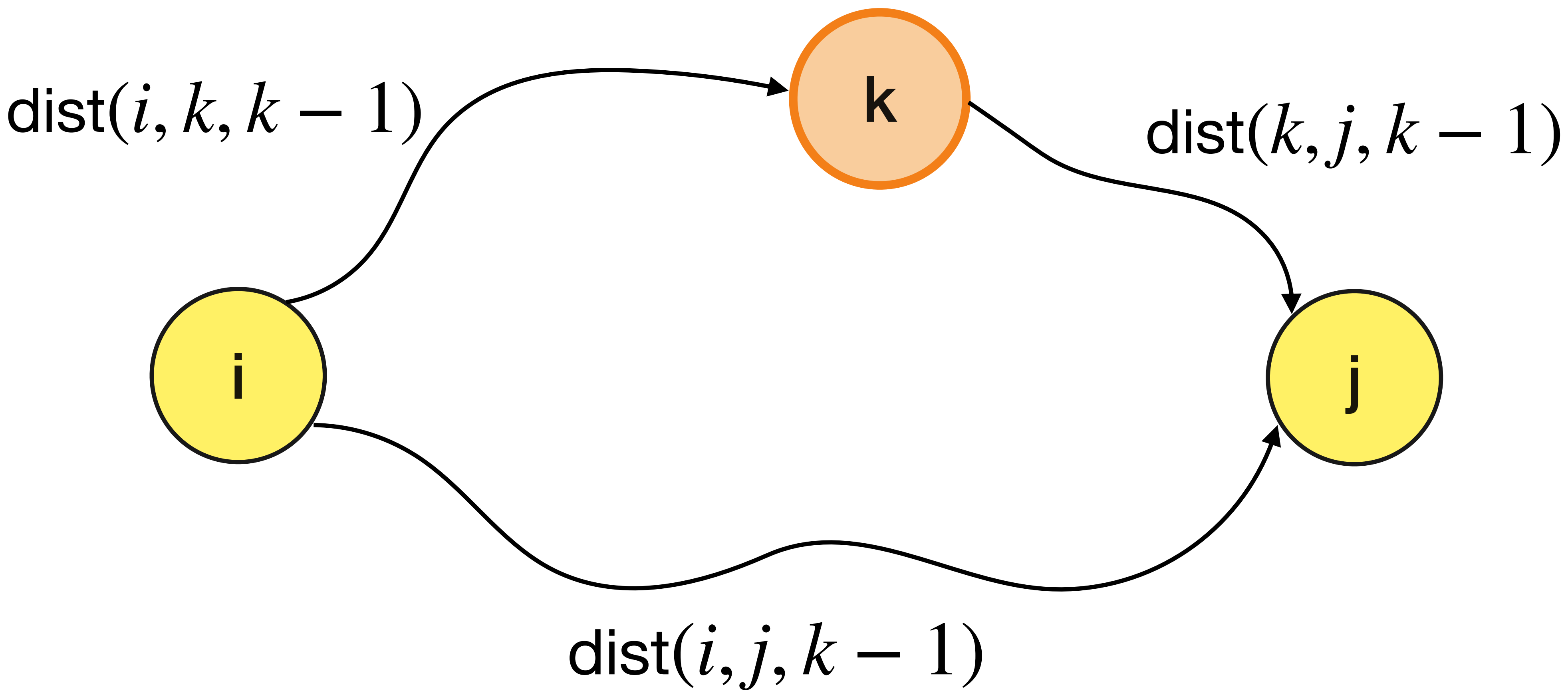
For the following graph, dist$(i, j, 2)$ is …

Review



1. 9
2. 10
3. 11
4. 12
5. 15

All-Pairs: Recursion on index of intermediate nodes



$$\text{dist}(i, j, k) = \min \begin{cases} \text{dist}(i, j, k-1) \\ \text{dist}(i, k, k-1) + \text{dist}(k, j, k-1) \end{cases}$$

- Base case: $\text{dist}(i, j, 0) = l(i, j)$ if $(i, j) \in E$, otherwise ∞
- Correctness: If $i \to j$ shortest walk goes through k then k occurs only once on the path — otherwise there is a negative length cycle

All-Pairs: Recursion on index of intermediate nodes

If i can reach k and k can reach j and $\text{dist}(k, k, k-1) < 0$ then G has a negative length cycle containing k and $\text{dist}(i, j, k) = -\infty$.

Recursion below is valid only if $\text{dist}(k, k, k-1) \geq 0$. We can detect this during the algorithm or wait till the end.

$$\text{dist}(i, j, k) = \min \begin{cases} \text{dist}(i, j, k-1) \\ \text{dist}(i, k, k-1) + \text{dist}(k, j, k-1) \end{cases}$$

Floyd-Warshall Algorithm

Floyd - Warshall Algorithm

For All-Pairs Shortest Paths

```
for i = 1 to n do
    for j = 1 to n do
        d(i,j,0) = l(i,j)

(* l(i, j) = ∞ if (i, j) ∉ E, 0 if i = j *)
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
```

$$dist(i,j,k) = min \begin{cases} dist(i,j,k-1) \\ dist(i,k,k-1) + dist(k,j,k-1) \end{cases}$$

```
for i = 1 to n do
    if (dist(i,i,n) < 0) then
        Output ∃ negative cycle in G
```

Running time: $\Theta(n^3)$

Space: $\Theta(n^3)$

Correctness: via induction and recursive definition

Floyd - Warshall Algorithm

Finding the Paths

Question: Can we find the paths in addition to the distances?

- Create a $n \times n$ array Next that stores the next vertex on shortest path for each pair of vertices
- With array Next, for any pair of given vertices i, j can compute a shortest path in $O(n)$ time.

Floyd - Warshall Algorithm

Finding the Paths

```
for i = 1 to n do
    for j = 1 to n do
        d(i,j,0) = l(i,j)

(* l(i, j) = ∞ if (i, j) ∉ E, 0 if i = j *)
       Next(i, j) = -1
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
          if (d(i,j,k − 1) > d(i,k,k − 1) + d(k,j,k − 1)) then
             d(i,j,k) = d(i,k,k − 1) + d(k,j,k − 1)
             Next(i, j) = k

for i = 1 to n do
    if (dist(i,i,n) < 0) then
       Output ∃ negative cycle in G
```

Exercise:

Given *Next* array and any two vertices i, j describe an $O(n)$ algorithm to find a $i - j$ shortest path.

Summary of shortest path algorithms

Summary of results on shortest paths

Single source		
No negative edges	Dijkstra	$O(n \log n + m)$
Edge lengths can be negative	Bellman Ford	$O(nm)$

All Pairs Shortest Paths		
No negative edges	n * Dijkstra	$O(n^2 \log n + nm)$
No negative cycles	n * Bellman Ford	$O(n^2m) = O(n^4)$
No negative cycles	Johnson's [1]	$O(nm + n^2 \log n)$
No negative cycles	Floyd-Warsh	$O(n^3)$
Unweighted	Matrix multiplication [2]	$O(n^{2.38})$, $O(n^{2.58})$

Summary of results on shortest paths

(1) The algorithm for the case that there are no negative cycles, and doing all shortest paths, works by computing a potential function using **Bellman-Ford** and then doing **Dijkstra**. It is mentioned for the sake of completeness, but it outside the scope of the class.

(2) https://resources.mpi-inf.mpg.de/ departments/d1/teaching/ss12/AdvancedGraphAlgorithms/Slides14.pdf