Minimum spanning trees (MSTs)

Sides based on material by Kani, Erickson, Chekuri, et. al.

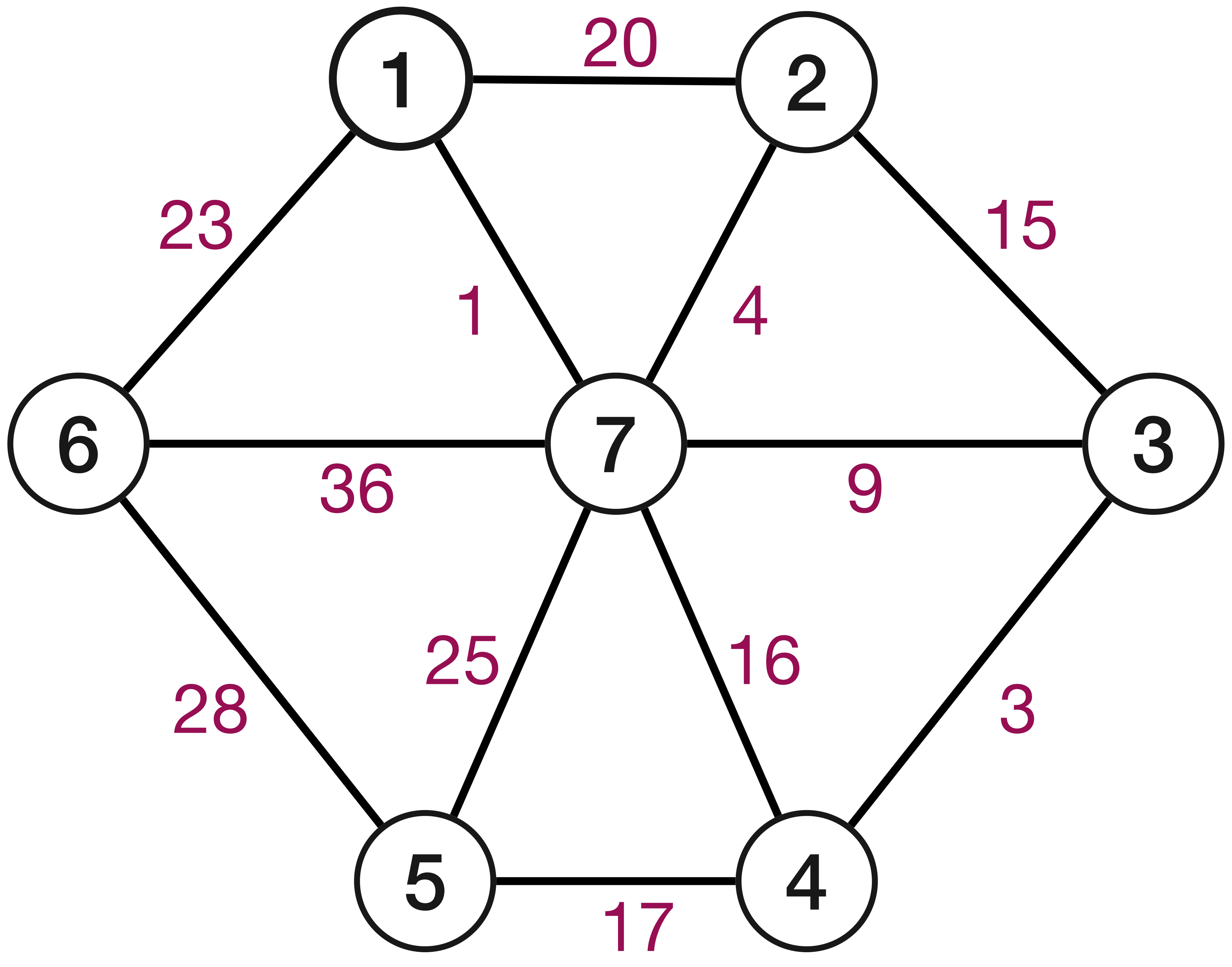
All mistakes are my own! - Ivan Abraham (Fall 2024)

Image by ChatGPT (probably collaborated with DALL-E)

Minimum Spanning Tree

Input: Connected graph $G = (V, E)$ with edge costs

Goal: Find $T \subseteq E$ such that (V, T) is connected and total cost of all edges in T is smallest



Minimum Spanning Tree

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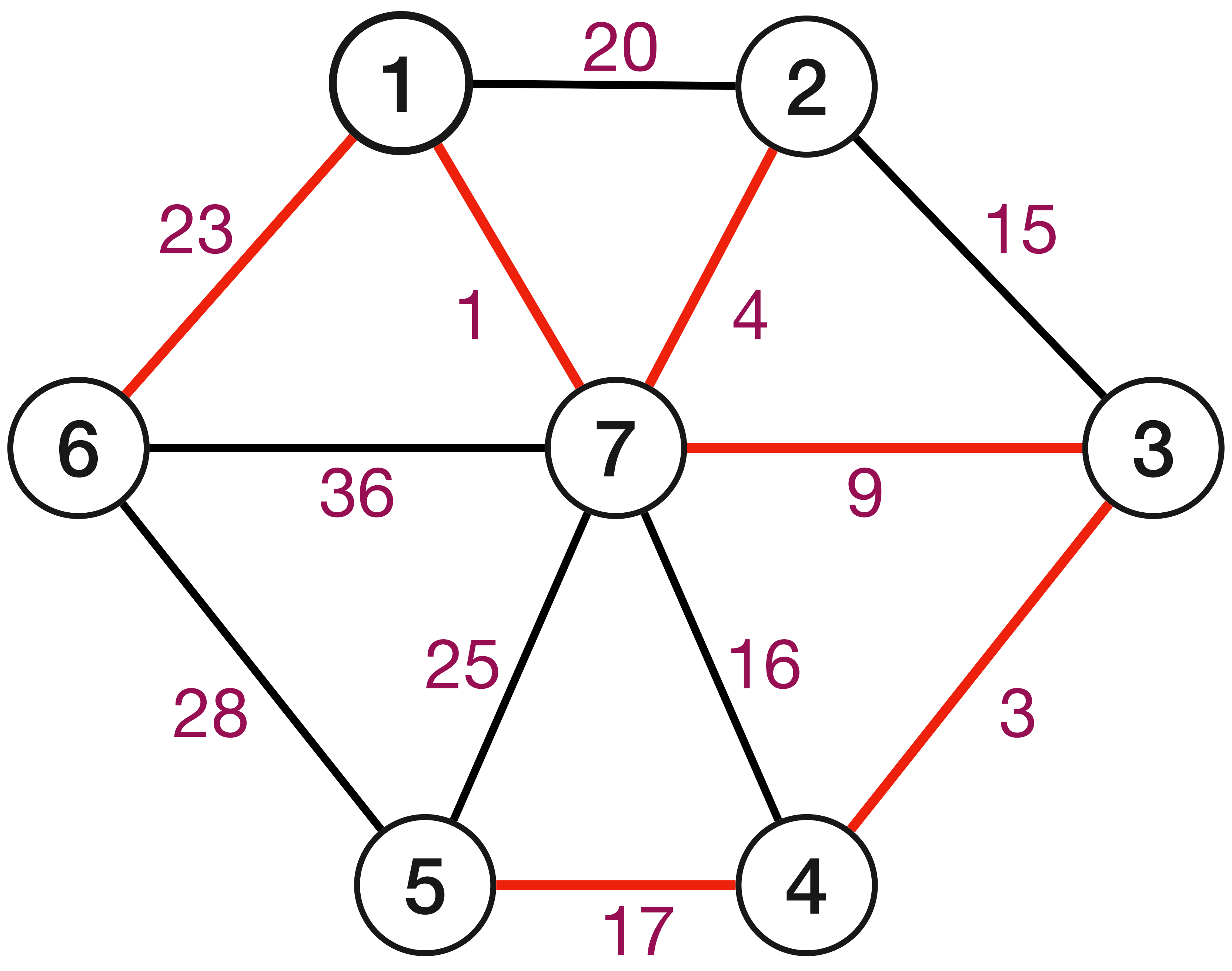
Goal: Find $T \subseteq E$ such that (V, T) is connected and total cost of all edges in T is smallest

T is the **minimum spanning tree (MST)** of G.



Minimum Spanning Tree

Applications

- Network design
  - Designing networks with minimum cost but maximum connectivity
- Approximation algorithms
  - Can be used to bound the optimality of algorithms to approximate Traveling Salesman Problem, Steiner Trees, etc.
- Cluster analysis

Spanning Trees

Basic properties

- Subgraph H of G is **spanning** for G, if G and H have same connected components.
- ***Tree***: undirected graph in which any two vertices are connected by exactly one path $\sim$ a connected (undirected) graph with no cycles.
  - Every tree has a leaf (i.e., vertex of degree one).
- A tree T on a graph G is **spanning** if T includes every node of G.
- Every ***spanning tree*** of a graph on n nodes has $n-1$ edges.
- A graph G is connected $\iff$ it has a spanning tree.

Minimum Spanning Tree

Some history

The first algorithm for MST was first published in 1926 by Otakar Borůvka as a method of constructing an efficient electricity network for Moravia. From his memoirs:

> *My studies at poly-technical schools made me feel very close to engineering sciences and made me fully appreciate technical and other applications of mathematics. Soon after the end of World War I, at the beginning of the 192Os, the Electric Power Company of Western Moravia, Brno, was engaged in rural electrification of Southern Moravia. In the framework of my friendly relations with some of their employees, I was asked to solve, from a mathematical standpoint, the question of the most economical construction of an electric power network. I succeeded in finding a construction-as it would be expressed today-of a maximal(ly) connected subgraph of minimum length, which I published in 1926 (i.e., at a time when the theory of graphs did not exist).*

There is some work in 1909 by a Polish anthropologist Jan Czekanowski on clustering, which is a precursor to MST.

Exchanging an edge in a spanning tree

Useful lemma

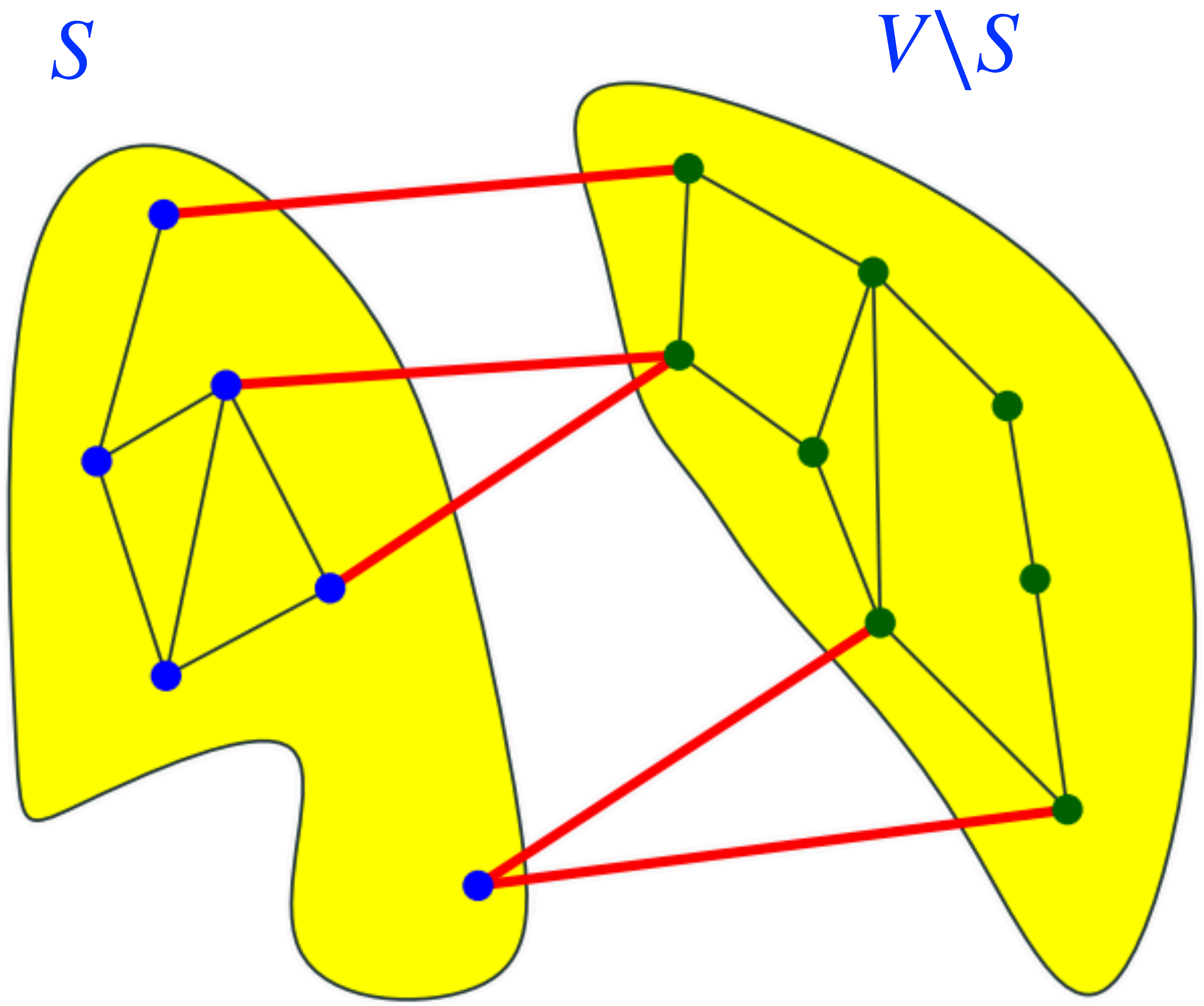
Let $T = (V, E_T)$ be a spanning tree of $G = (V, E)$. Then,

- For every non-tree edge $e \in E \backslash E_T$ there is a unique cycle C in $T + e$.
- For every edge $f \in C - \{e\}$, $T - f + e$ is another spanning tree of G.

Cuts

Definition

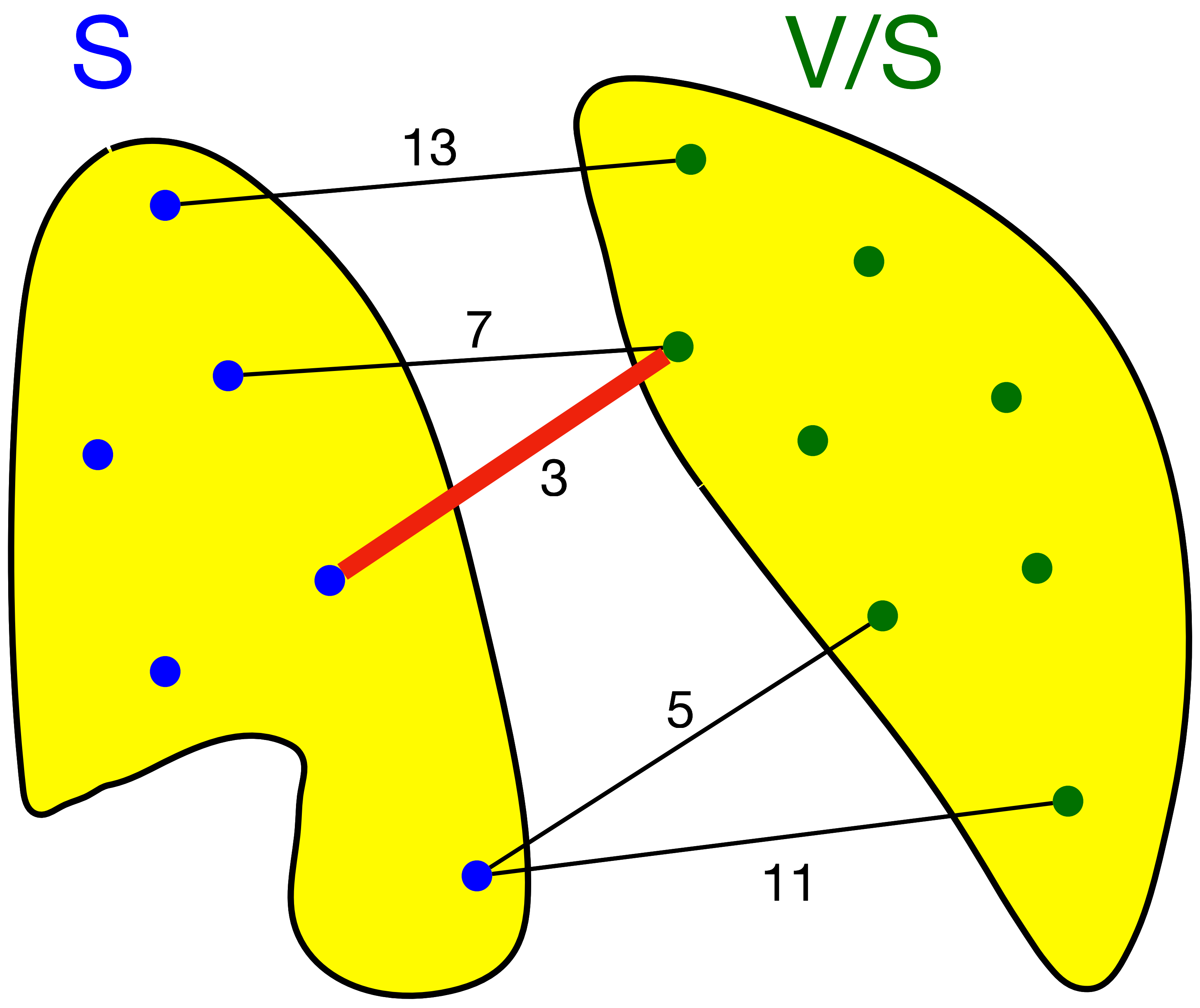
- Given a graph $G = (V, E)$, a ***cut*** is a partition of the vertices of the graph into two sets $(S, V \backslash S)$.
- Edges having an endpoint on both sides are the *edges of the cut.*
- A cut edge is *crossing* the cut.



Safe edge

Example

- Every cut identifies one safe edge …
- … the cheapest edge in the cut.
- Note: An edge e may be a safe edge for many cuts!

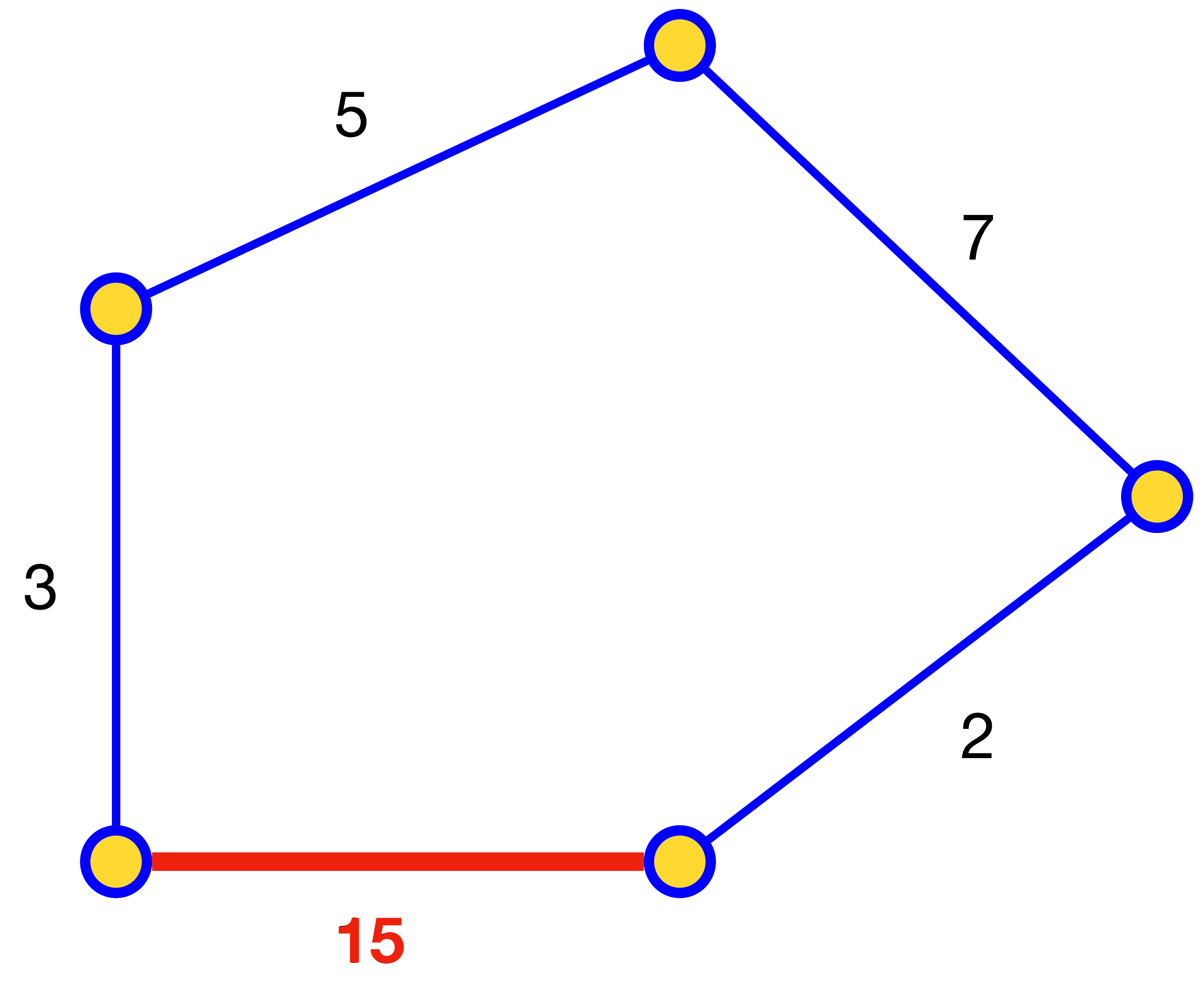


Safe edge in the cut (S, V/S)

Unsafe edge

Example

- Every cycle identifies one unsafe edge …
- … the most expensive edge in the cycle.



Safe and unsafe edges

Assumption: Edge costs are distinct, that is no two edge costs are equal.

Safe edge:

An edge $e = (u, v)$ is a *safe edge* if there is some partition of V into S and $V \backslash S$ and e is the unique minimum cost edge crossing S (one end in S and the other in $V \backslash S$).

Unsafe edge

An edge $e = (u, v)$ is an *unsafe edge* if there is some cycle C such that e is the unique maximum cost edge in C.

Safe and unsafe edges

Every edge is either safe or unsafe

Proposition: If edge costs are distinct then every edge is either safe or unsafe.

Proof: Consider any edge $e = uv$. Let $G_{<w(e)} = \big(V, \{xy \in E \,|\, w(xy) < w(e)\}\big)$

- If x, y in same connected component of $G_{<w(e)}$, then $G_{<w(e)} + e$ contains a cycle where e is most expensive $\implies$ e is unsafe.

- If x and y are in different connected components of $G_{<w(e)}$, let S the vertices of connected component of $G_{<w(e)}$ containing x . The edge e is cheapest edge in cut $(S, V \backslash S) \implies$ e is safe.

Example

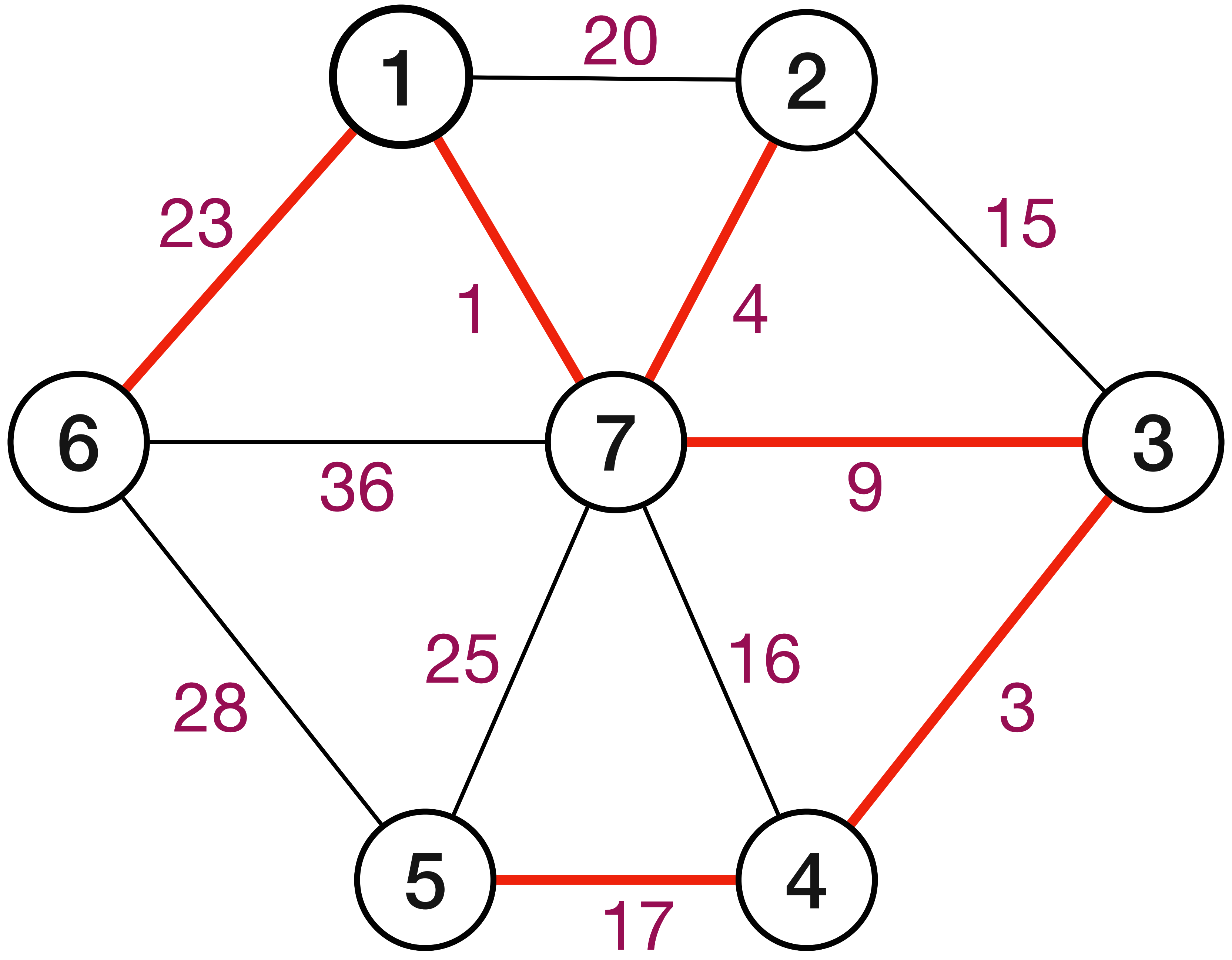


Figure 1: Graph with unique edge costs.

Safe edges are red, rest are unsafe.

And all safe edges are in the MST in this case …

Why do we care about safety?

Lemma: (a) If e is a safe edge then every minimum spanning tree contains e and **(b)** if e is an unsafe edge then no MST of G contains e.

- Many different MST algorithms
- All of them rely on some basic properties of MSTs, in particular the ***Cut Property*** (part one of the lemma).
- Part two of the lemma is called the ***Cycle Property***.

Is the proof correct?

Key observation

Cut property

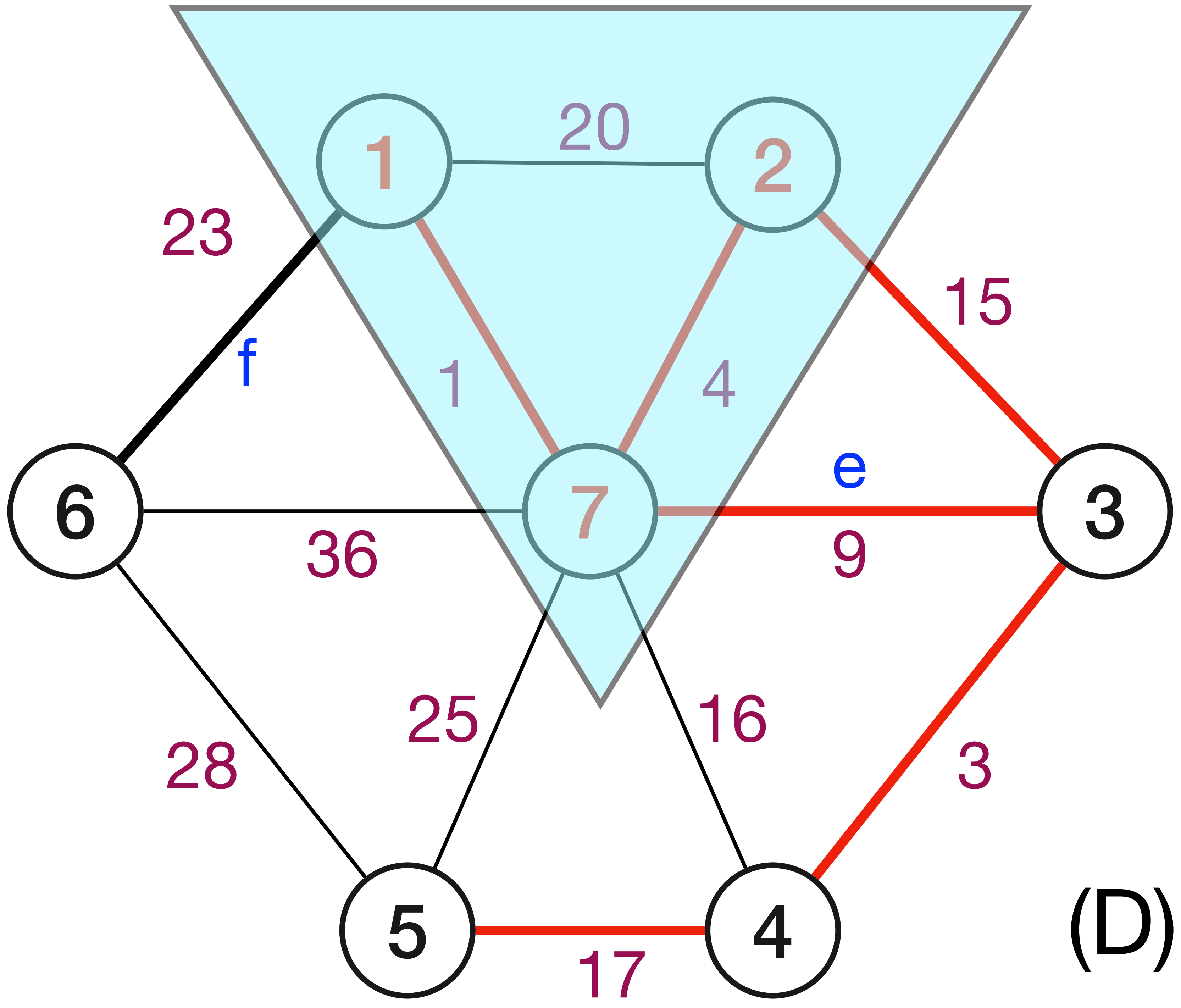
Lemma: If e is a safe edge then ***every*** minimum spanning tree contains e.

Proof: Suppose (for contradiction) e is not in MST T.

- Since e is safe there is an $S \subset V$ such that e is the unique min cost edge crossing S.
- Since T is connected, there must be some edge f with one end in S and the other in $V \backslash S$
- Since $c_f > c_e$, $T' = \left(T \backslash \{f\}\right) \cup \{e\}$ is a spanning tree of lower cost!

Error in proof …

Problematic example. $S = \{1,2,7\}$, $e = (7,3)$, $f = (1,6)$. $T - f + e$ is not a spanning tree



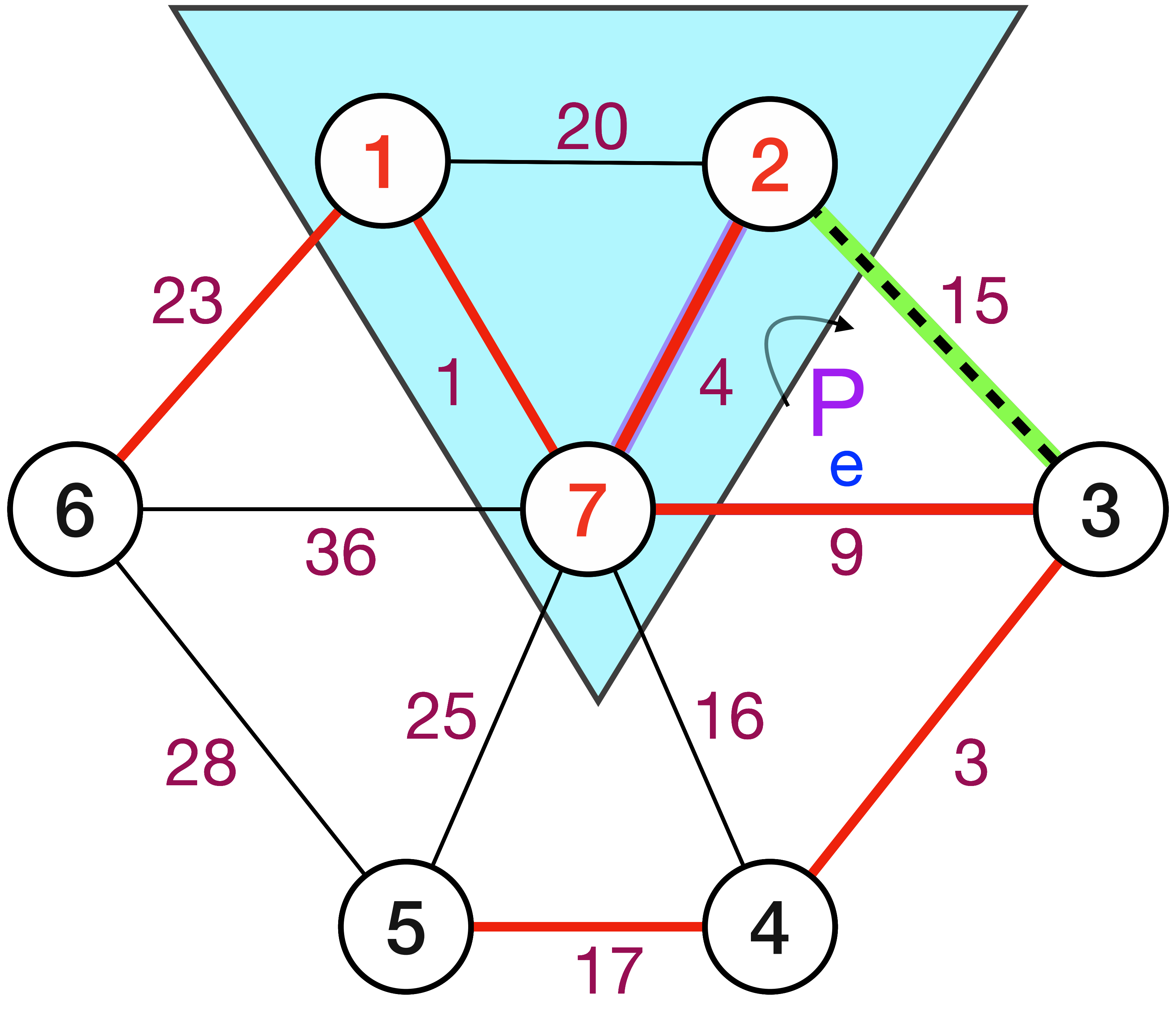
(A) Consider adding the edge e to MST.

(B) It is safe because it is the cheapest edge in the cut.

(C) Lets throw out the edge f currently in the spanning tree which is more expensive than e and is in the same cut. Put in e instead.

(D) New graph of selected edges is not a tree!

Proof of Cut Property



1. Suppose $e = (v, w)$ is not in MST T and e is min weight edge in cut $(S, V \backslash S)$. Assume $v \in S$. It is safe because it is the cheapest edge in the cut.

2. T is spanning tree: there is a unique path P from v to w in T.

3. Let w' be the first vertex in P belonging to $V \backslash S$; let v' be the vertex just before it on P, and let $e' = (v', w')$

4. $T' = (T \backslash \{e'\}) \cup \{e\}$ is spanning tree of lower cost. (Why?)

Proof of Cut Property

(contd)

Observation: $T' = (T \backslash \{e'\}) \cup \{e\}$ is a spanning tree.

Proof: T' is connected

Removed $e' = (v', w')$ from T but v' and w' are connected by the path $P - f + e$ in T' . Hence T' is connected if T is.

Proof: T' is a tree

T' is connected and has $n - 1$ edges (since T had $n - 1$ edges) and hence T' is a tree.

Safe edges form a connected graph

Lemma: Let G be a connected graph with distinct edge costs, then the set of safe edges form a connected graph.

Proof:

- Suppose not. Let S be a connected component in the graph induced by the safe edges.
- Consider the edges crossing S, there must be a safe edge among them since edge costs are distinct and so we must have picked it.

Safe edges, cycles and MST

Lemma: Let G be a connected graph with distinct edge costs, then the set of safe edges does not contain a cycle.

Corollary: Let G be a connected graph with distinct edge costs, then set of safe edges form the unique MST of G.

Consequence: Every correct MST algorithm when G has unique edge costs includes exactly the safe edges.

Borůvka's Algorithm

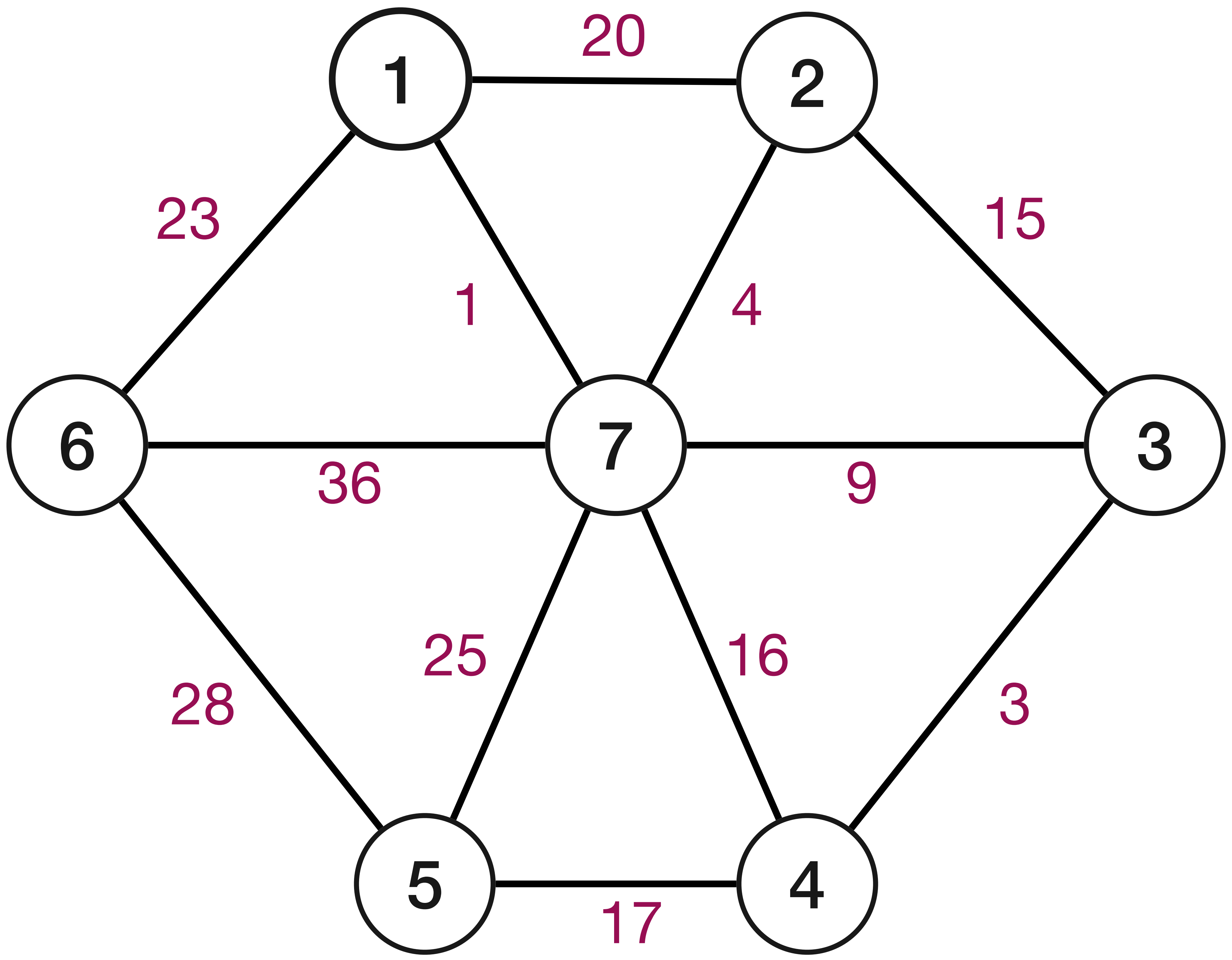
Simplest to implement. Assume G is a connected graph.

```
T is ∅ (* T will store edges of a MST *)

while T is not spanning do
    X ← ∅
    for each connected component S of T do
        add to X the cheapest edge between S and V\S
    Add edges in X to T

return the set T
```

Borůvka's Algorithm

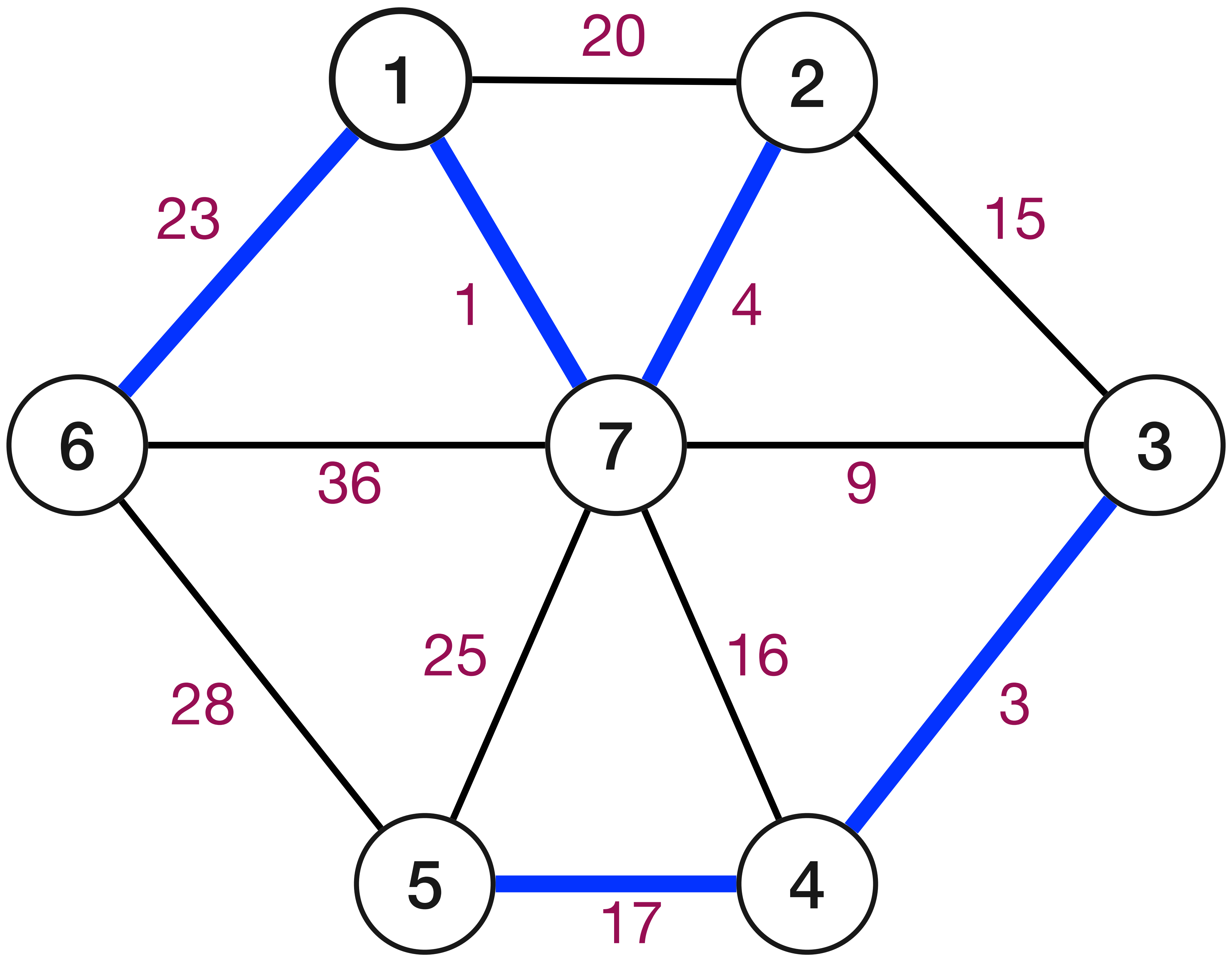


Initialize: All vertices are singleton connected components.

Heuristic: Each vertex tries to expand its "network" (connected component) by gaining the "least expensive friend."

Iterate until a spanning tree is formed.

Borůvka's Algorithm



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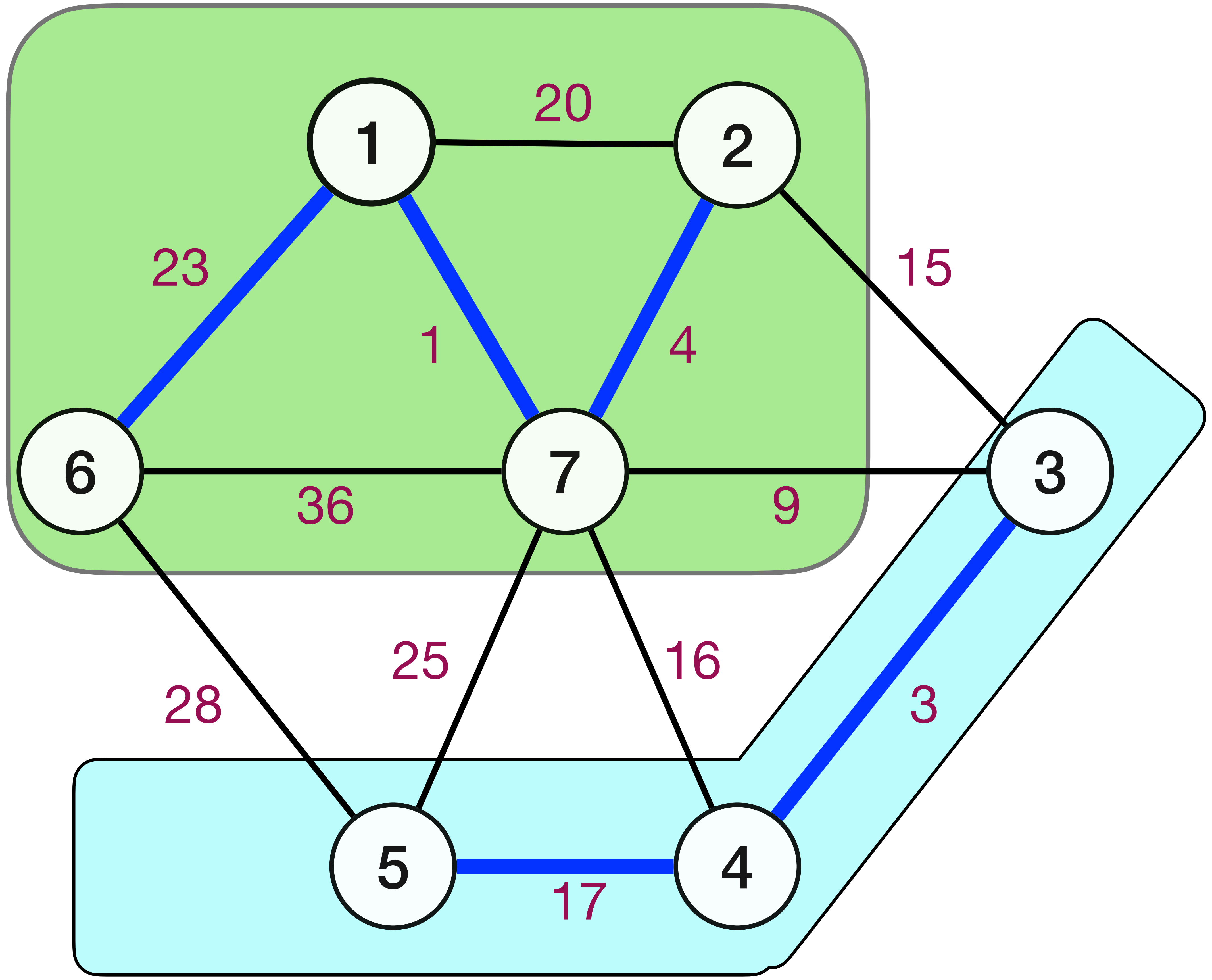

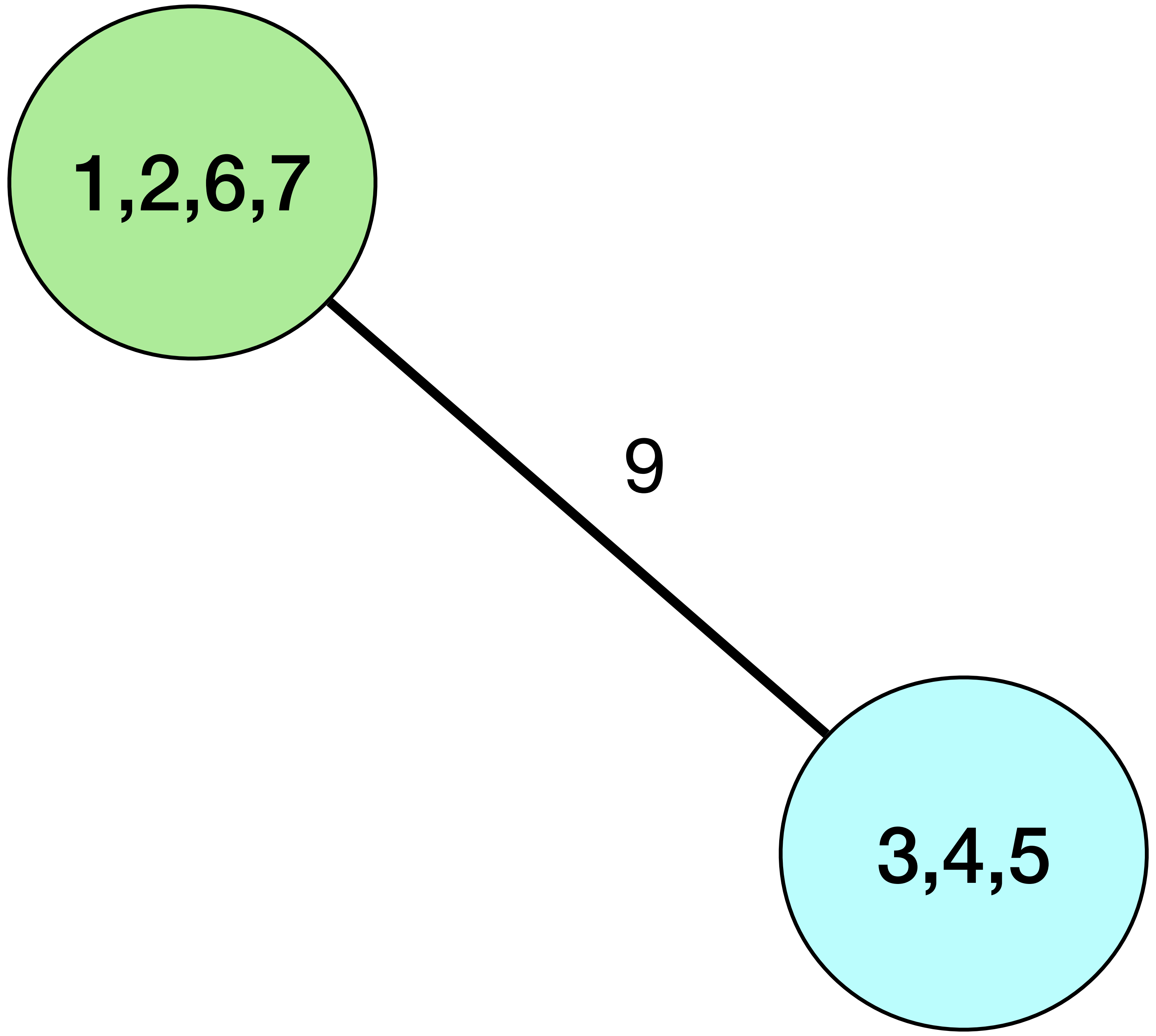

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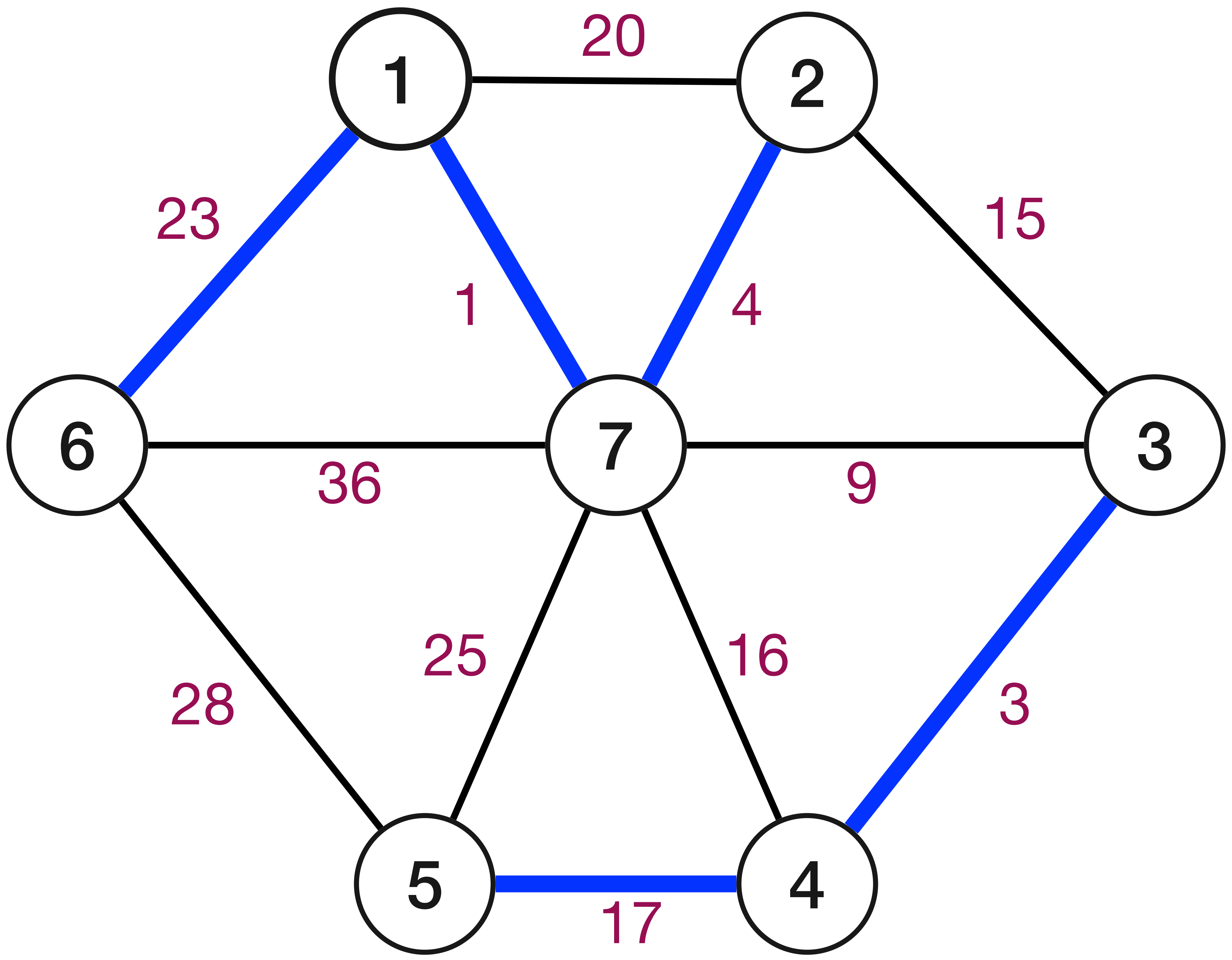

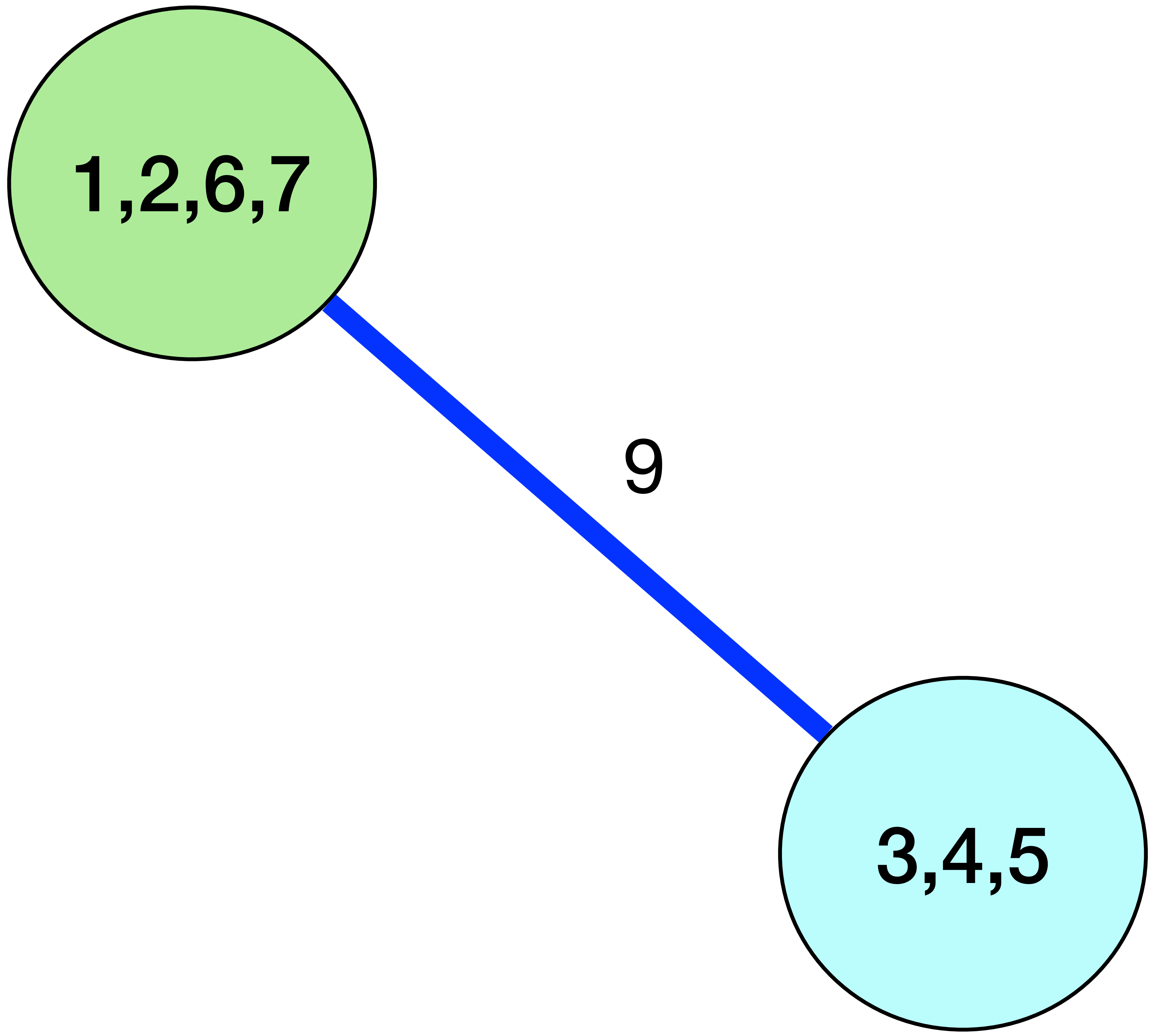

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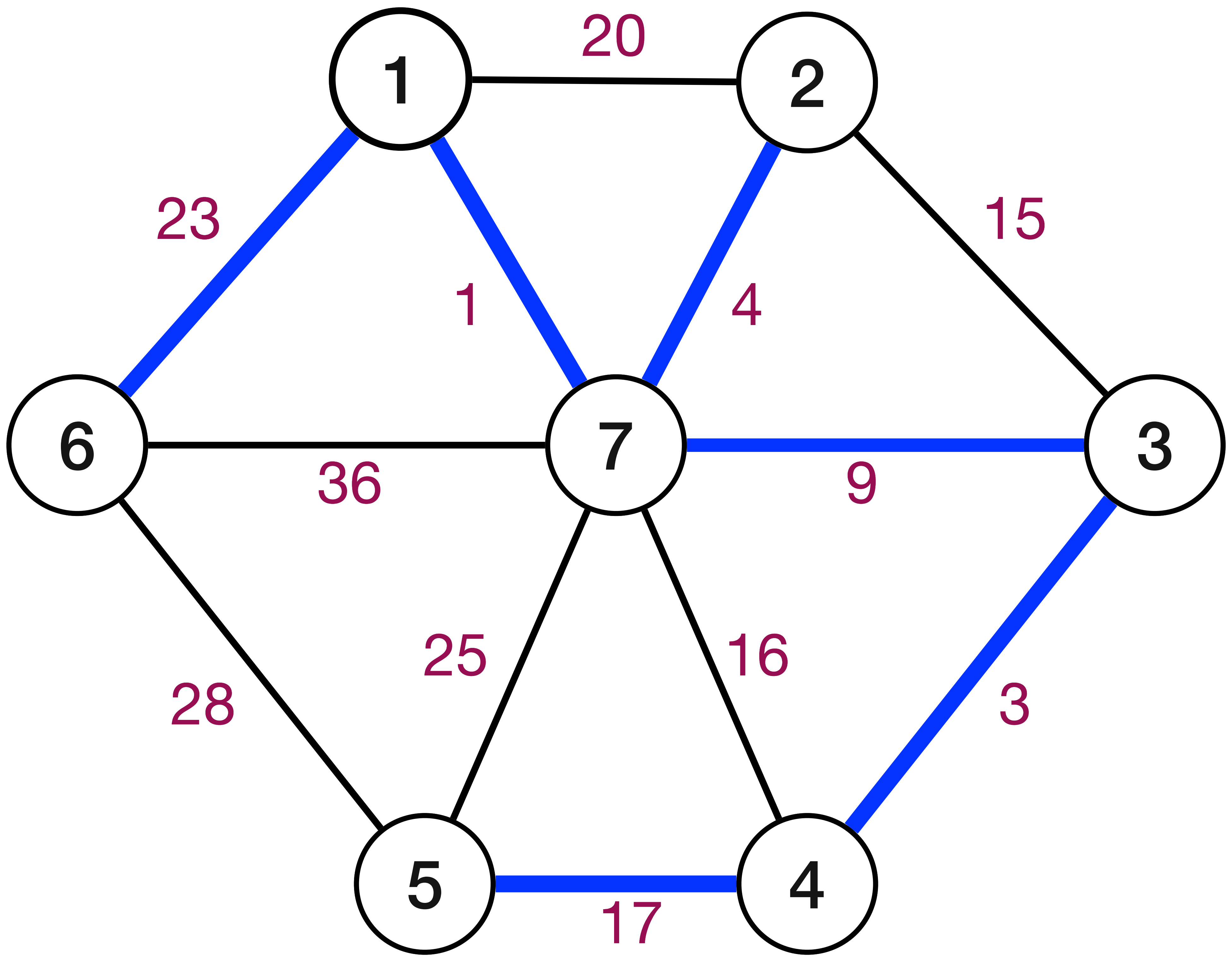


```
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while T is not spanning do
      X ← ∅
      for each connected component S of T
      do
          add to X the cheapest edge
          between S and V\S
      Add edges in X to T
return the set T
```

Implementing Borůvka's Algorithm

- $O(\log n)$ iterations of while loop. Why?
  - Number of connected components shrink by at least half since each component merges with one or more other components.
- Each iteration can be implemented in $O(m)$ time.
- Running time: $O(m \log n)$ time

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Mininimum Spanning Trees

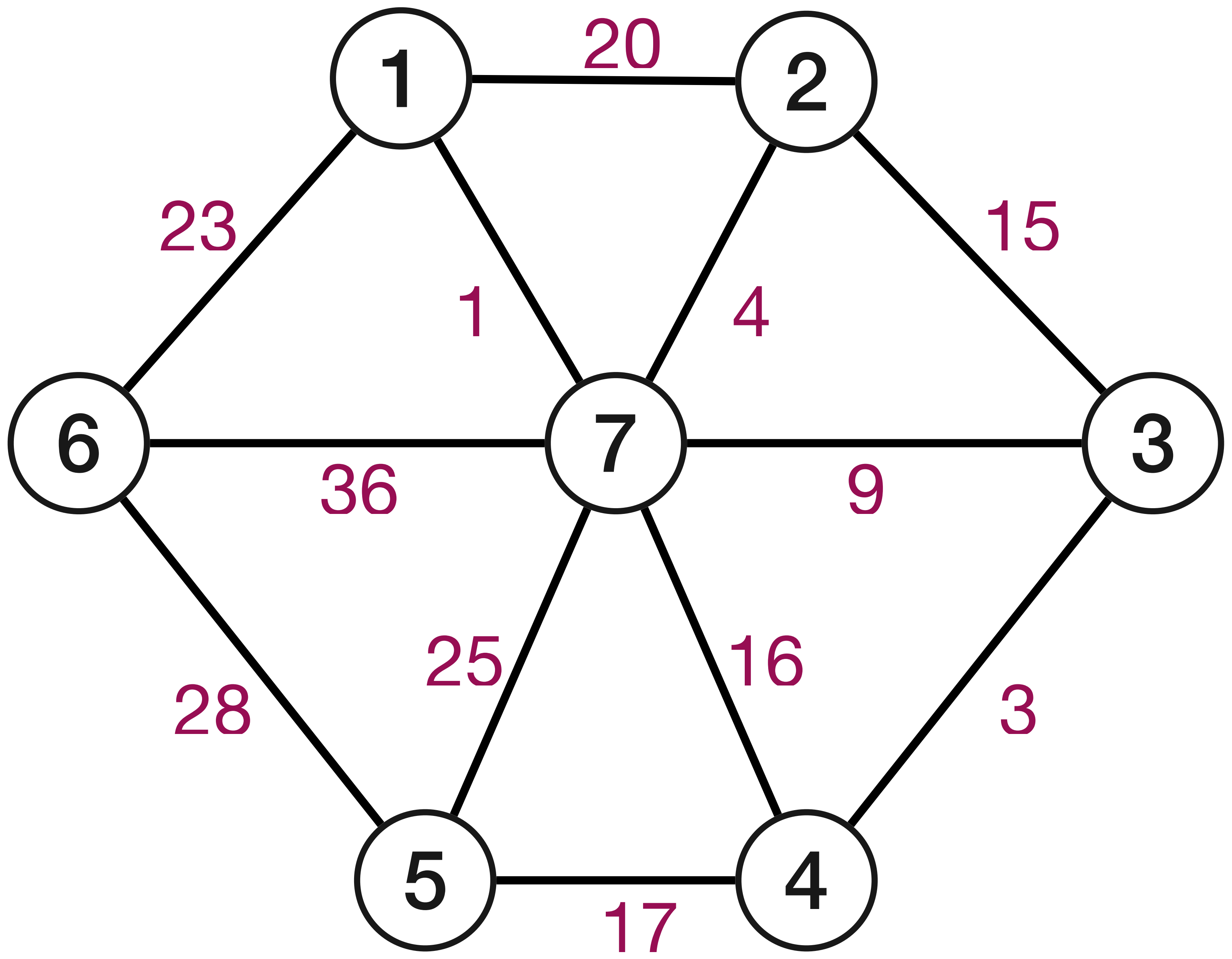
Greedy template

- In what order should the edges be processed?
- When should we add edget to spanning tree?
- Leads to Kruskal’s and Prim’s algorithms.

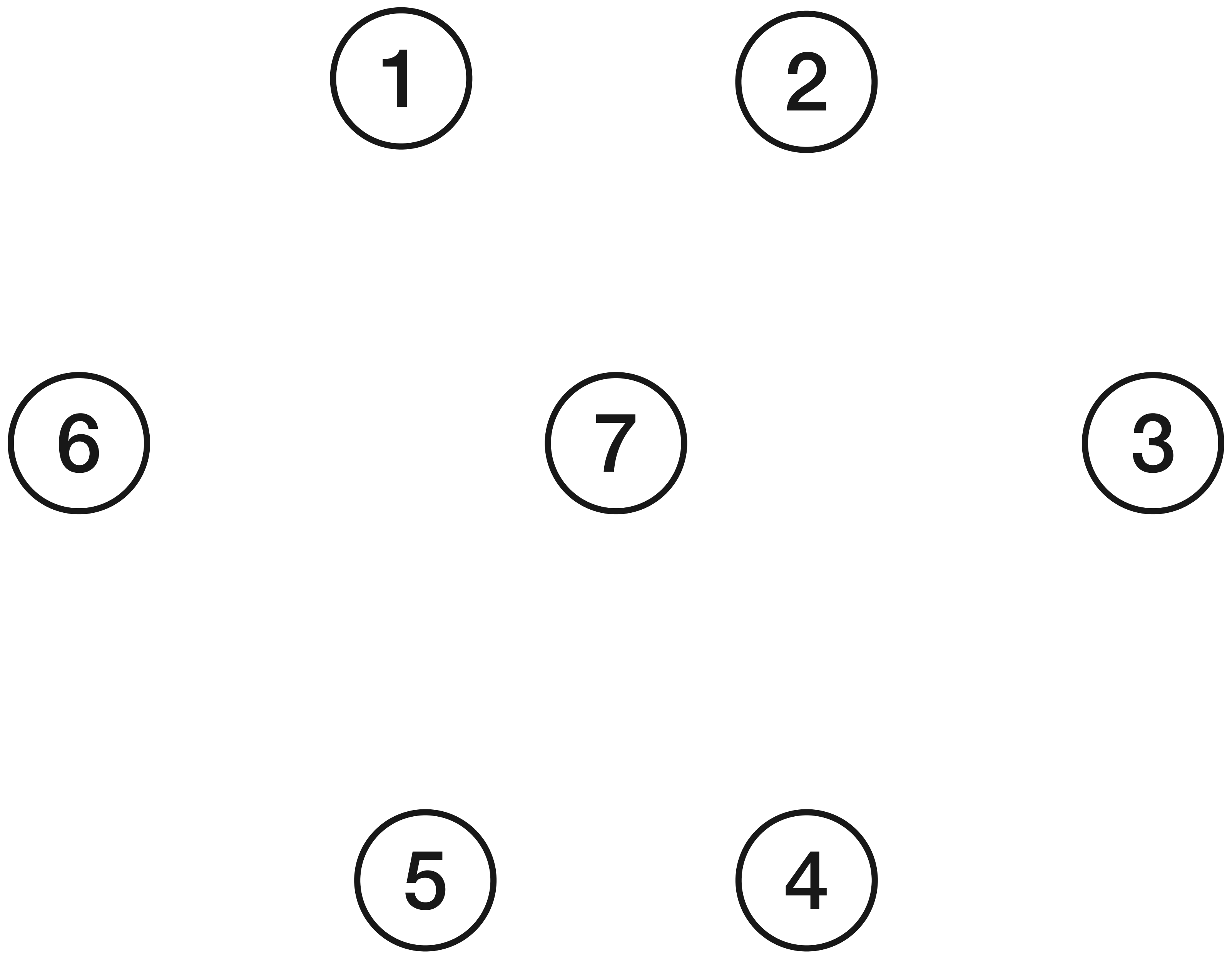
```
Initially E is the set of all edges in G

T is empty (*T will store edges of a MST*)

while E is not empty do
  choose e ∈ E
  if (e satisfies condition)
    add e to T

return the set T
```

Kruskal's Algorithm

Process edges in the order of their costs (starting from the least) and add edges to T as long as they don't form a cycle.



Graph G



MST of G

Kruskal's Algorithm

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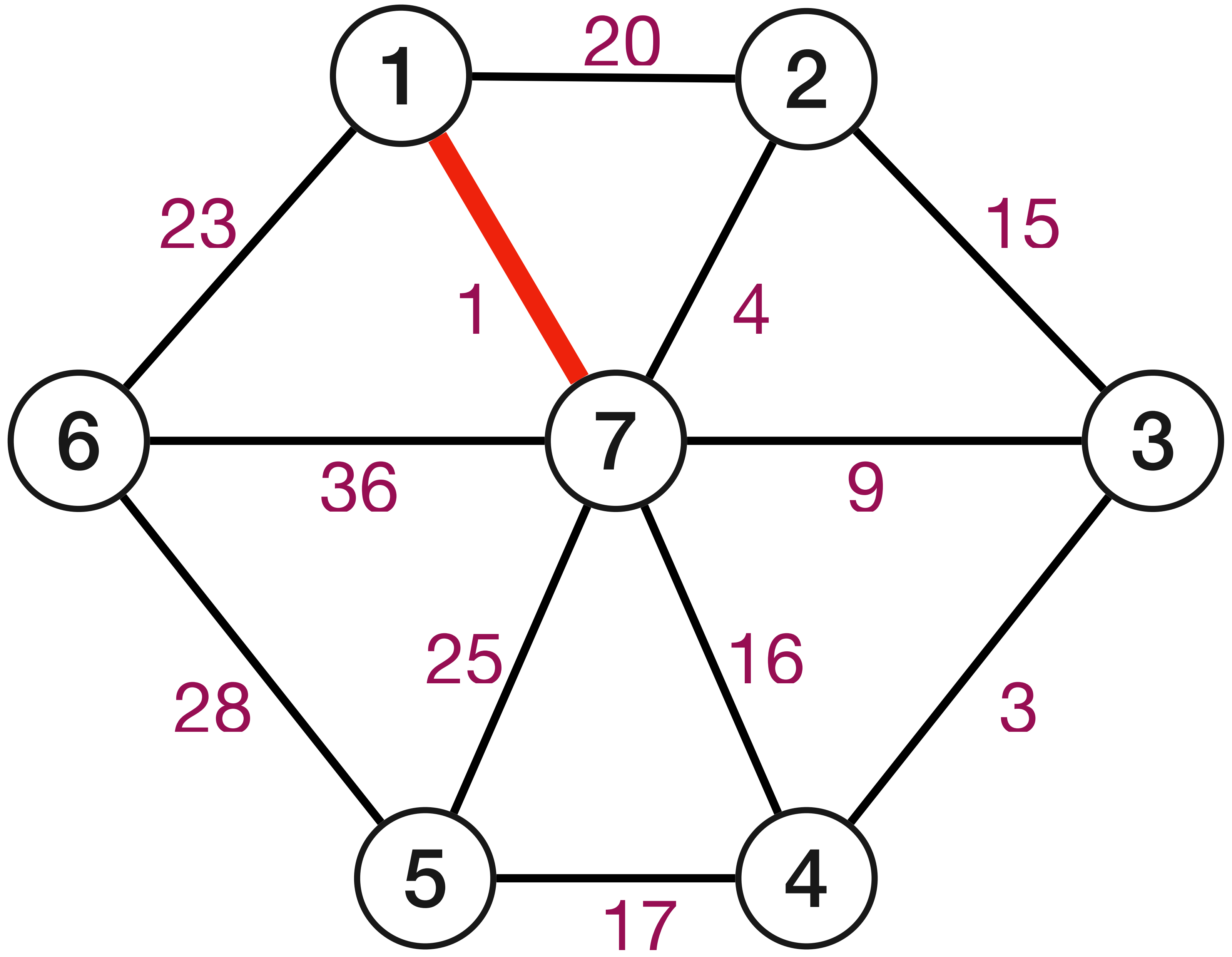

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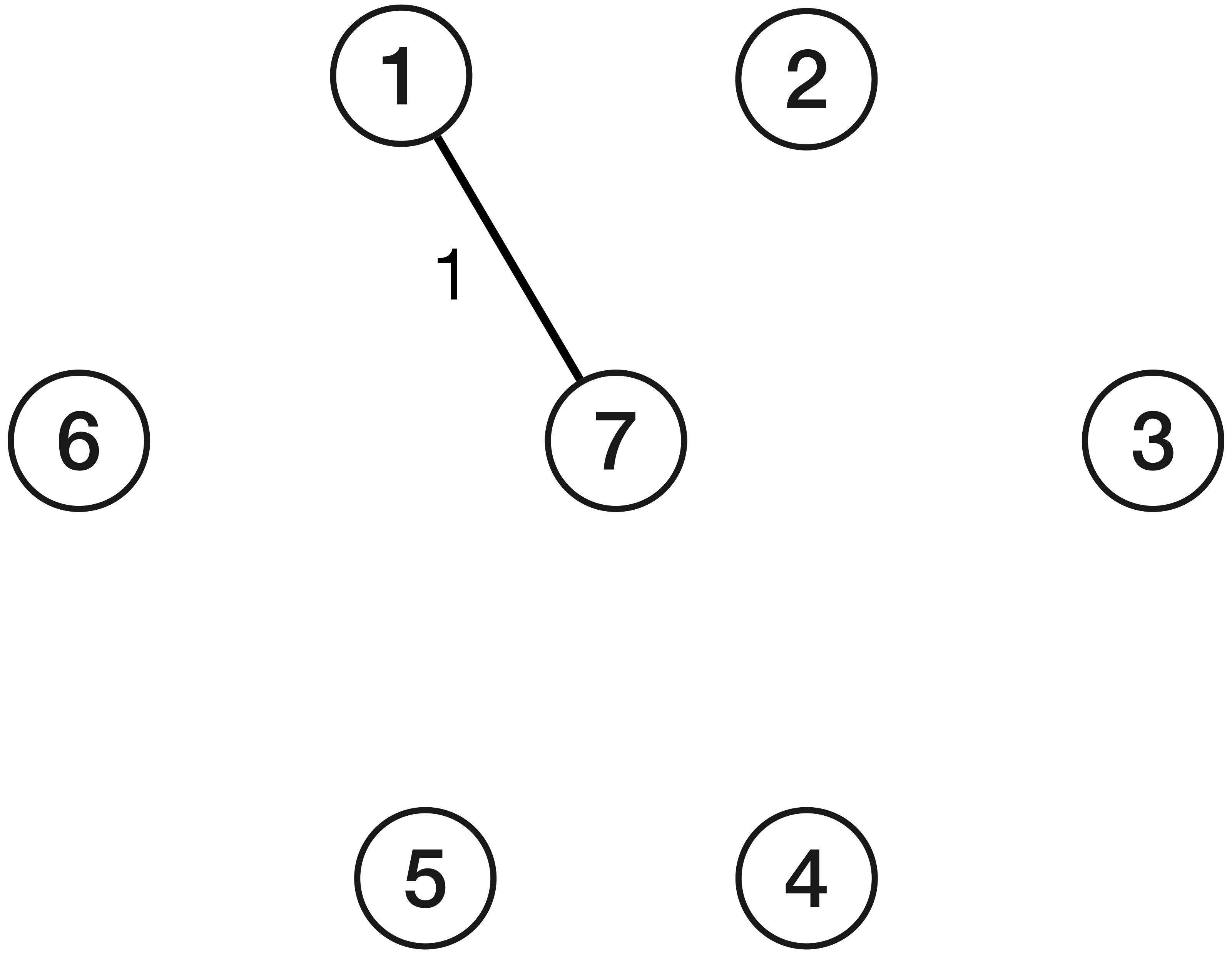

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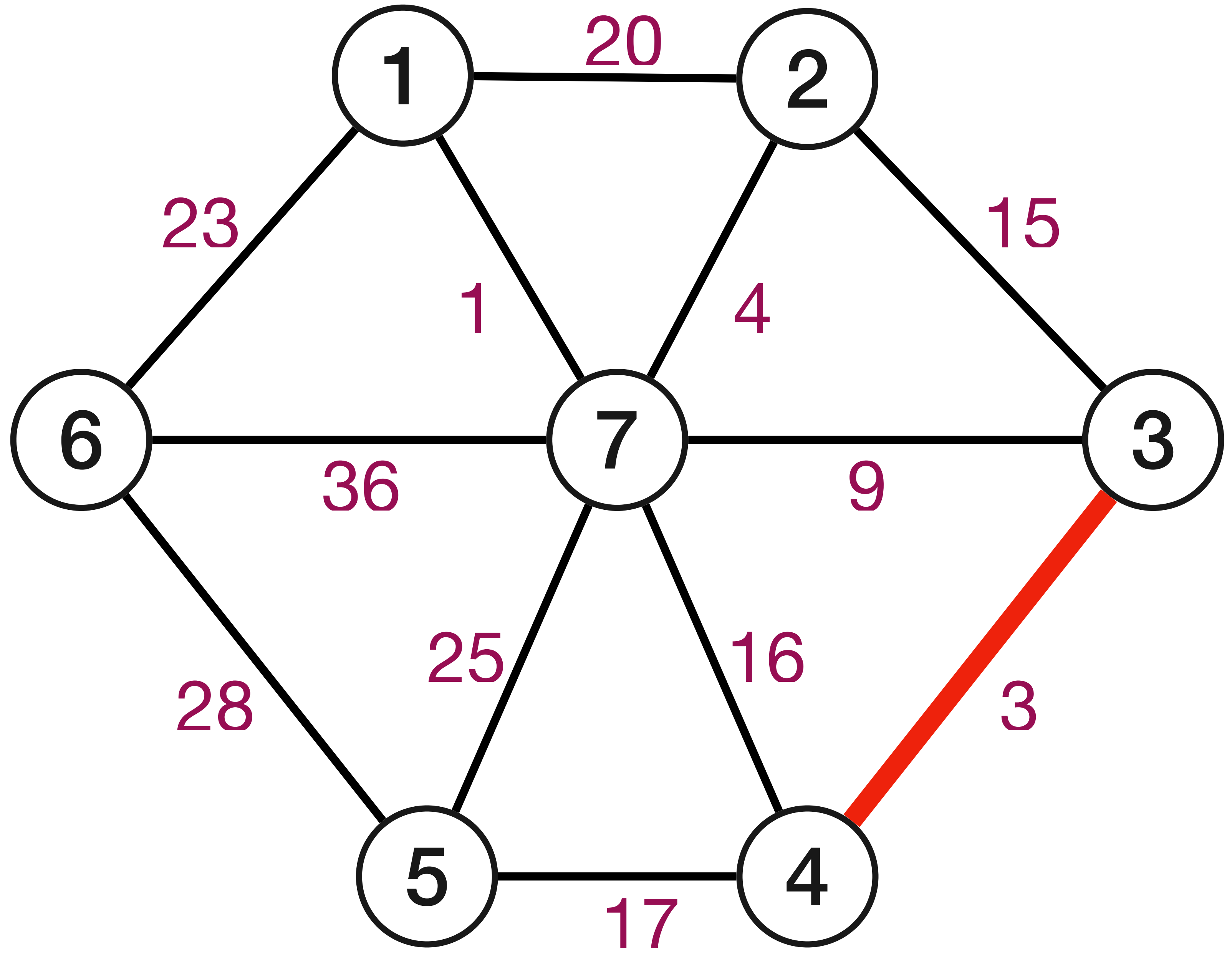

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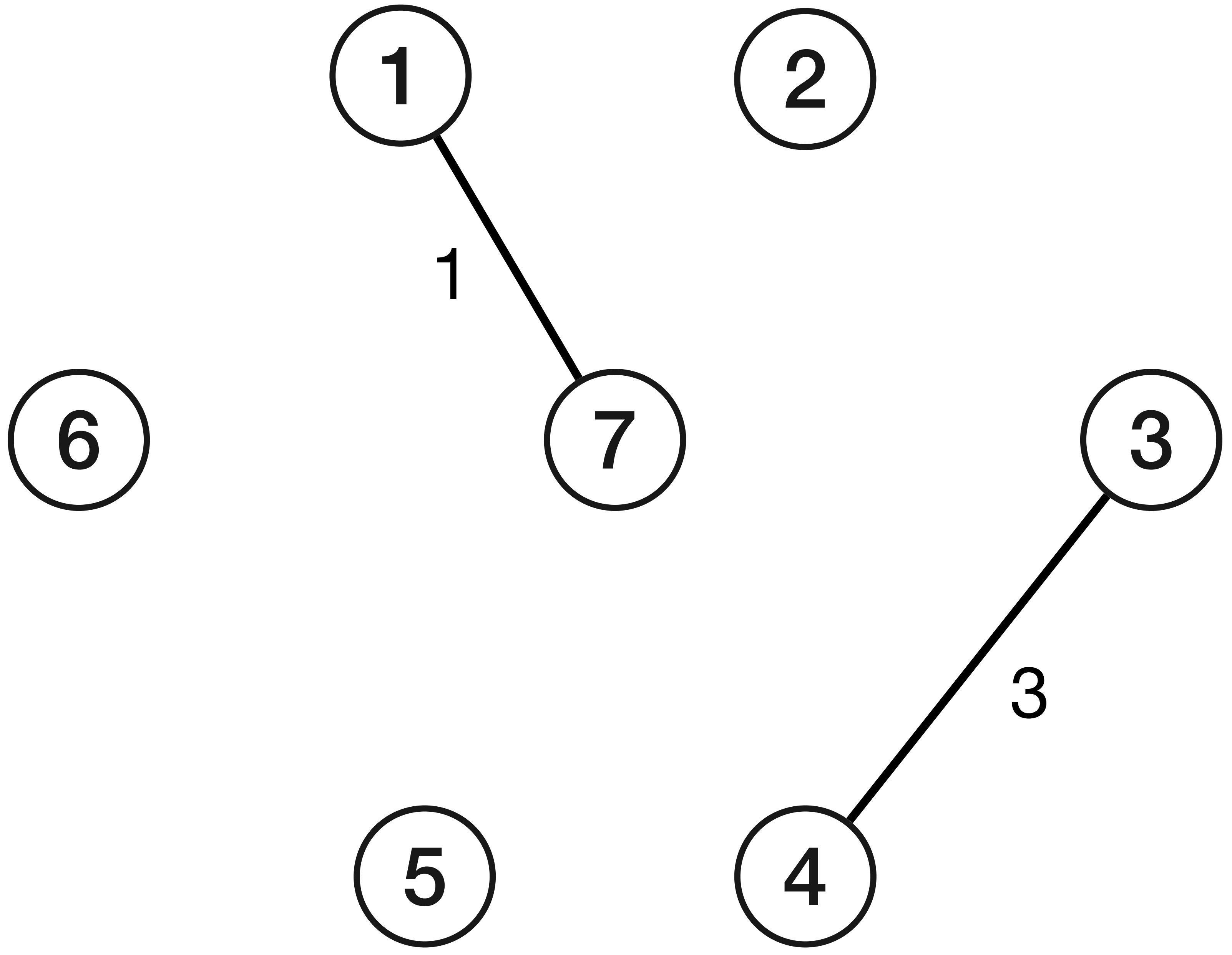

MST of G

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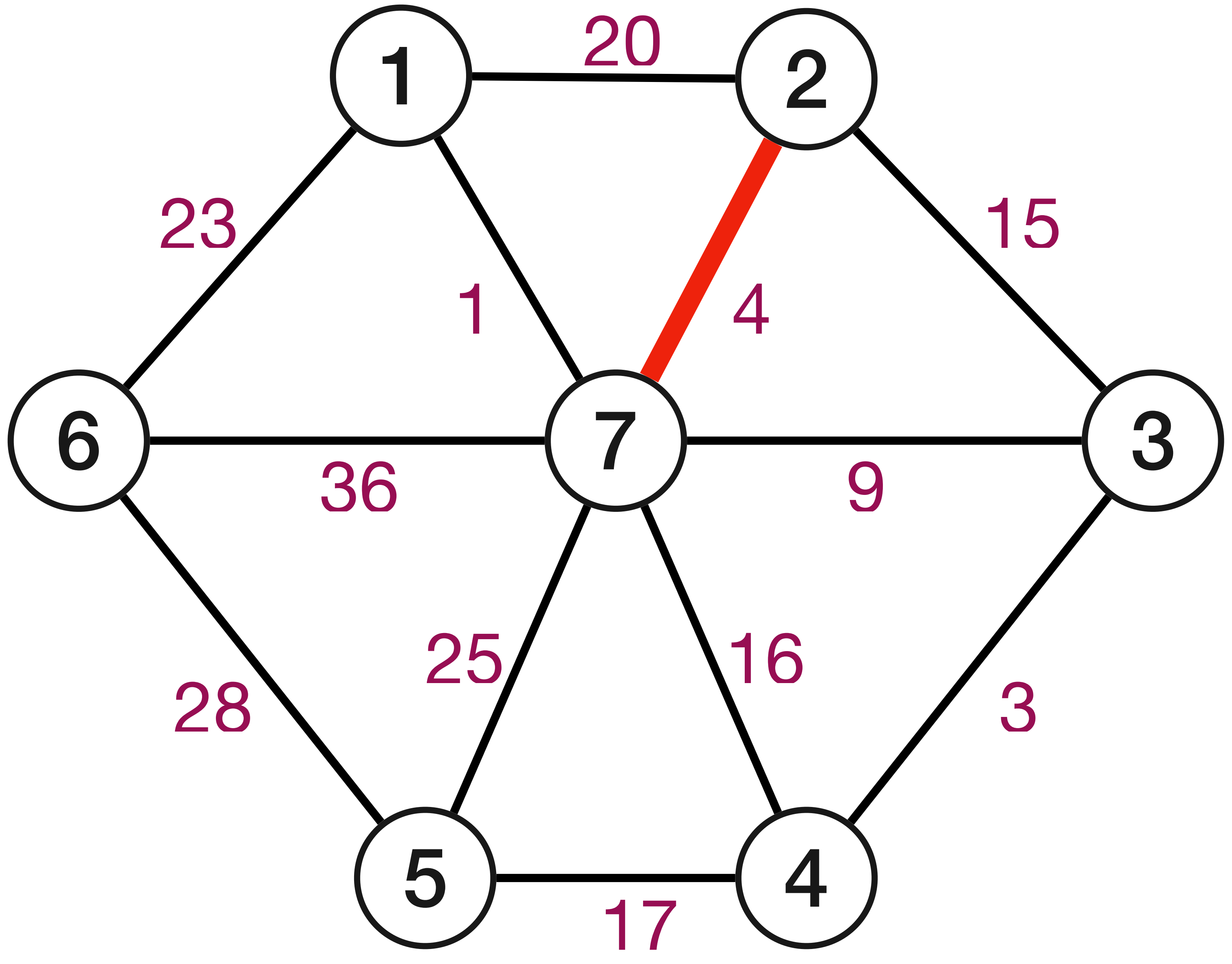

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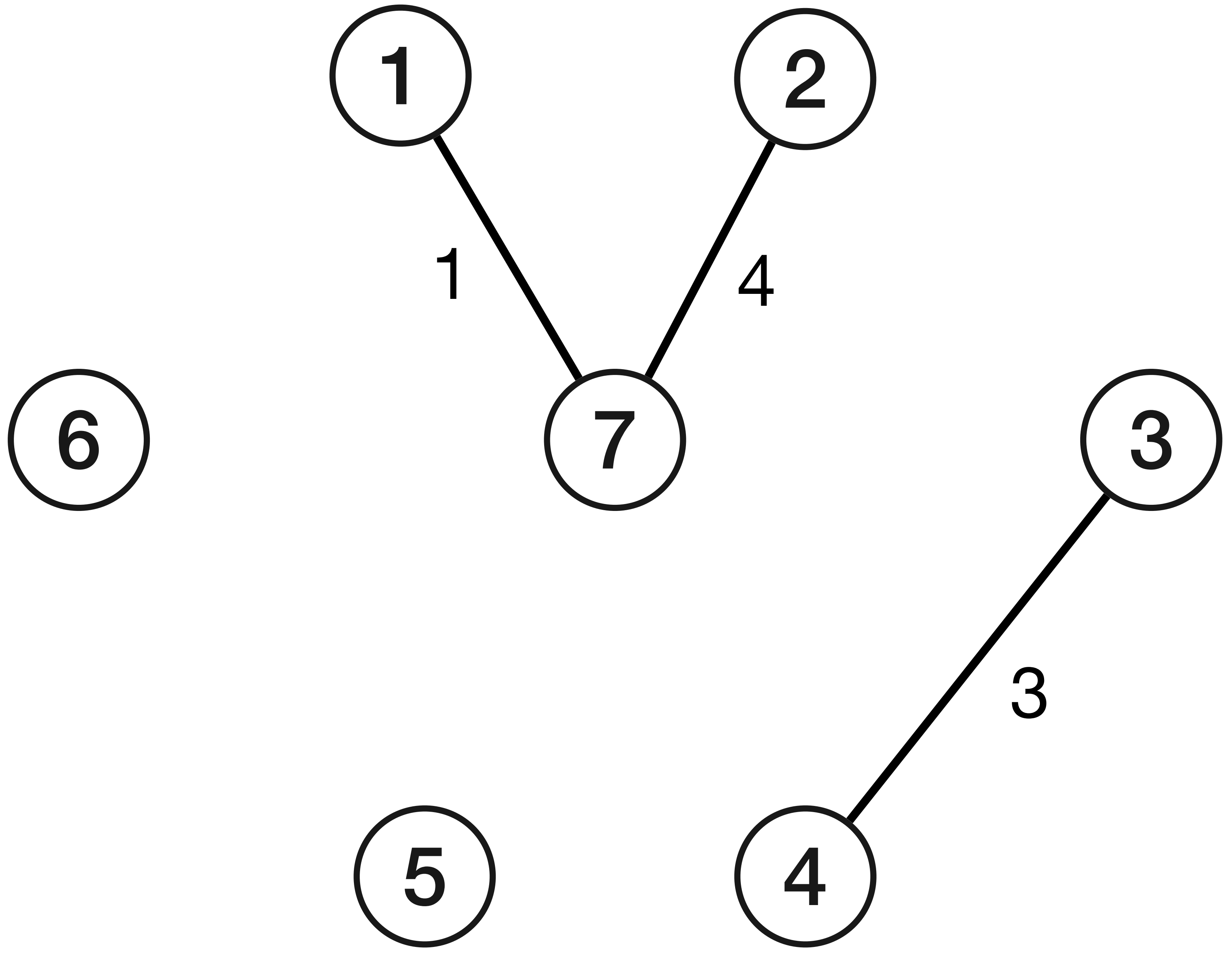

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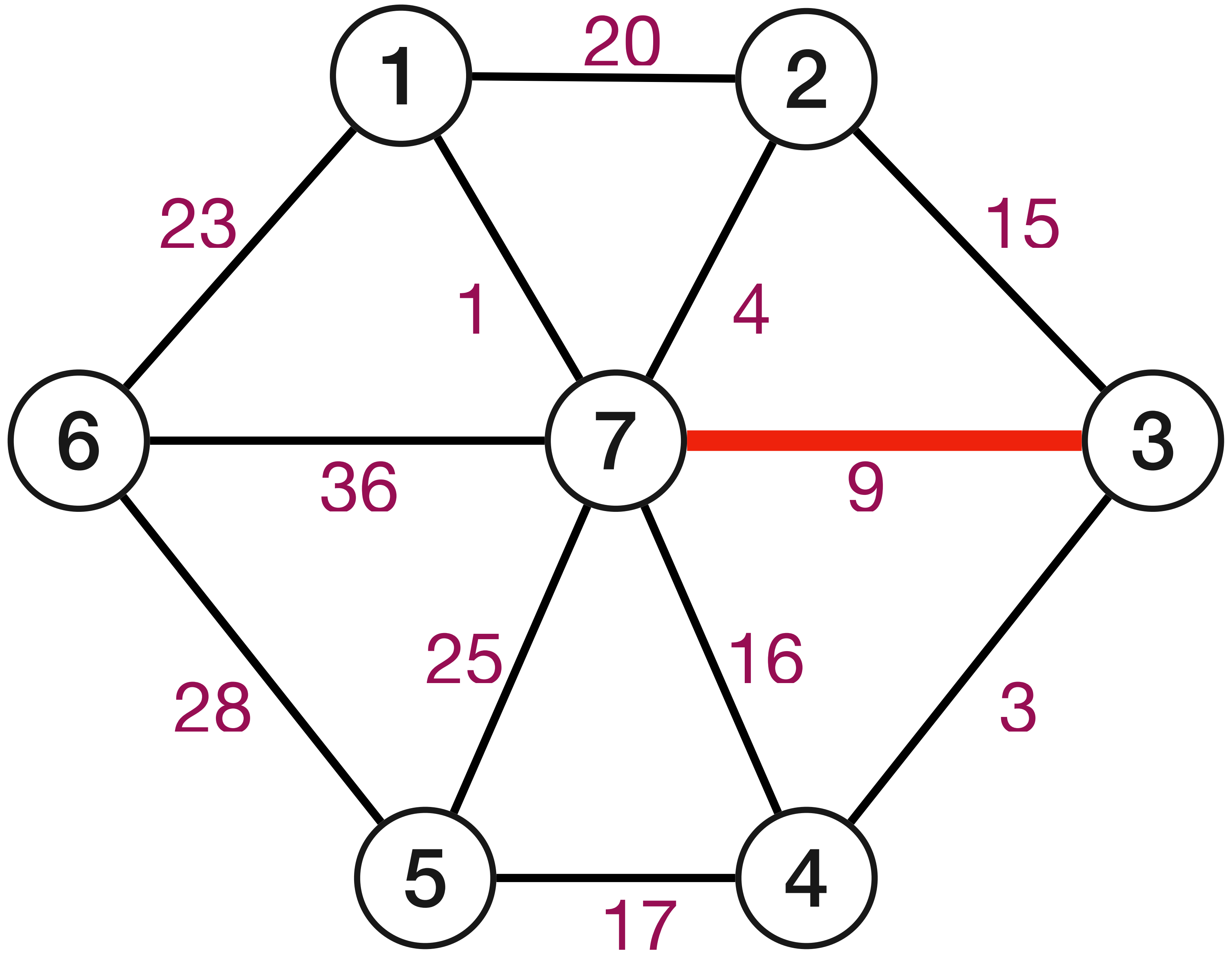

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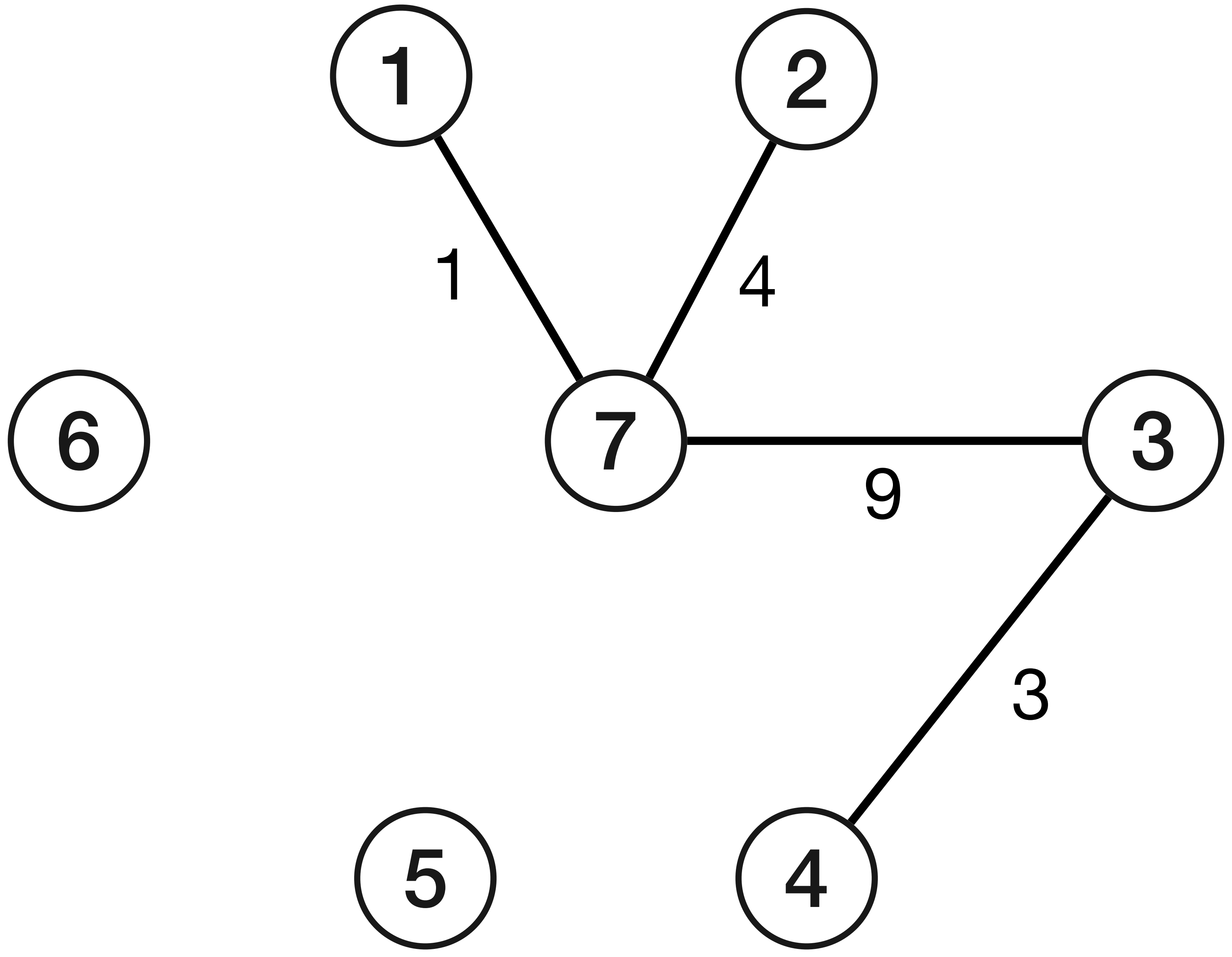

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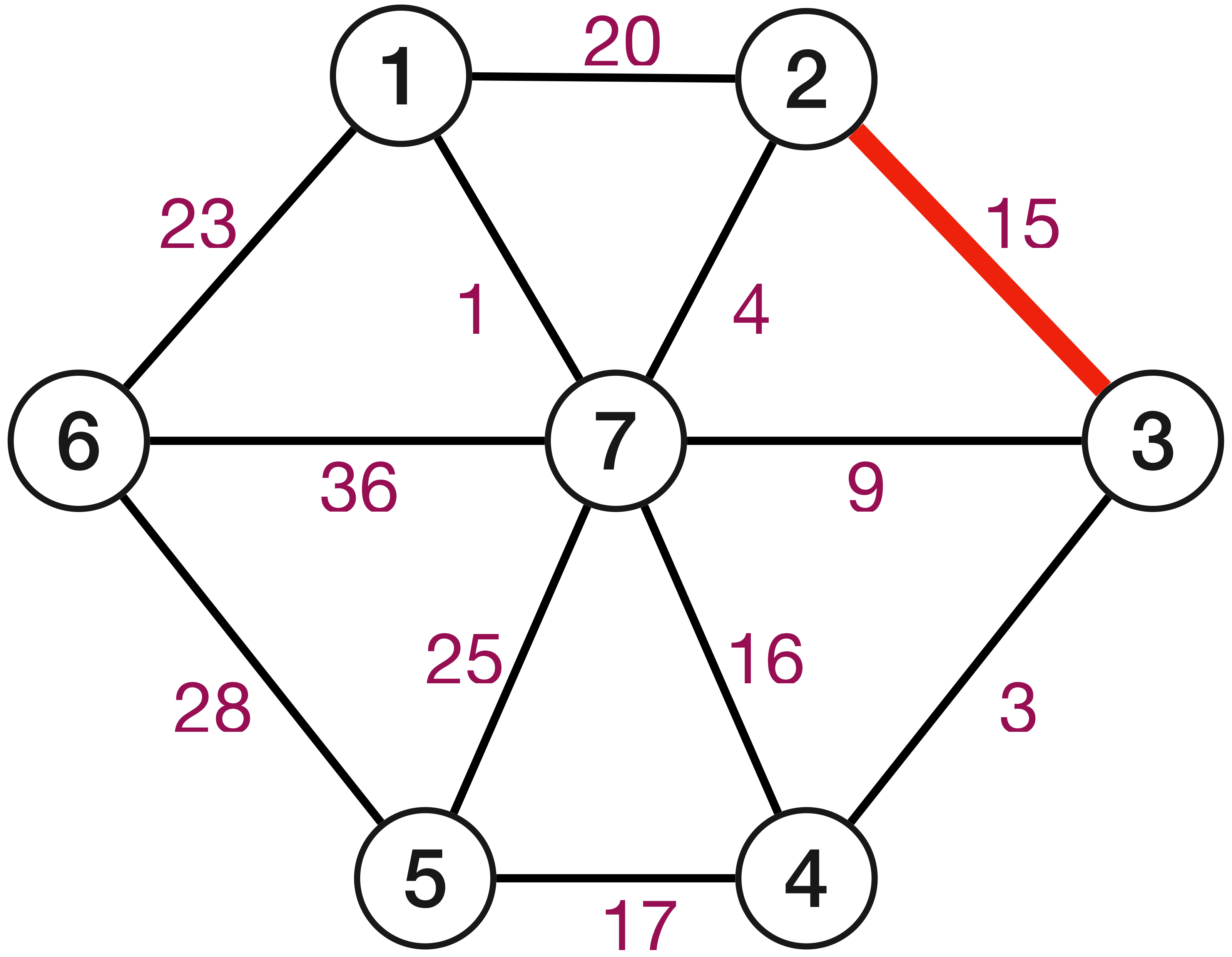

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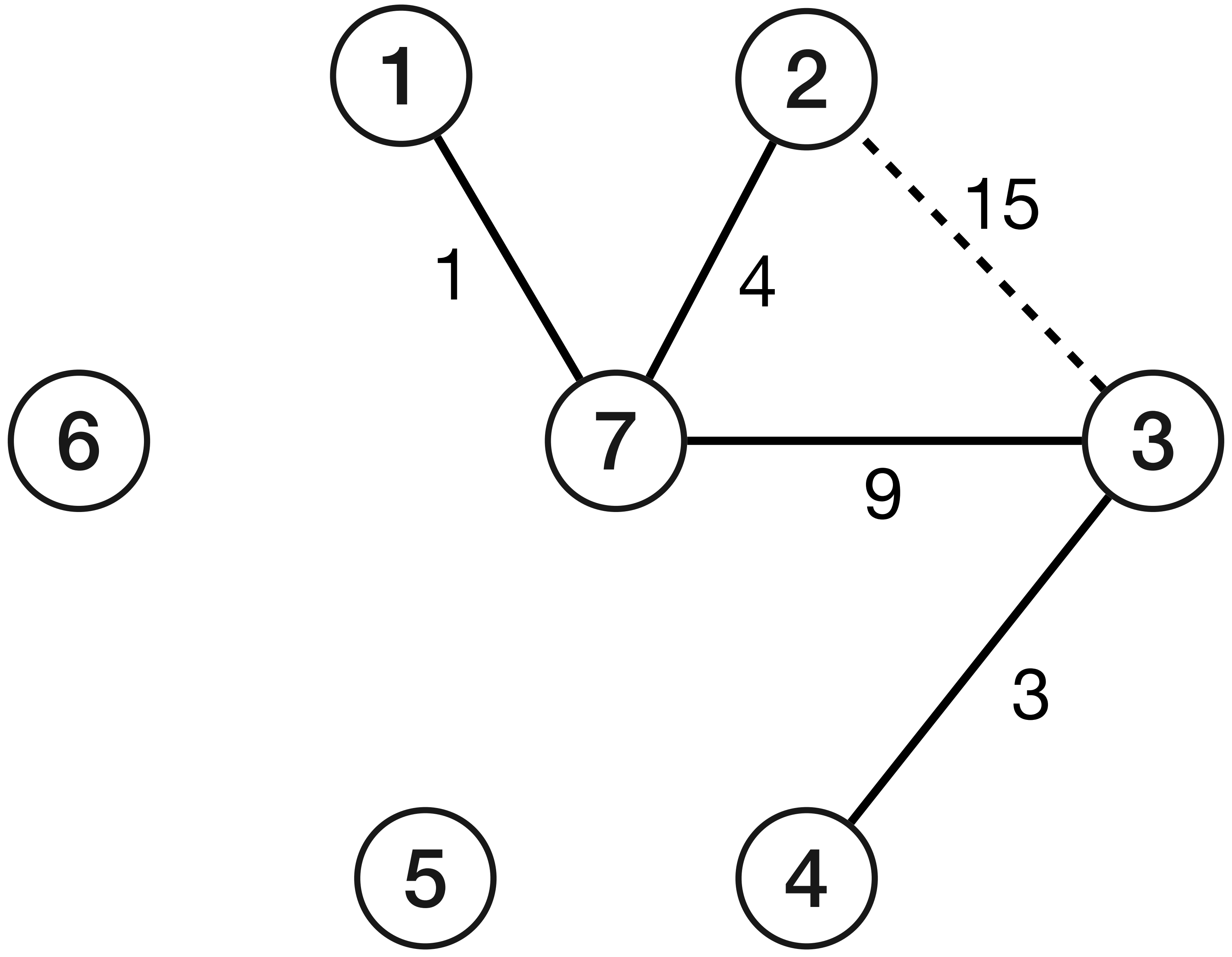

MST of G

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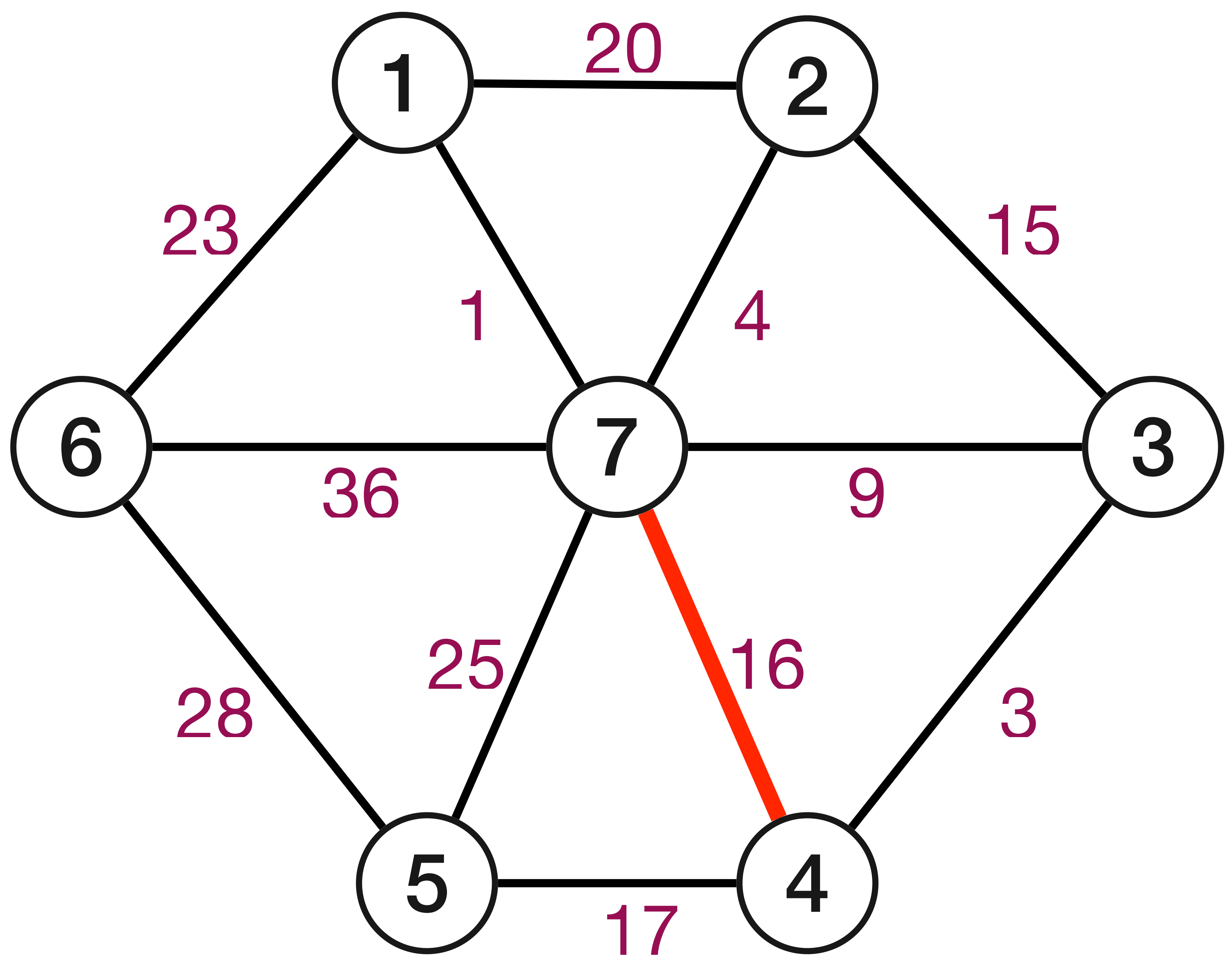

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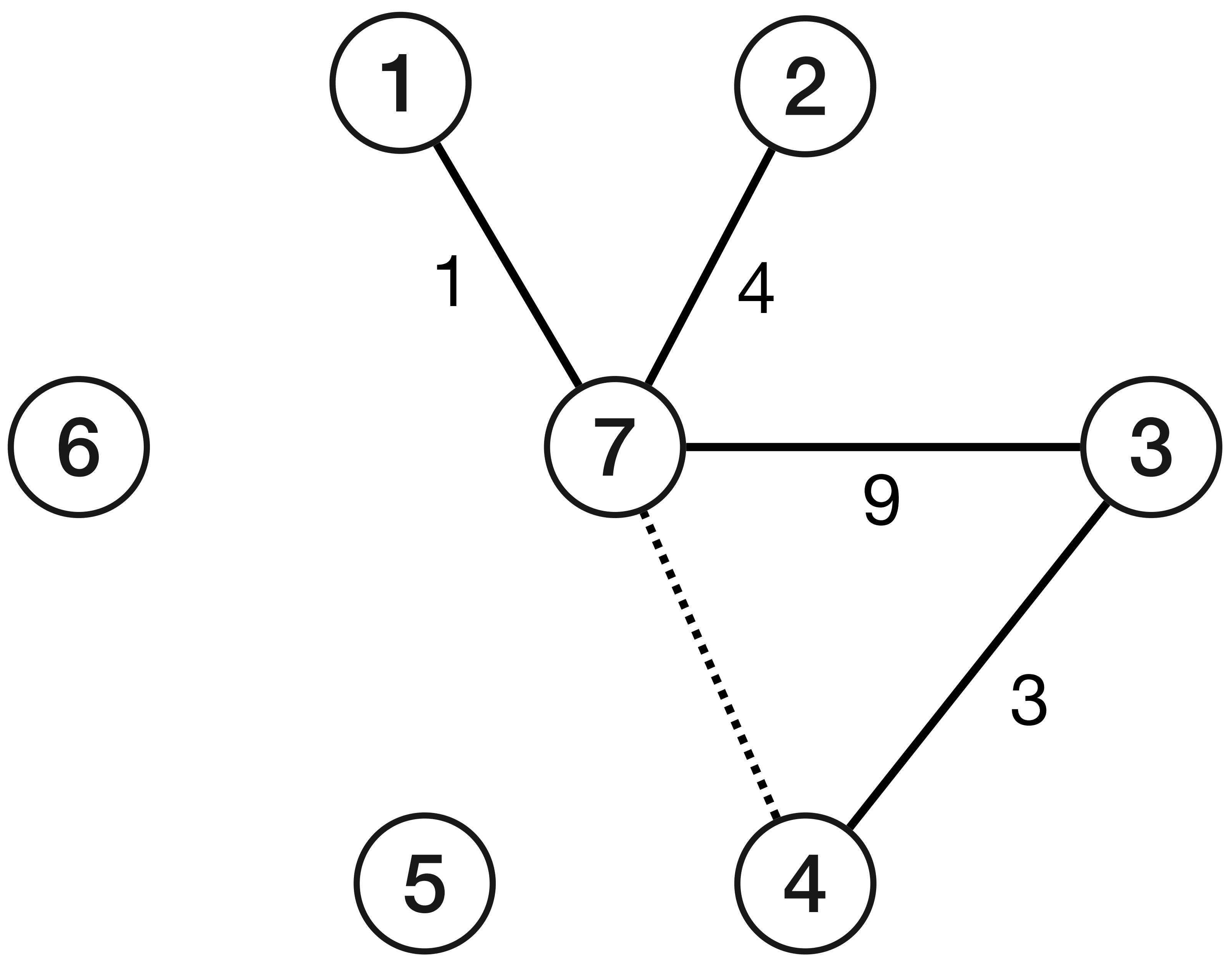

MST of G

Kruskal’s Algorithm

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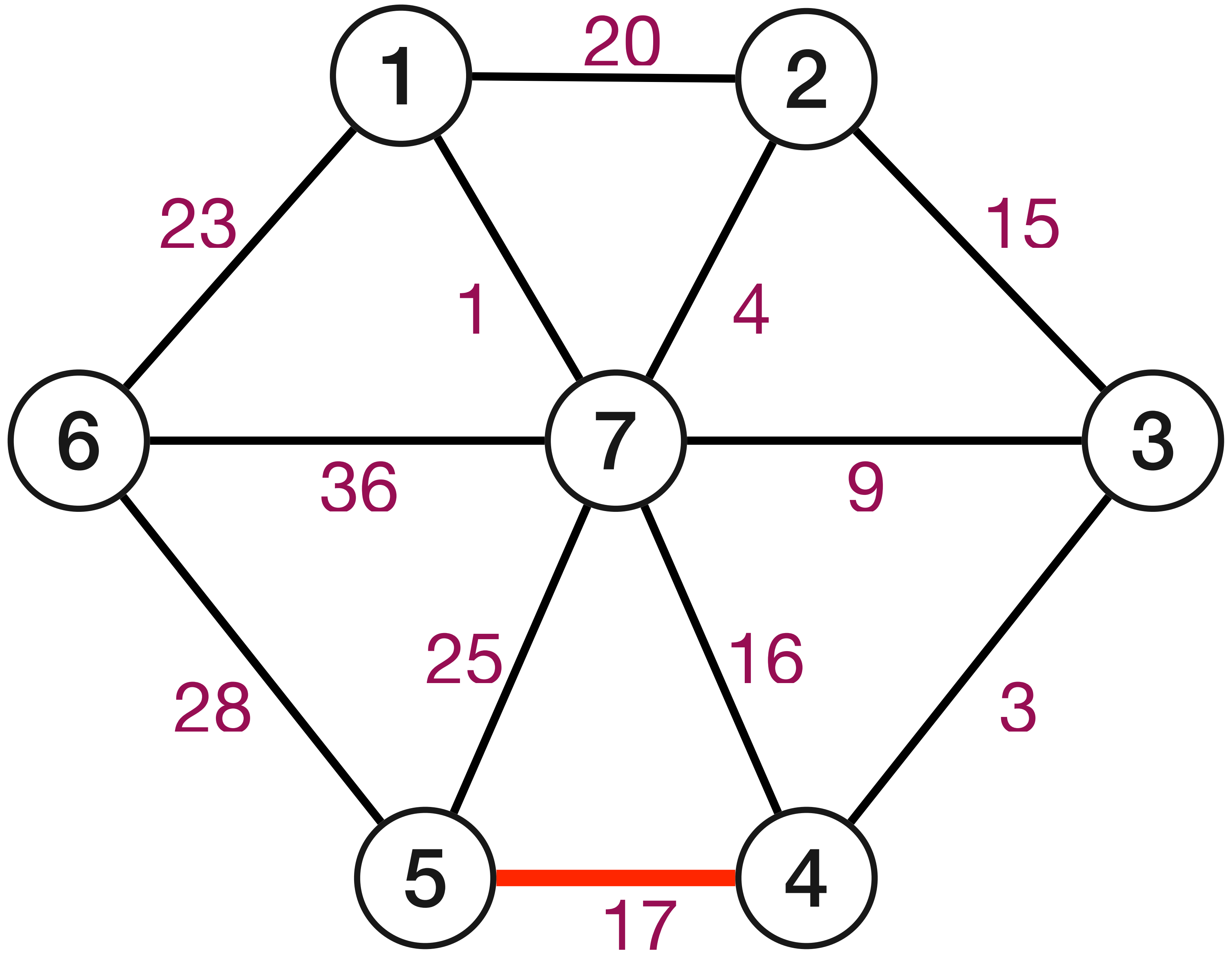

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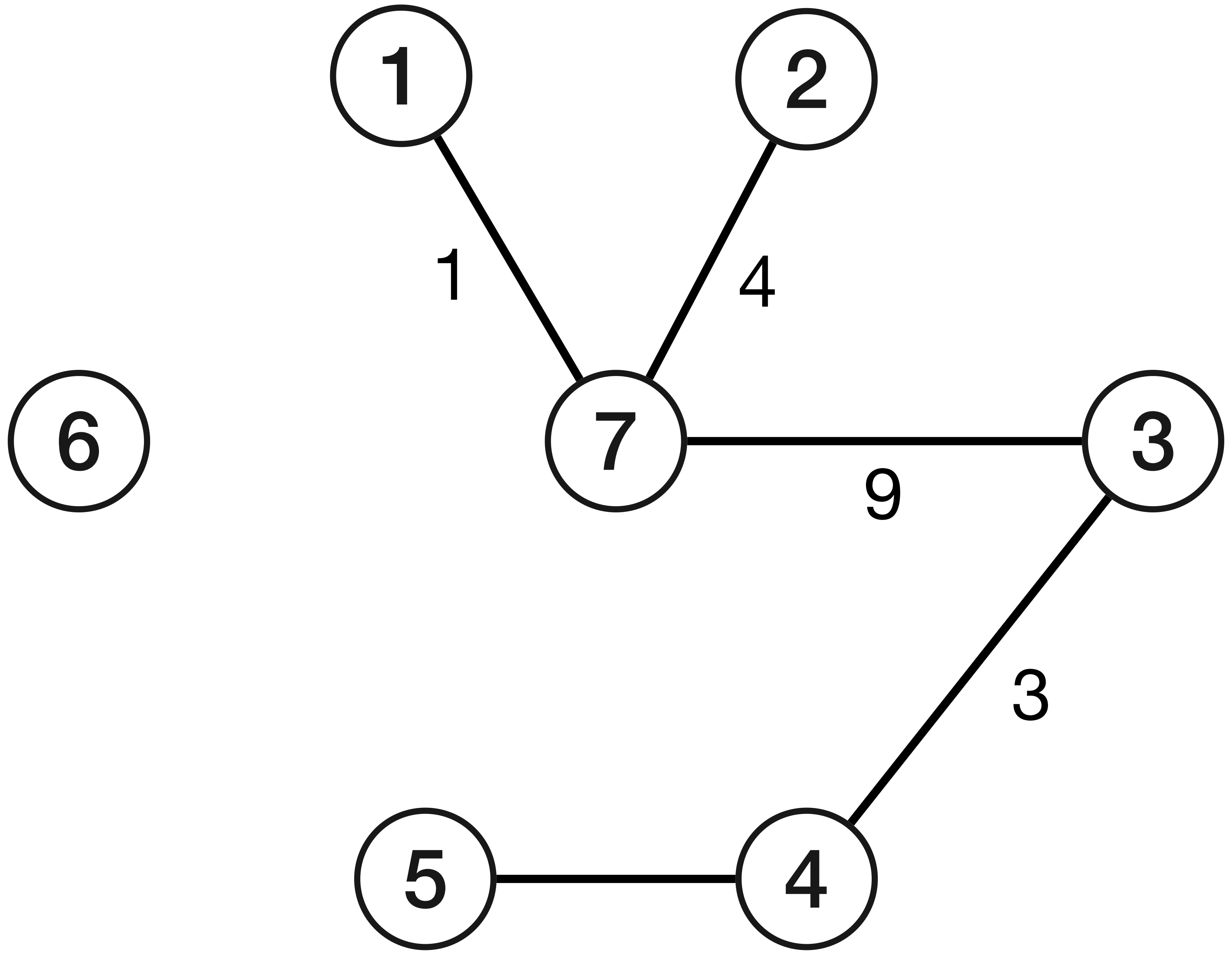

MST of G

Kruskal's Algorithm

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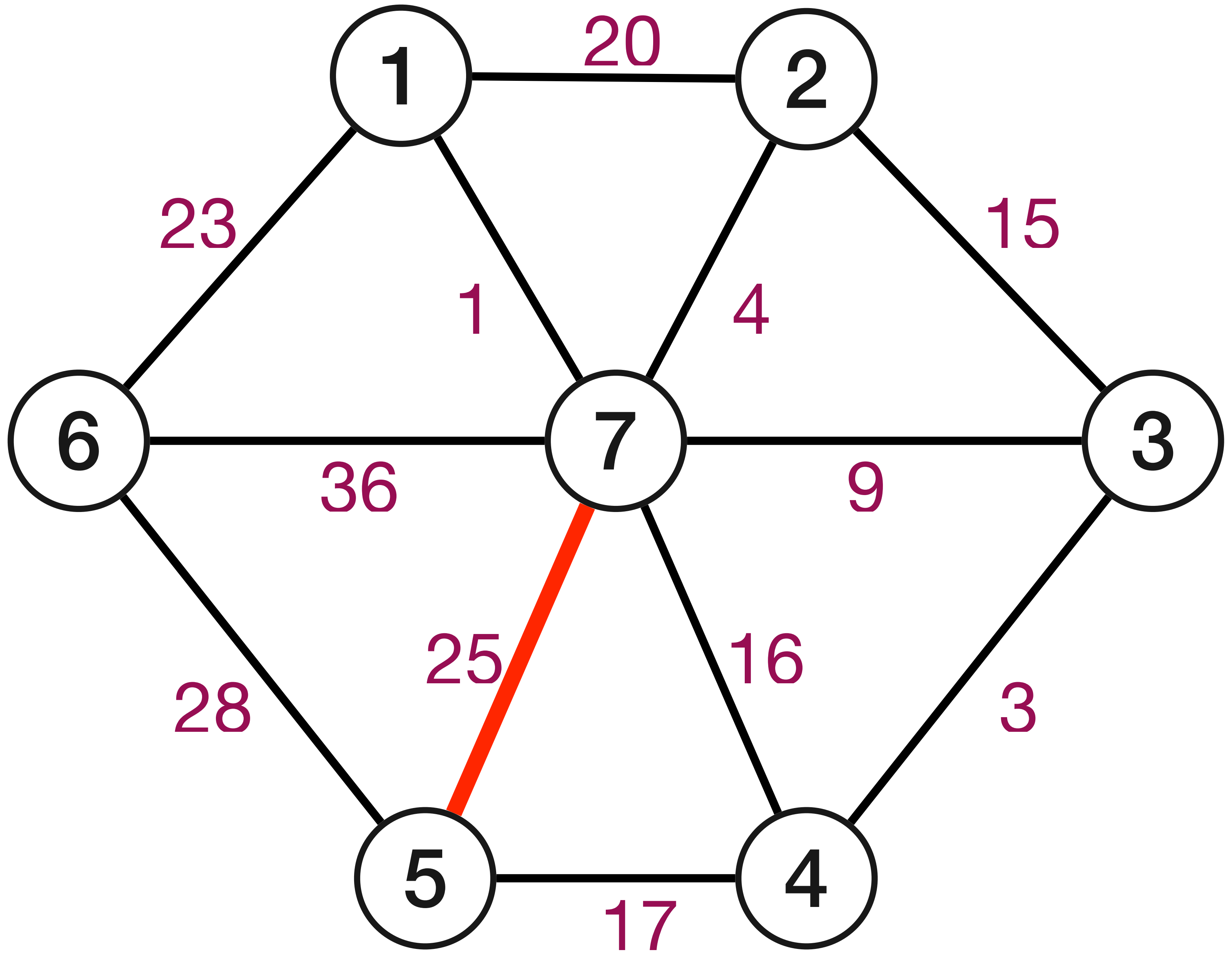

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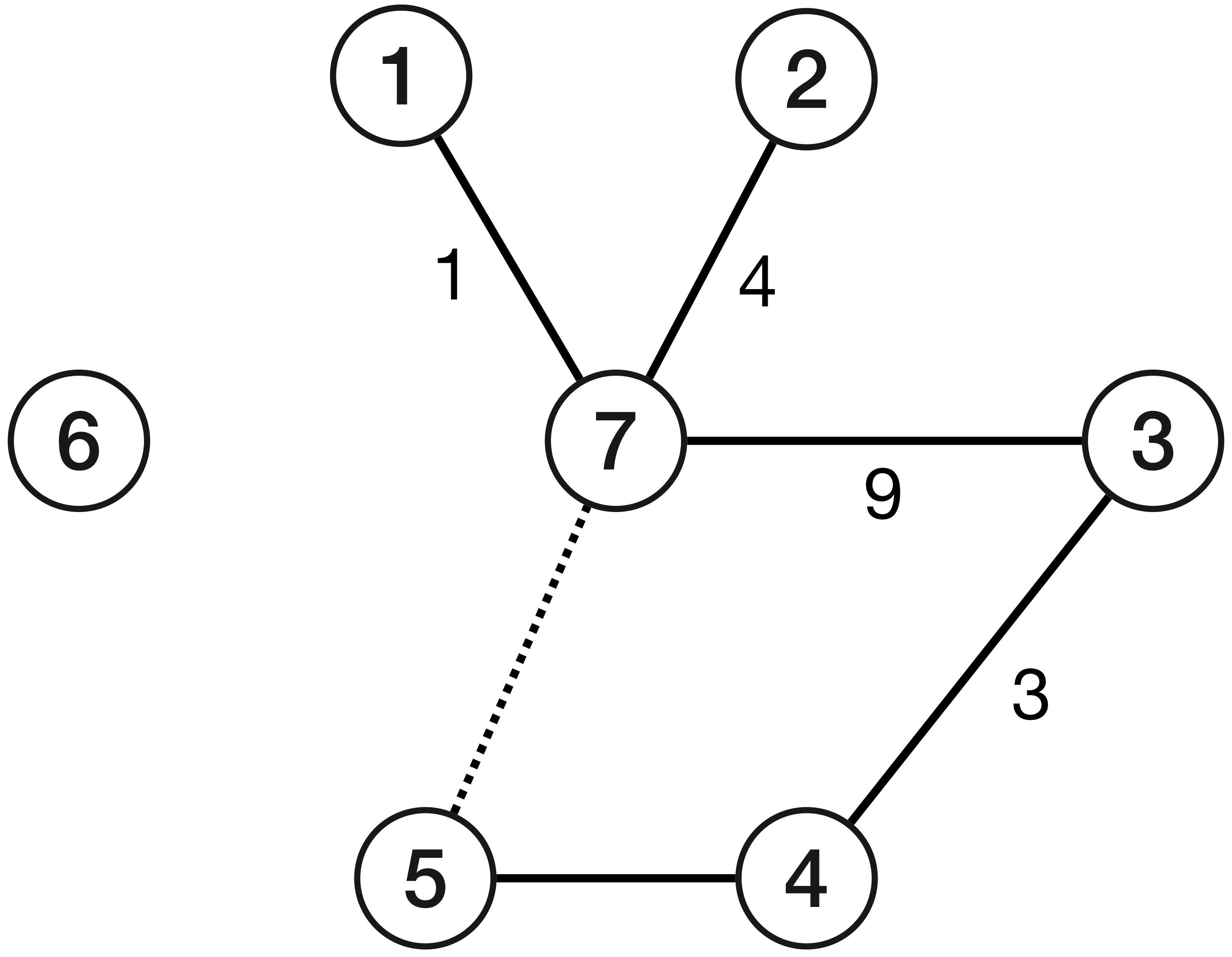

MST of G

Kruskal's Algorithm

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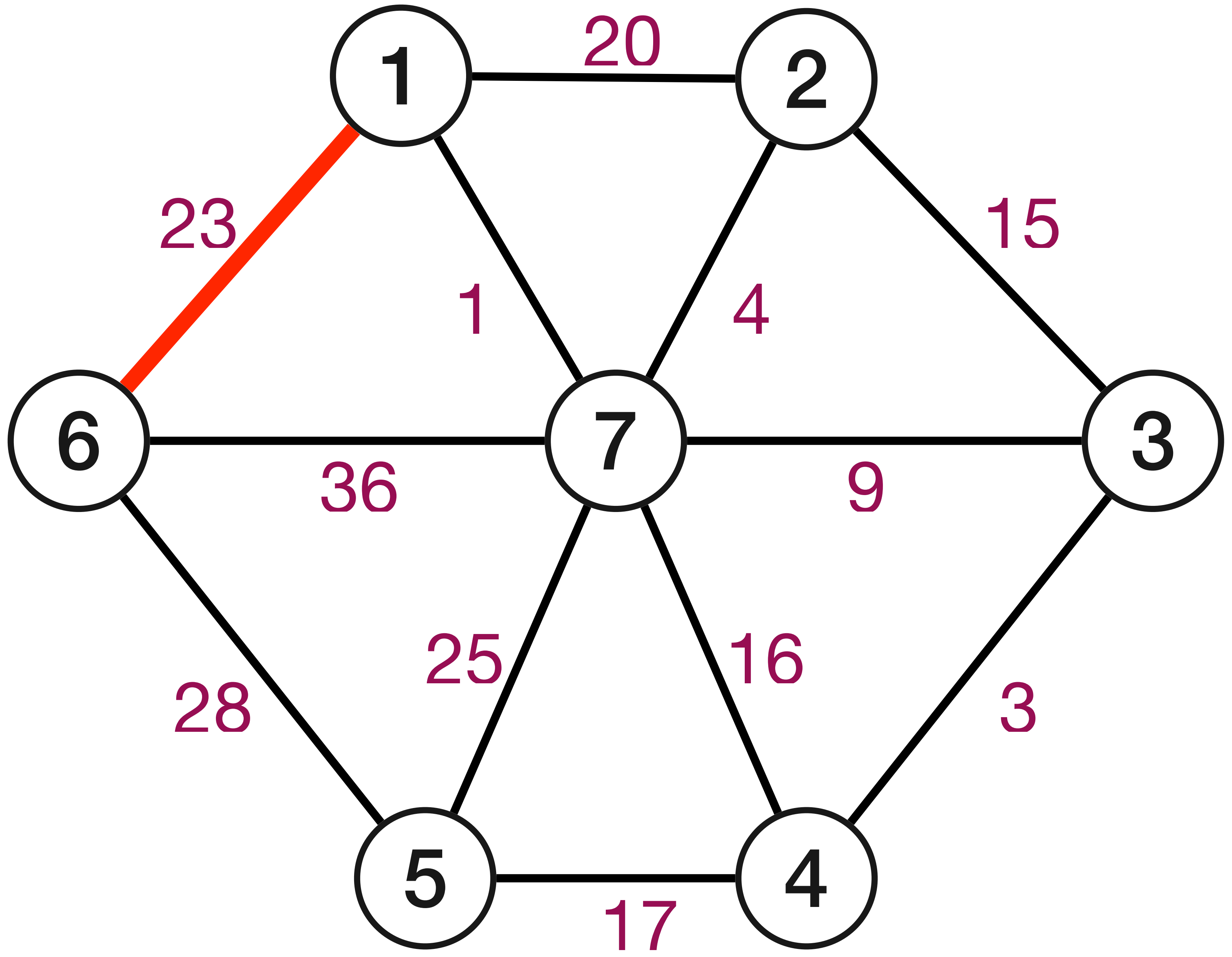
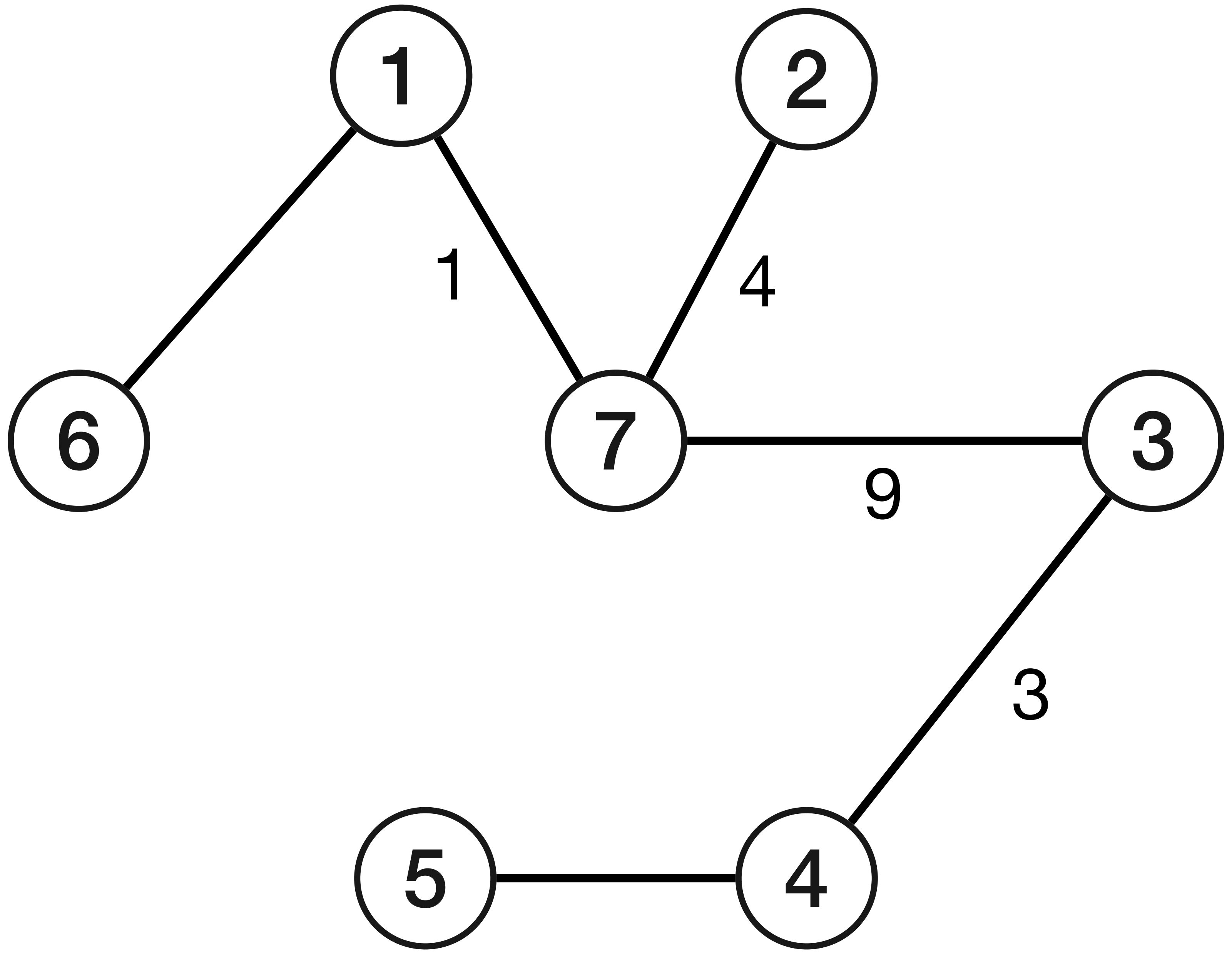


Graph G



MST of G

Correctness of Kruskal's Algorithm

Kruskal's Algorithm: Picking the edge of lowest cost and adding if it does not form a cycle with existing edges generates a MST.

Proof: If $e = (u, v)$ is added to tree, then e is safe

- When algorithm adds e let S and S' be the connected components containing u and v respectively
- e is the lowest cost edge crossing S (and also S').
- If there is an edge e' crossing S and has lower cost than e, then e' would come before e in the sorted order and would be added by the algorithm to T
- Set of edges output is a spanning tree

Kruskal's Algorithm

```
Kruskal_ComputeMST
    Initially E is the set of all edges in G
    T is empty (* T will store edges of a MST *)
    while E is not empty do
          choose e ∈ E of minimum cost
          remove e from E
          if (T ∪ {e} does not have cycles)
            add e to T
    return the set T
```

- Presort edges based on cost. Choosing minimum can be done in $O(1)$ time
- Do BFS/DFS on $T \cup \{e\}$. Takes $O(n)$ time
- Total time $O(m \log m) + O(mn) = O(mn)$

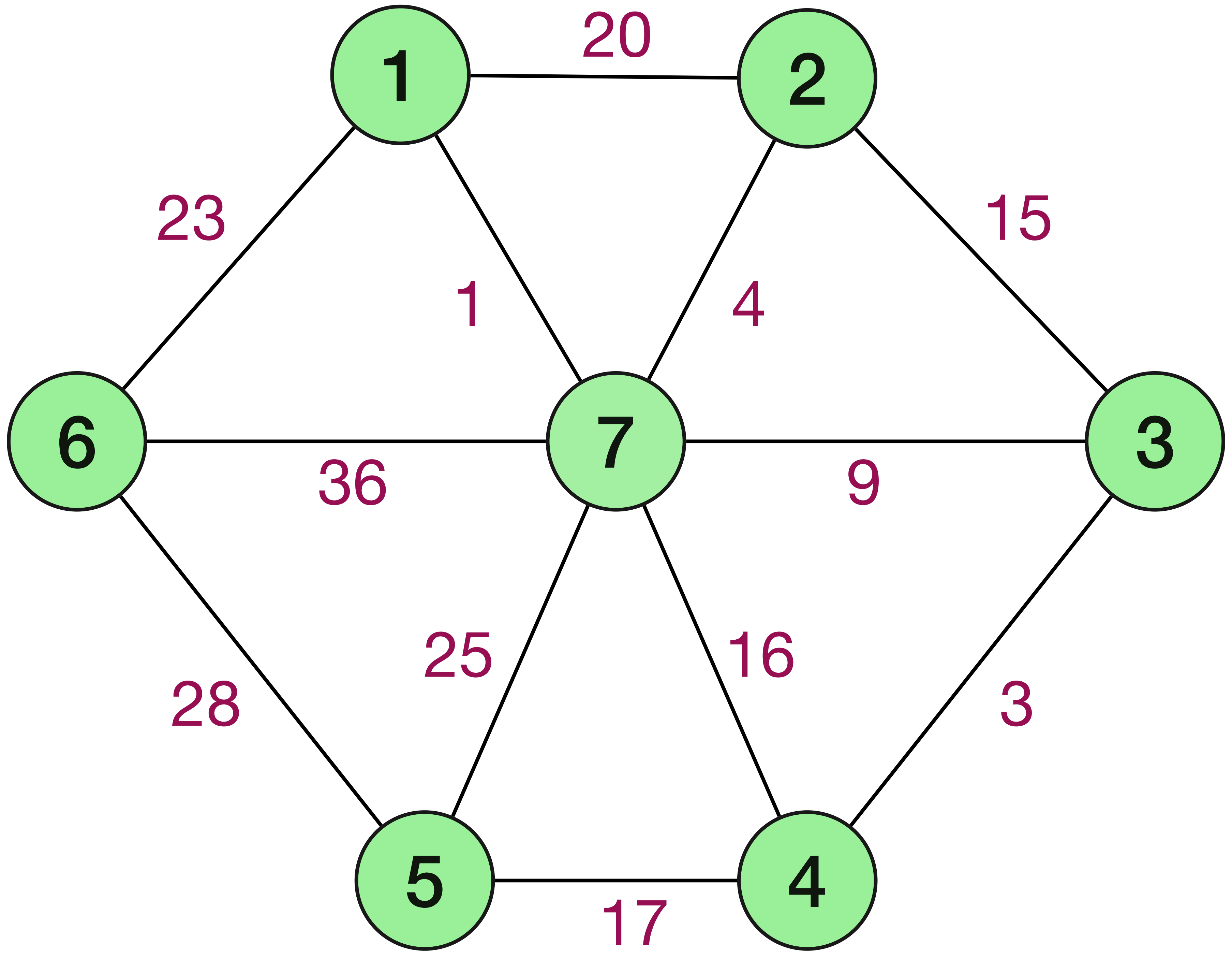
Kruskal's Algorithm (efficiently)

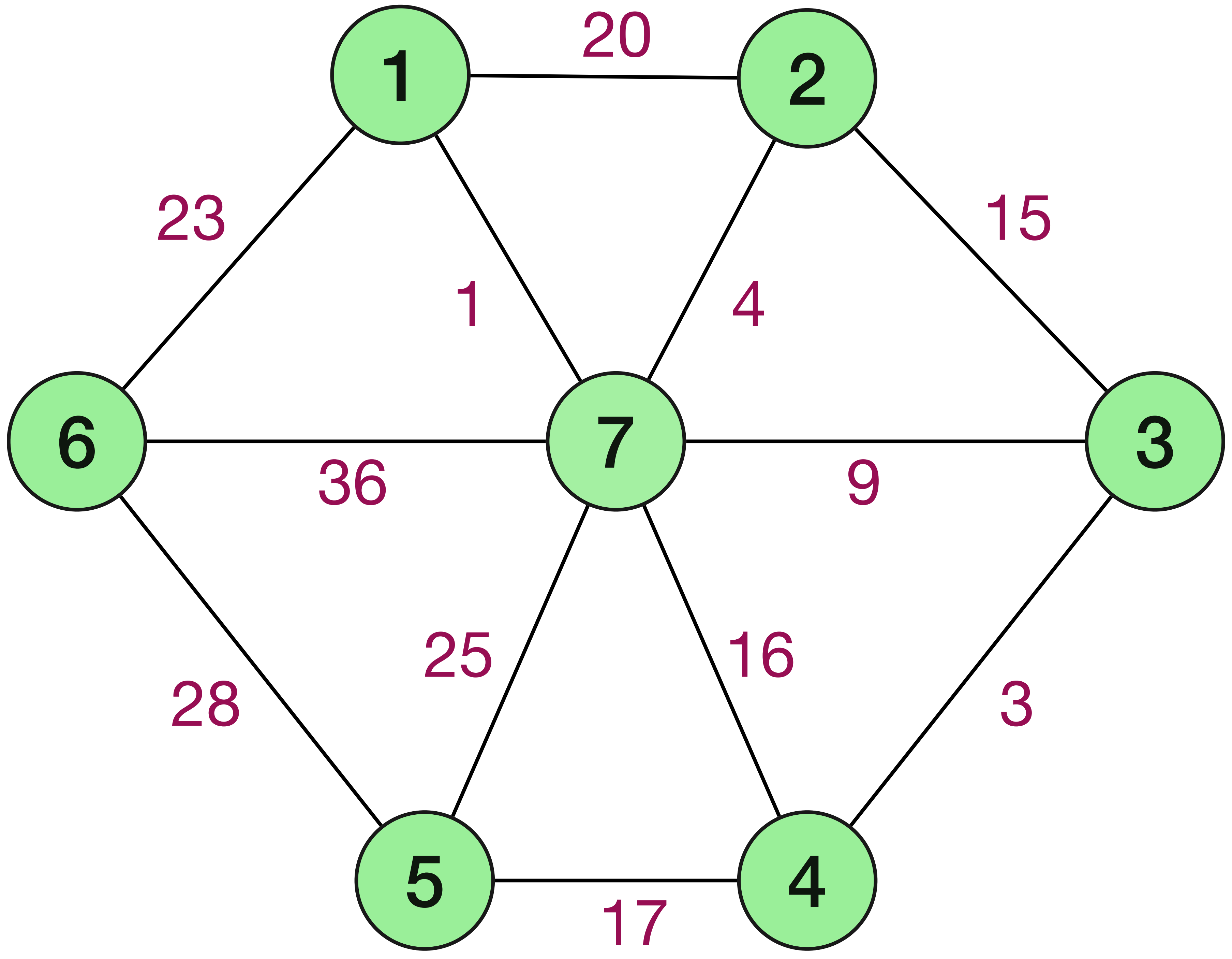
```
Kruskal_ComputeMST
    Sort edges in E based on cost
    T is empty (* T will store edges of a MST *)
    each vertex u is placed in a set by itself
    while E is not empty do
          pick e = (u,v)∈ E of minimum cost
          if u and v belong to different sets
            add e to T
            merge the sets containing u and v
    return the set T
```

- Need a data structure to check if two elements belong to same set and to merge two sets.
- Using Union-Find (disjoint-set) data structure can implement Kruskal's algorithm in $O((m+n)\log m)$ time.

Prim's algorithm

T maintained by algorithm will be a tree. Start with a node in T. In each iteration, pick edge with least attachment cost to T.





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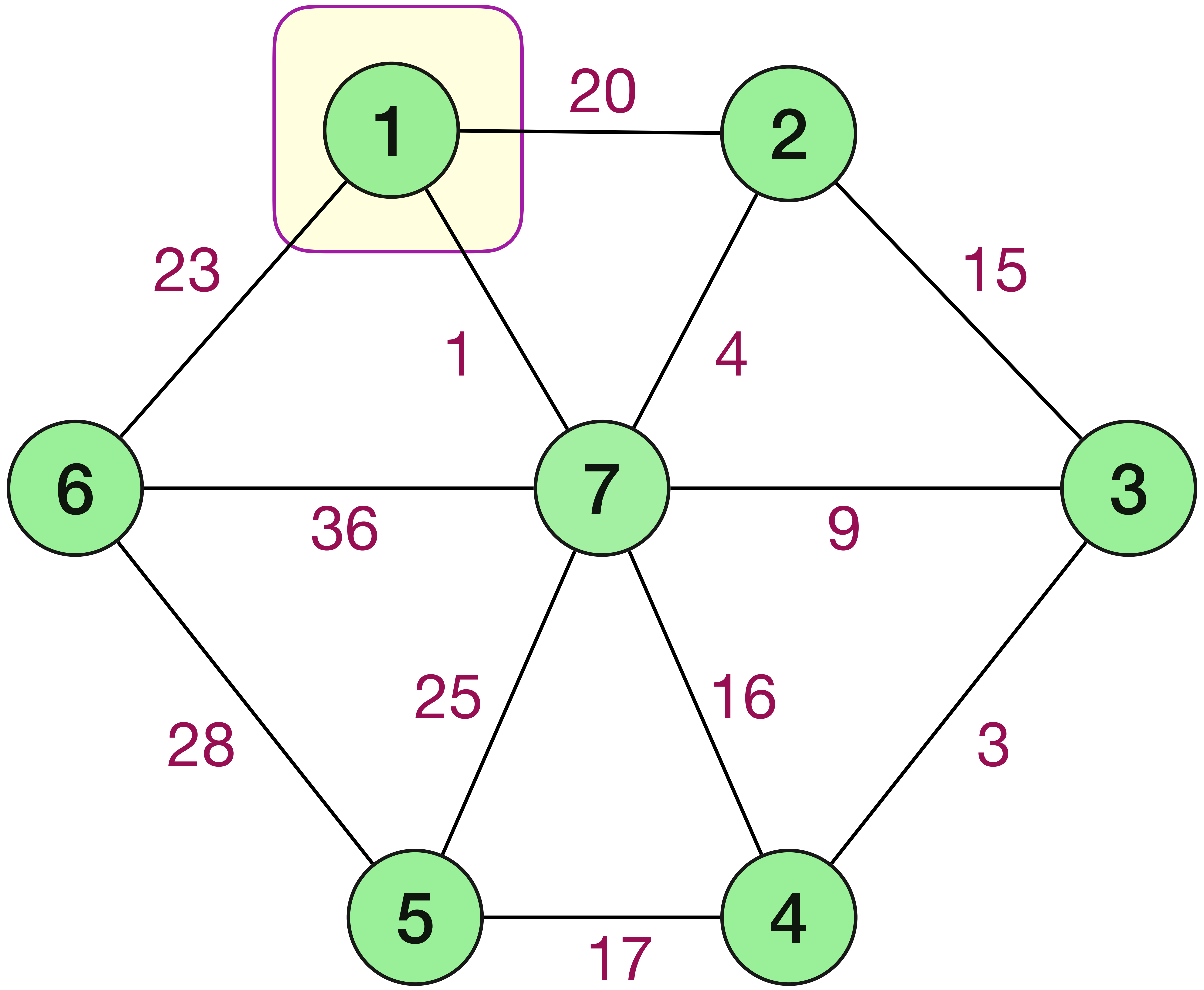

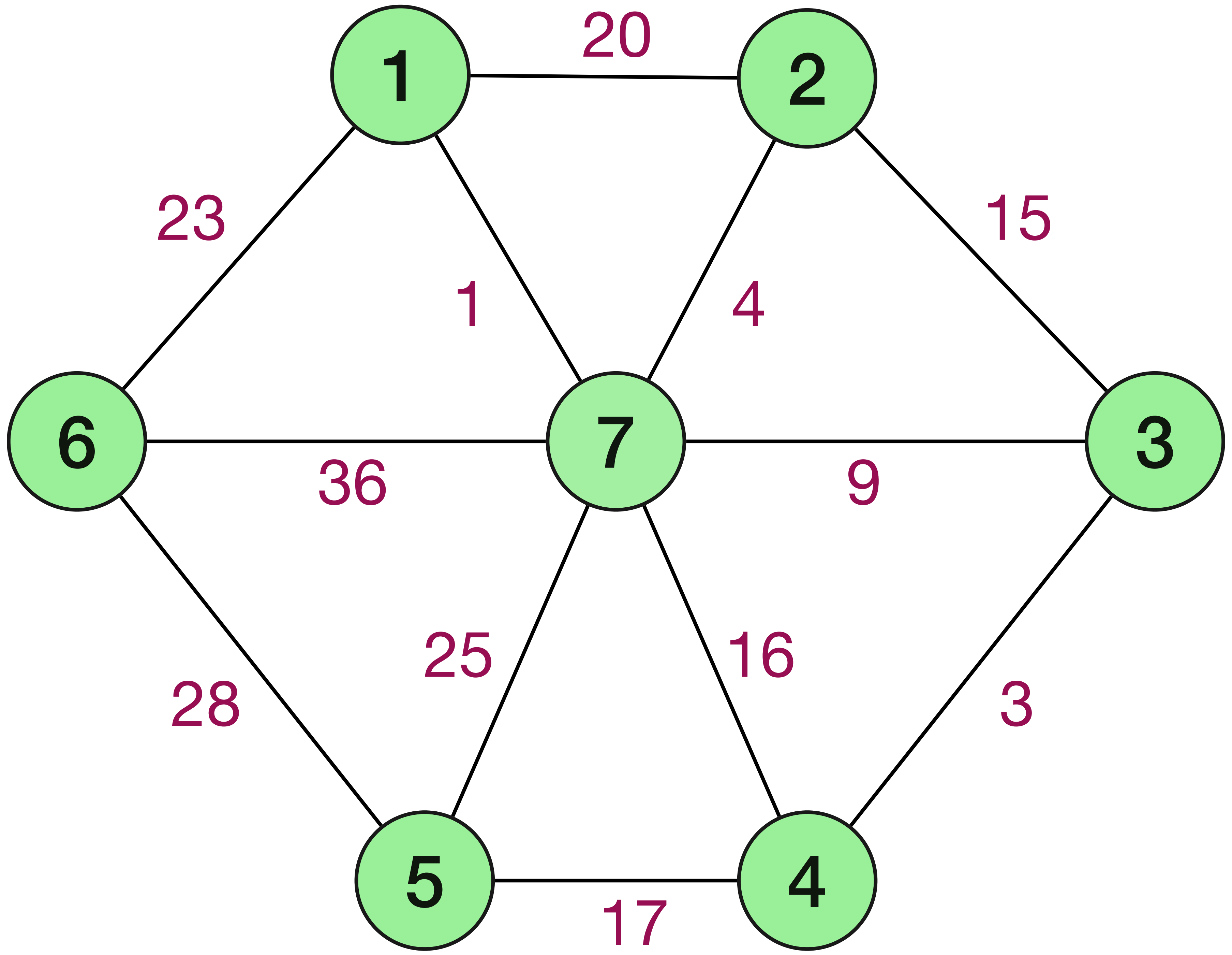

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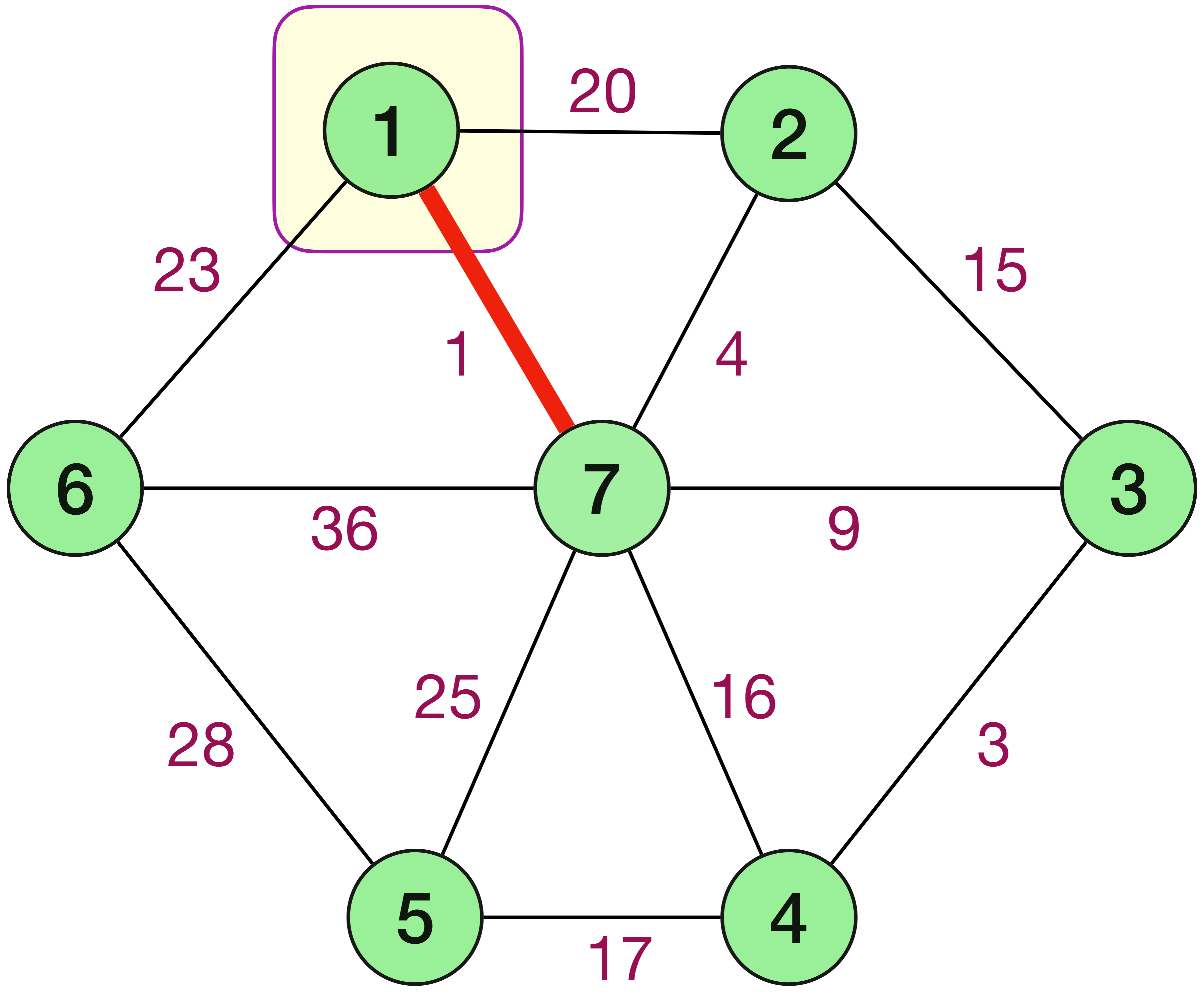

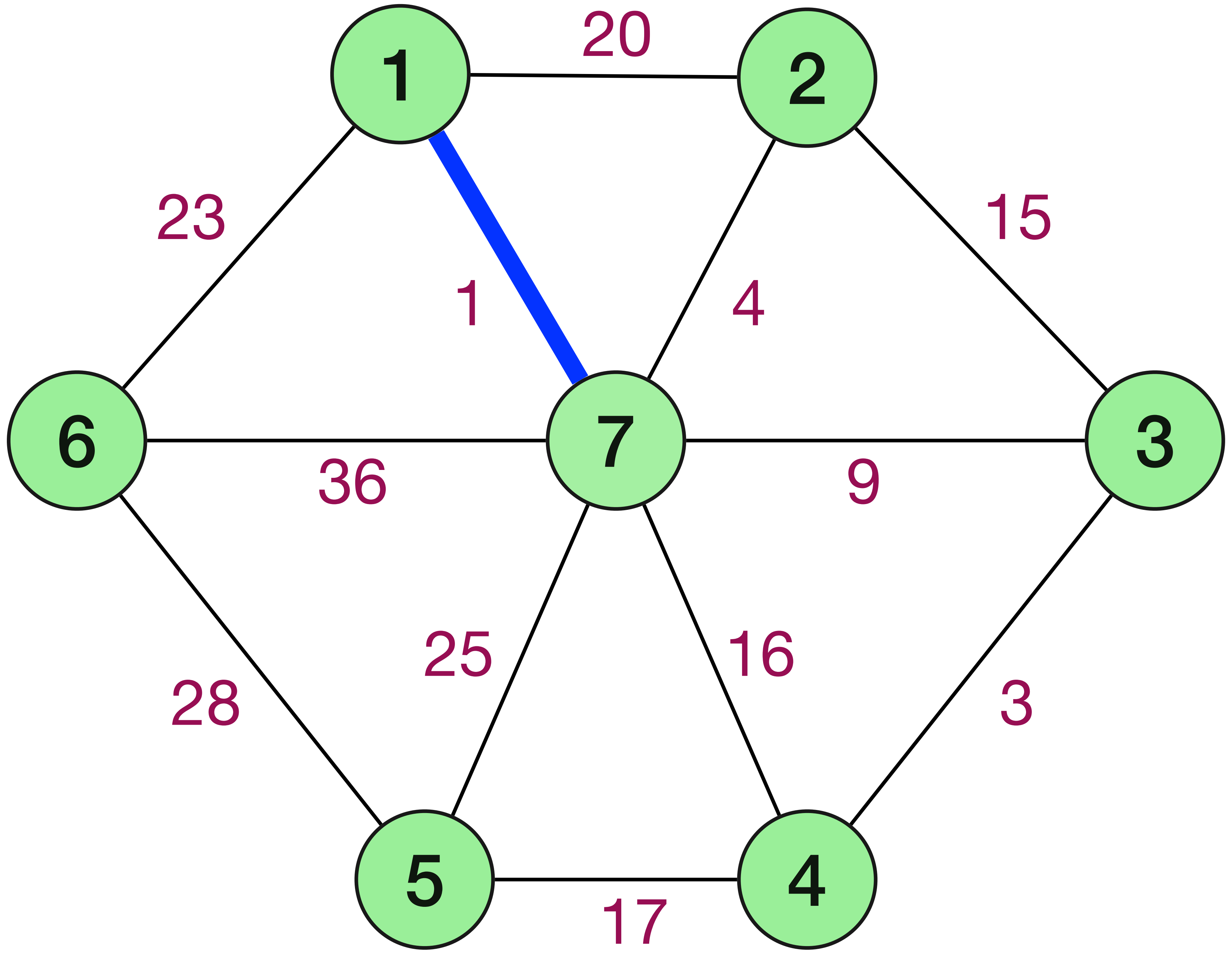

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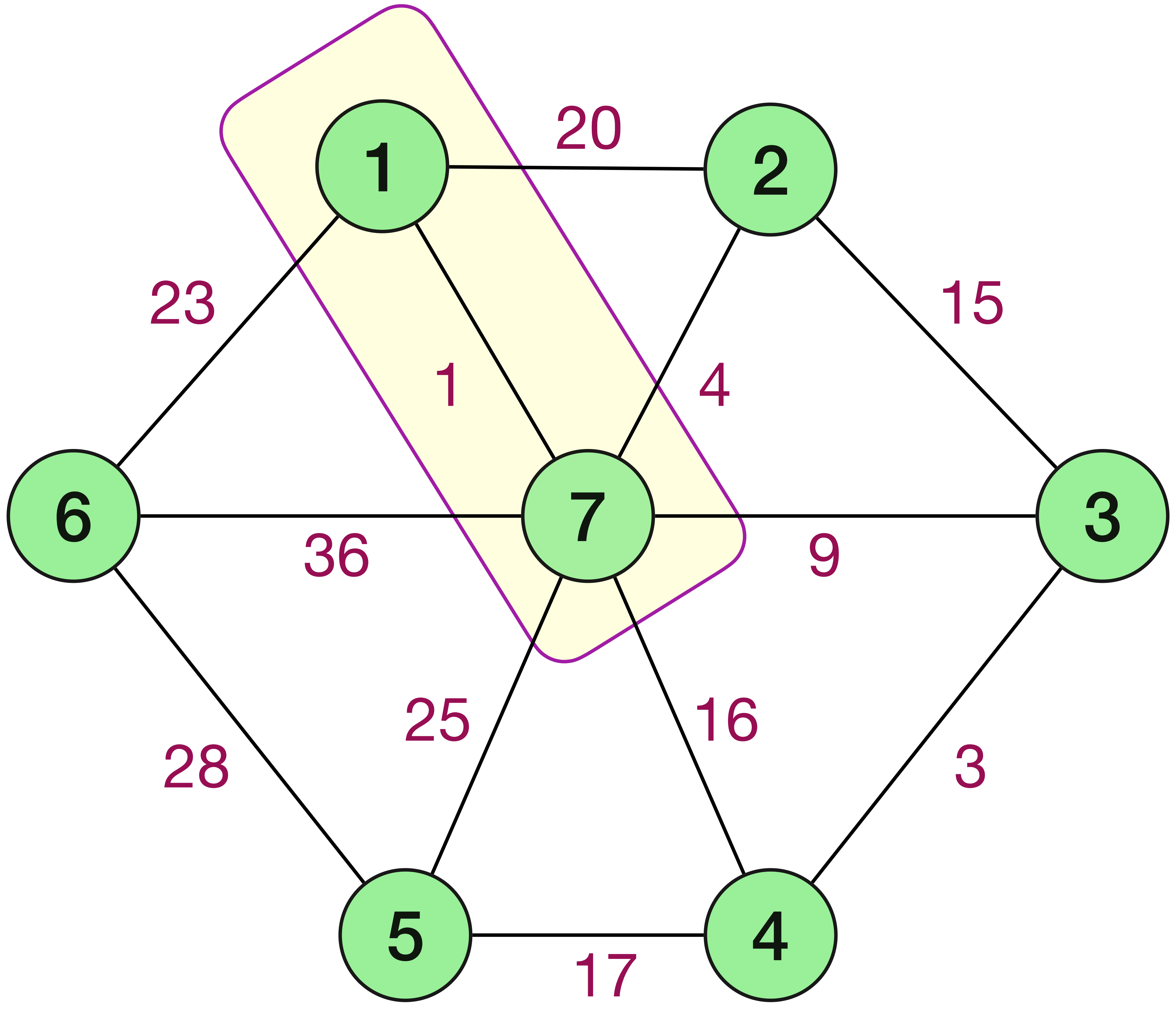

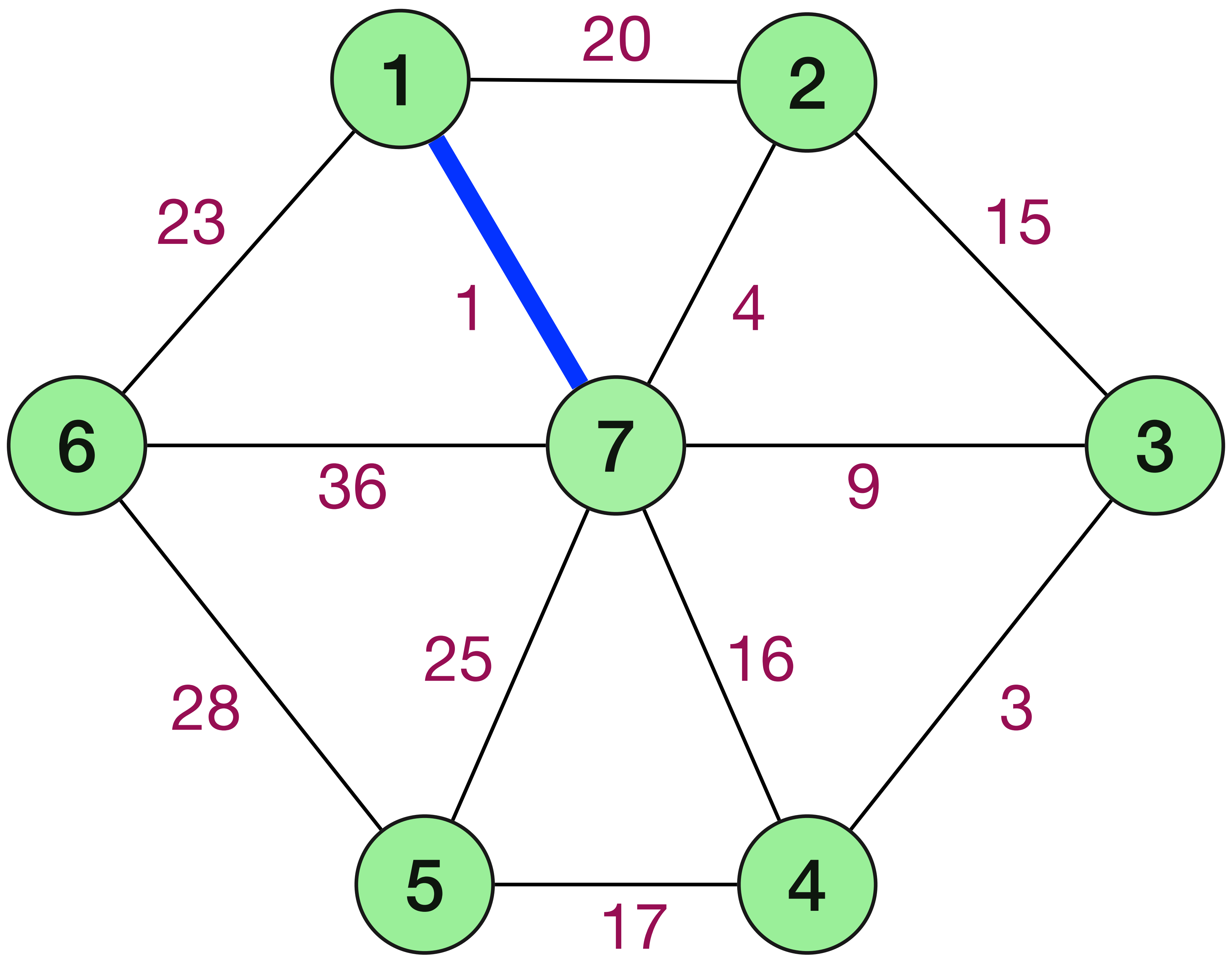

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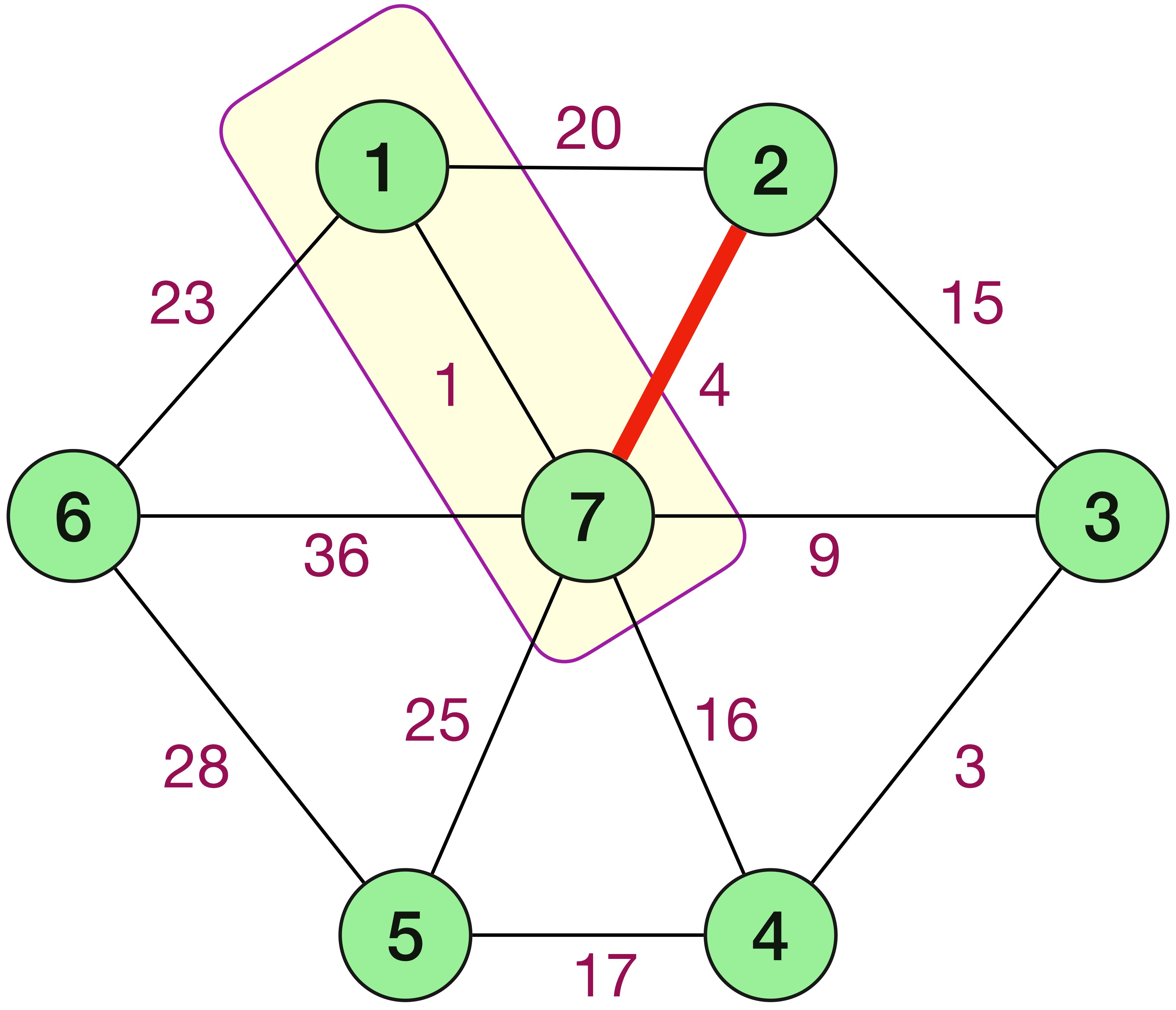

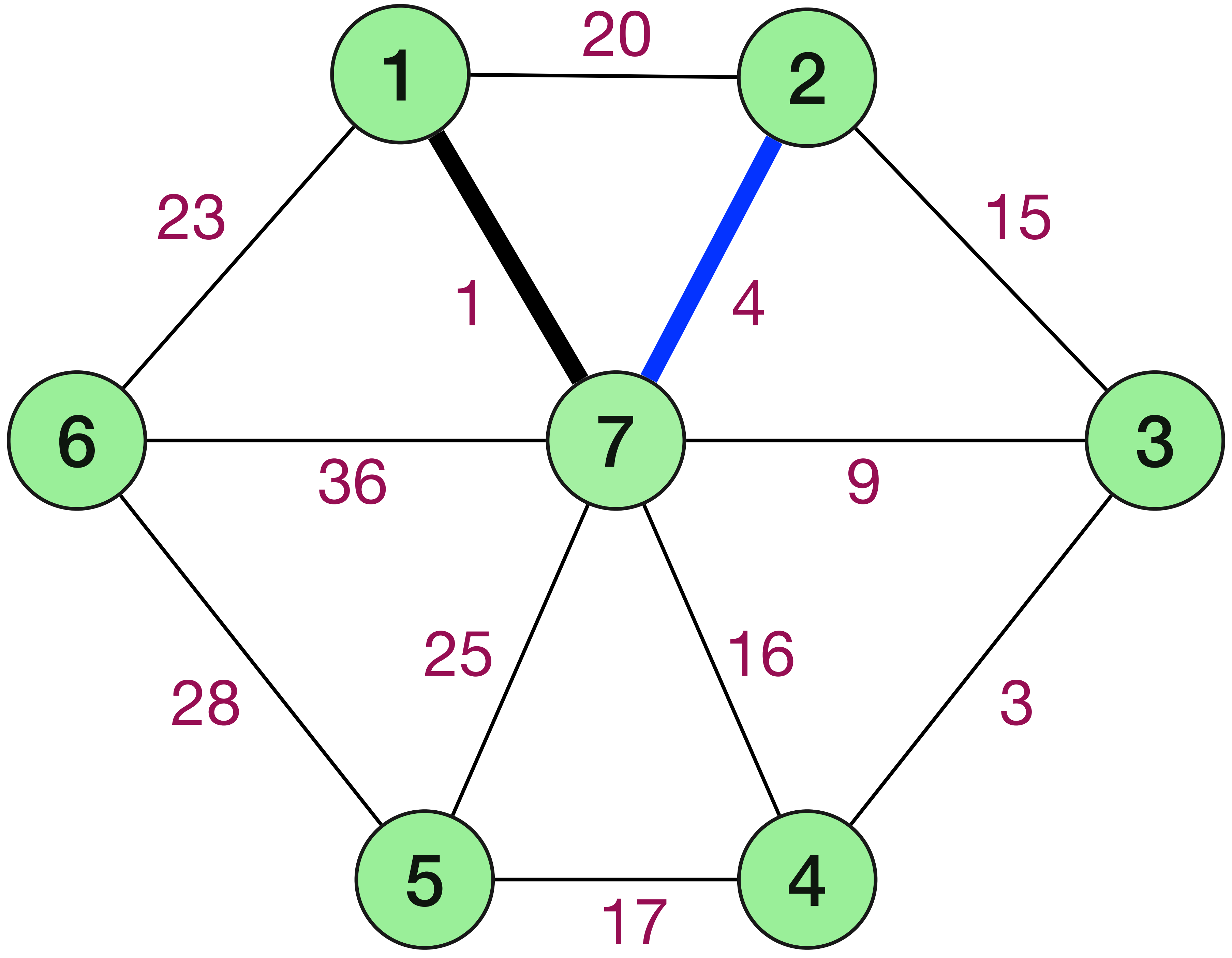

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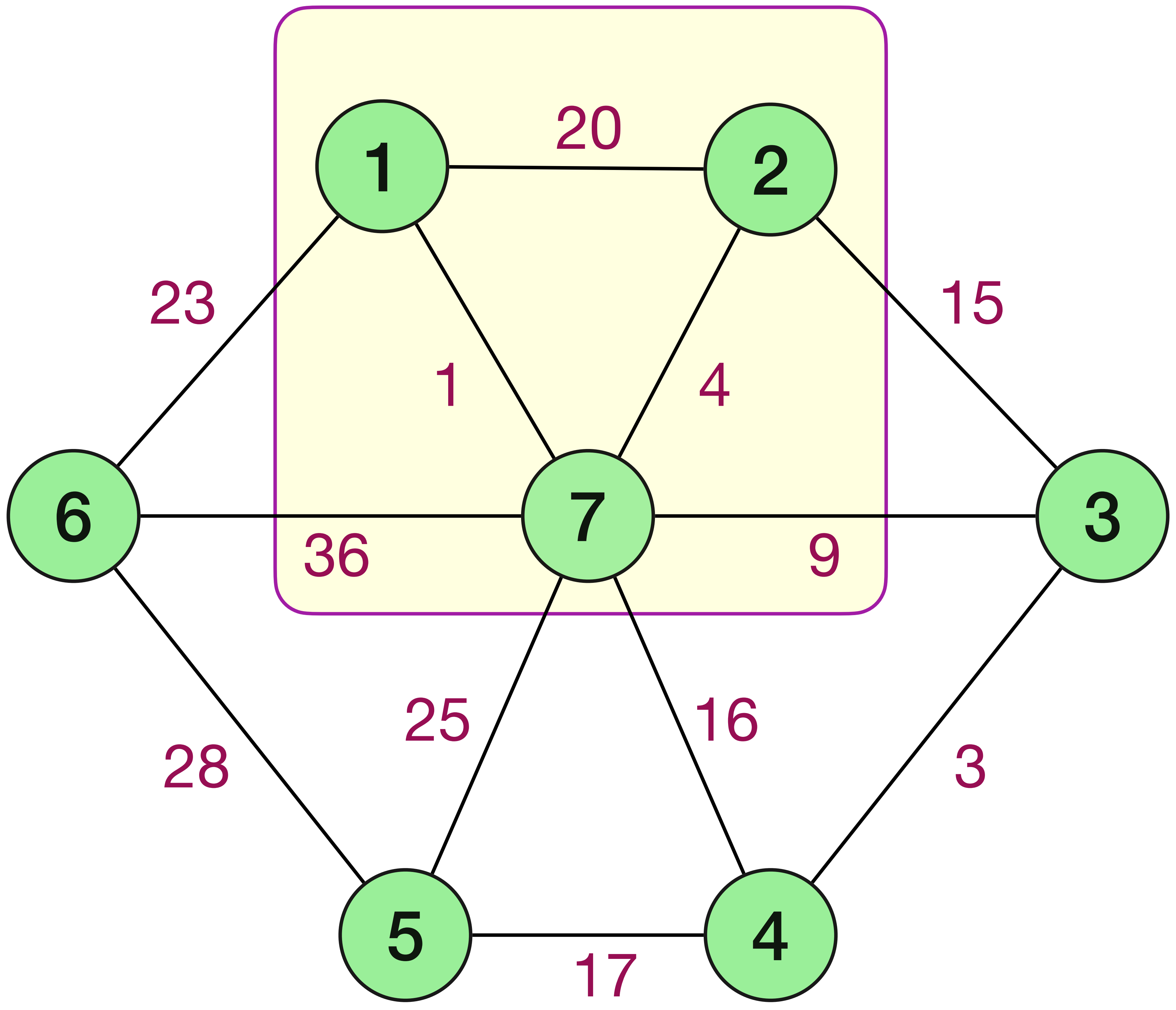

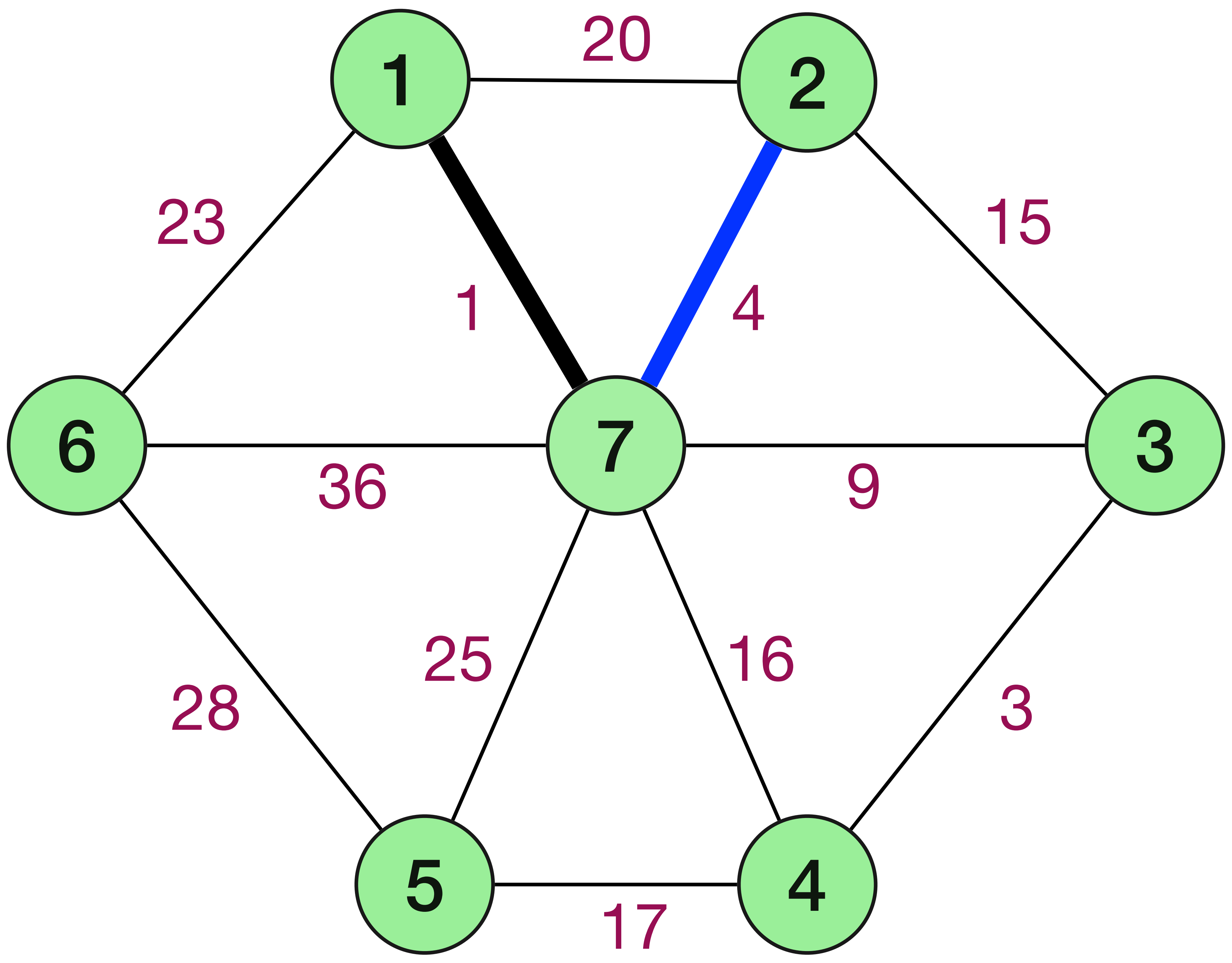

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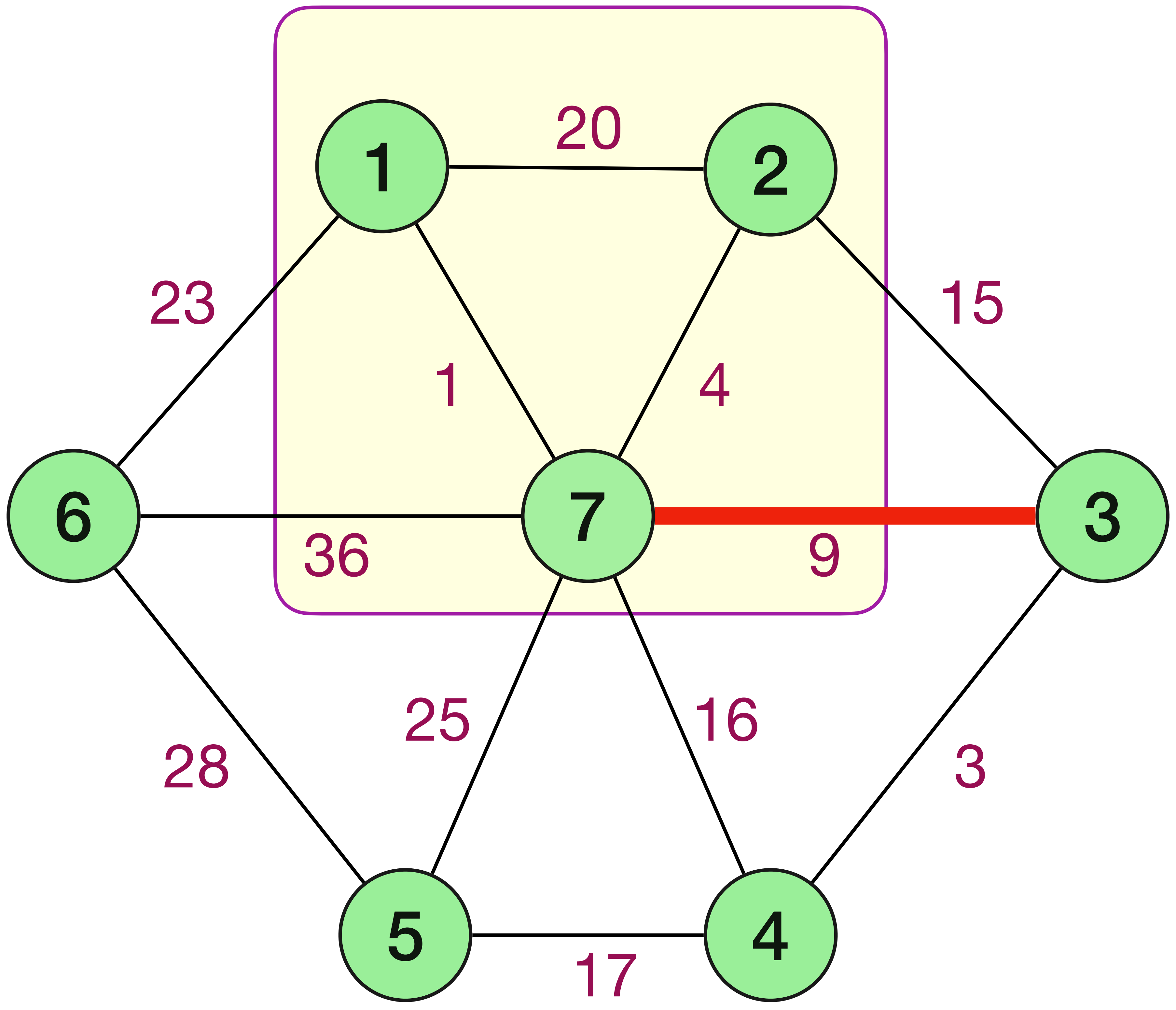

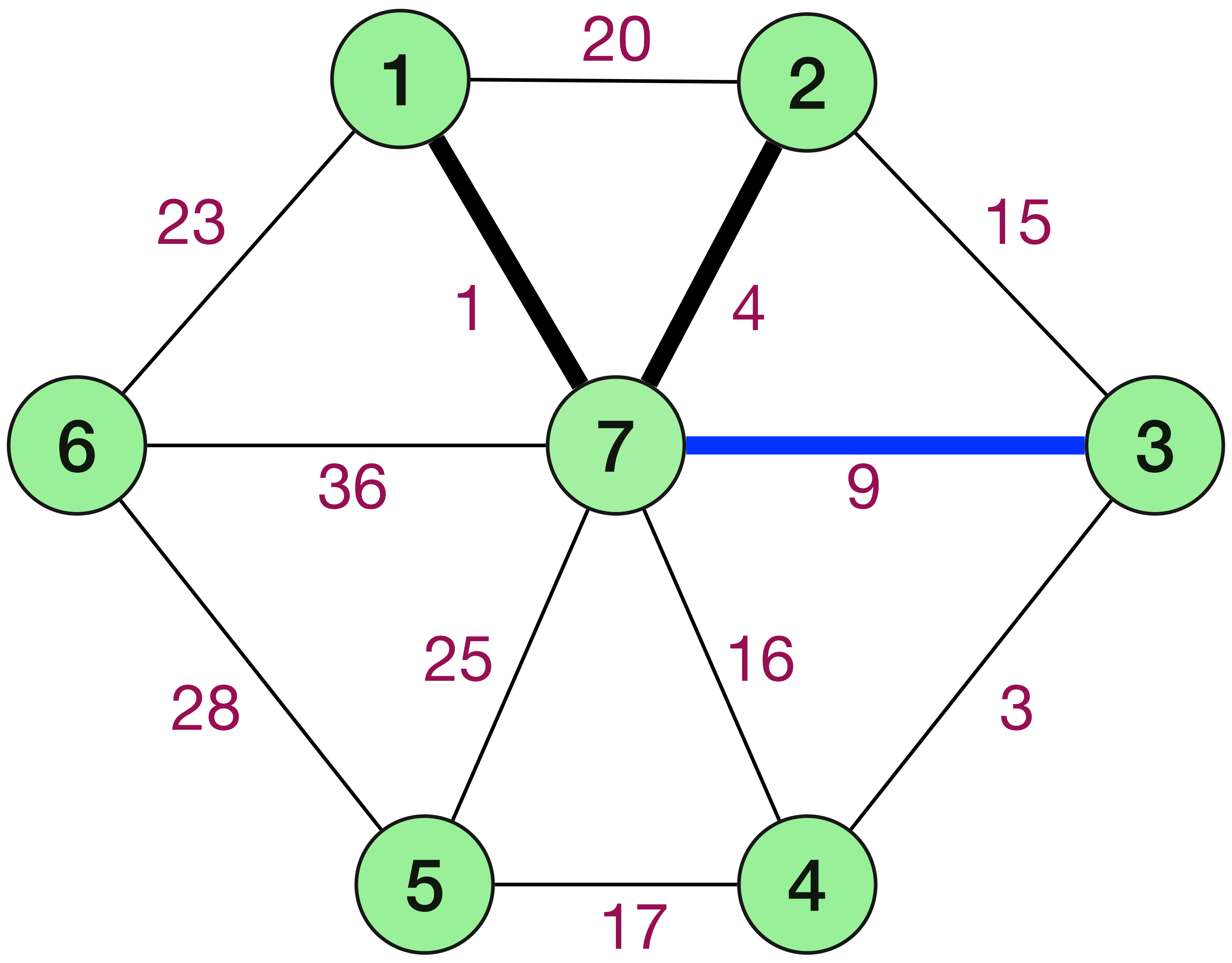

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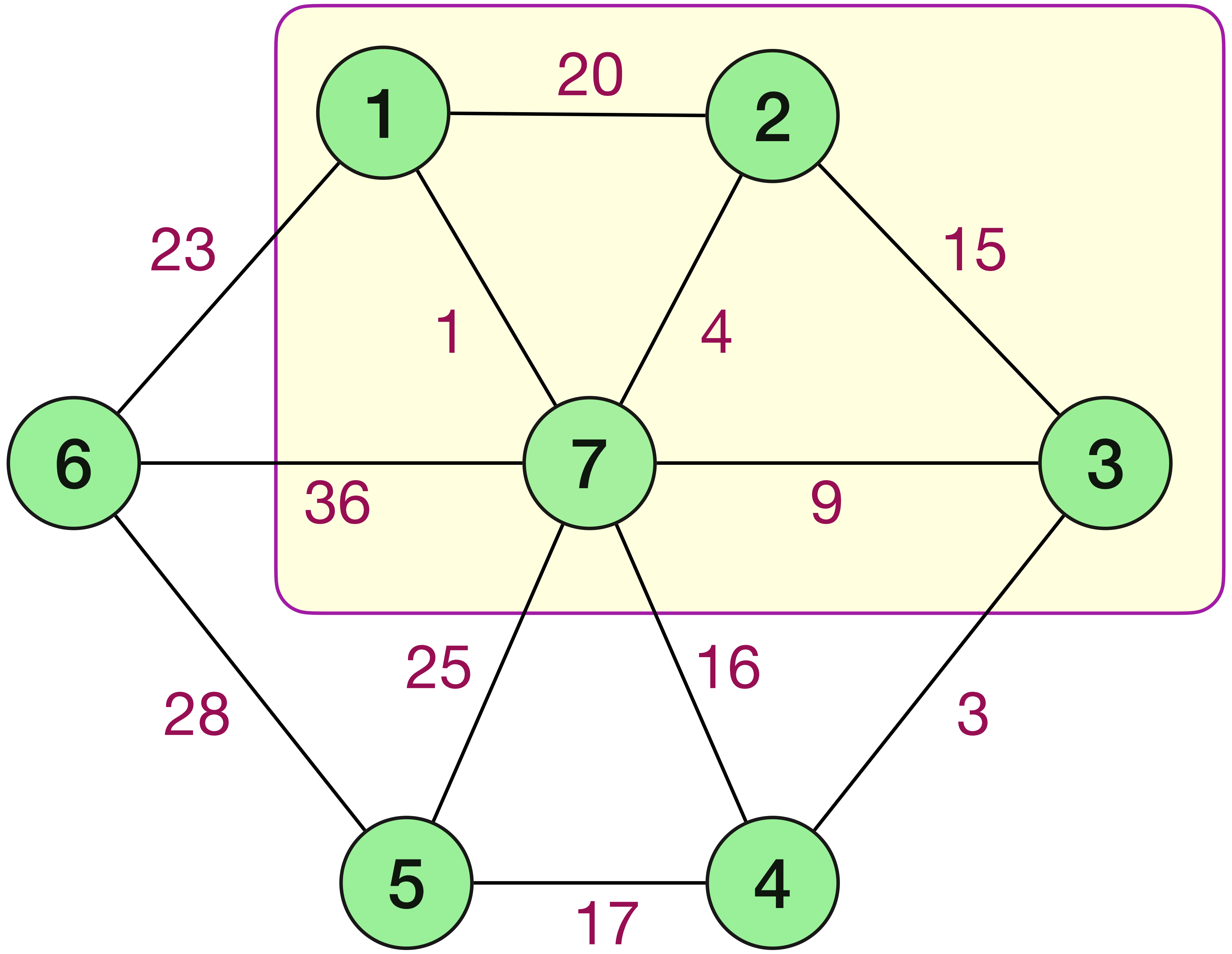

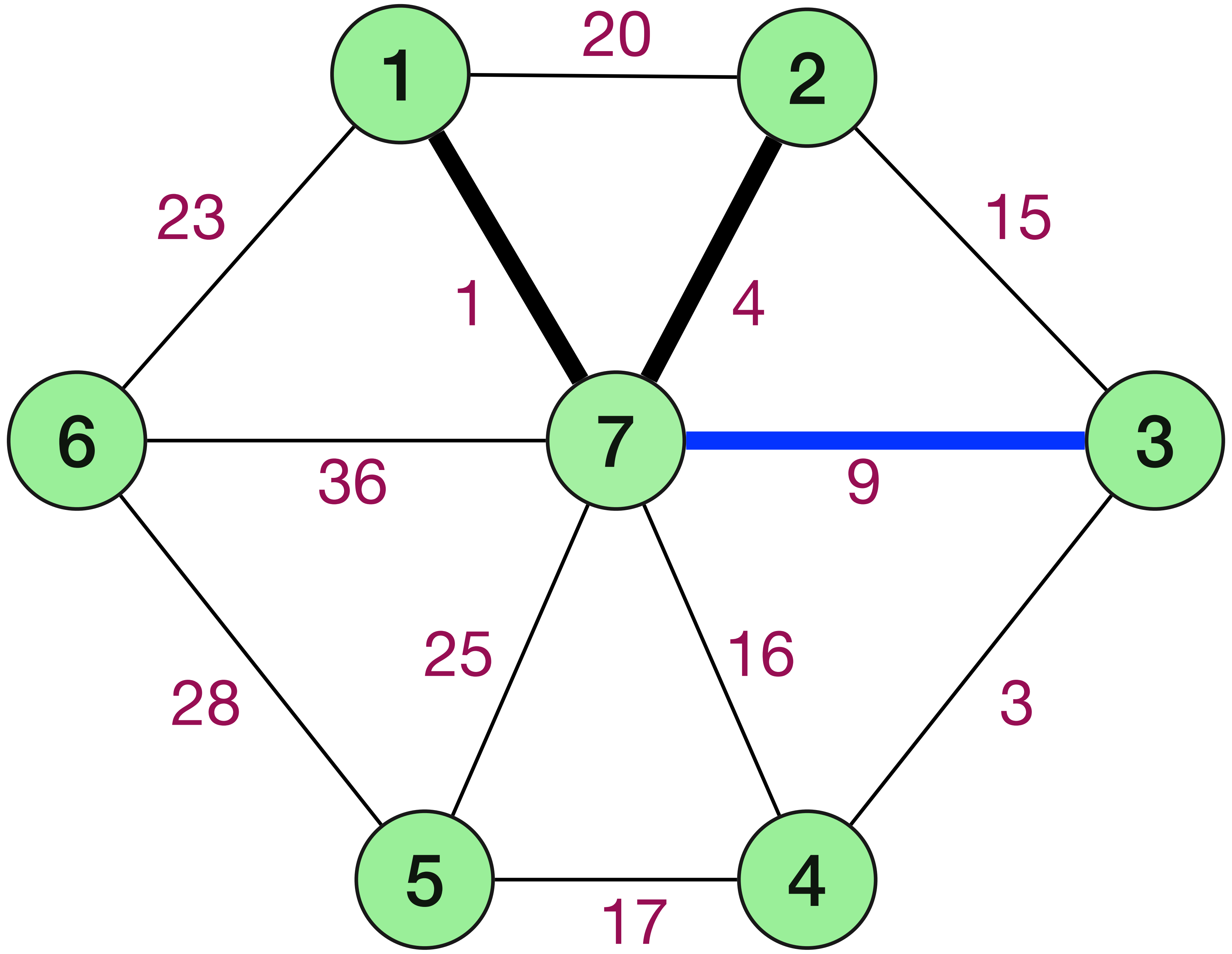

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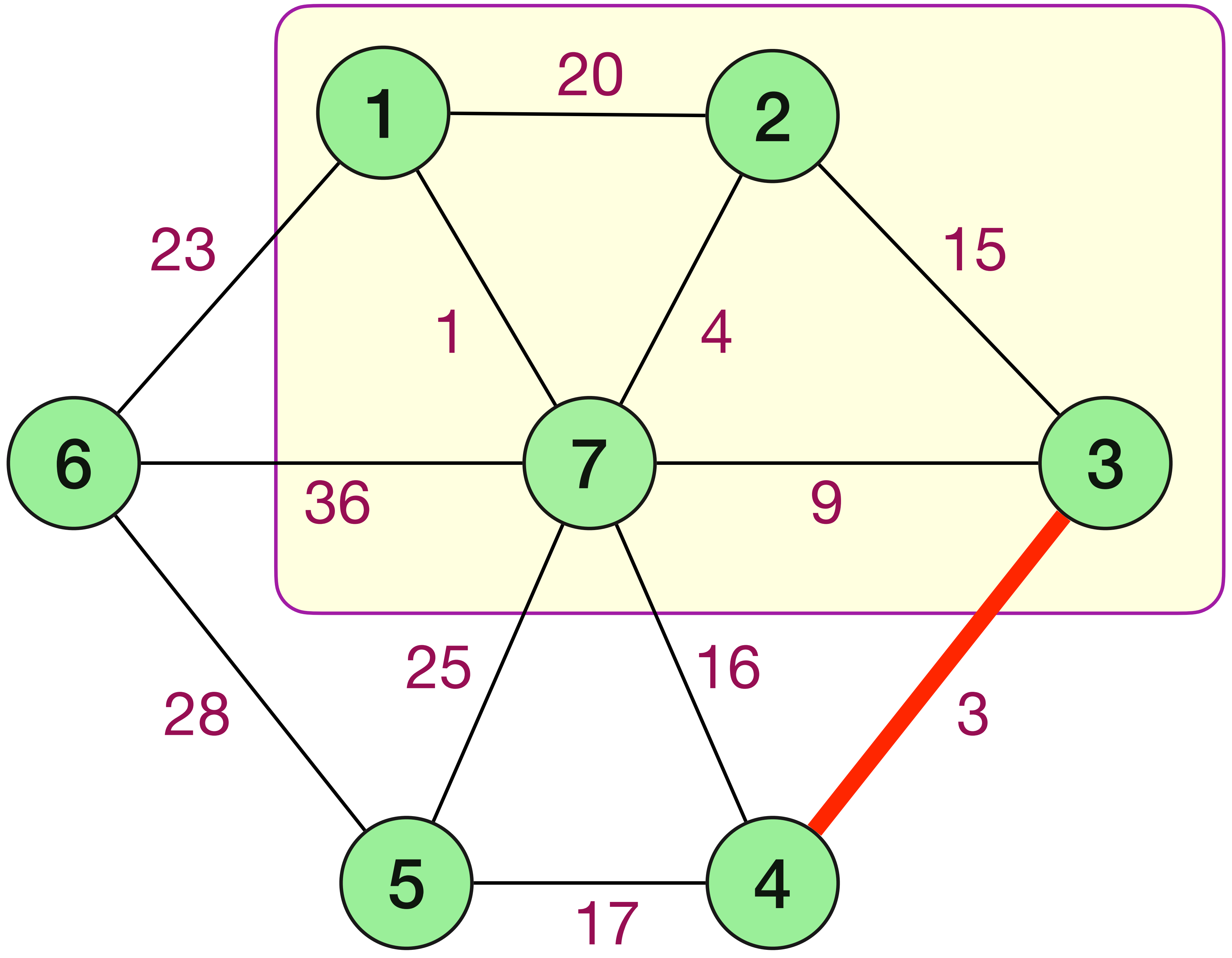

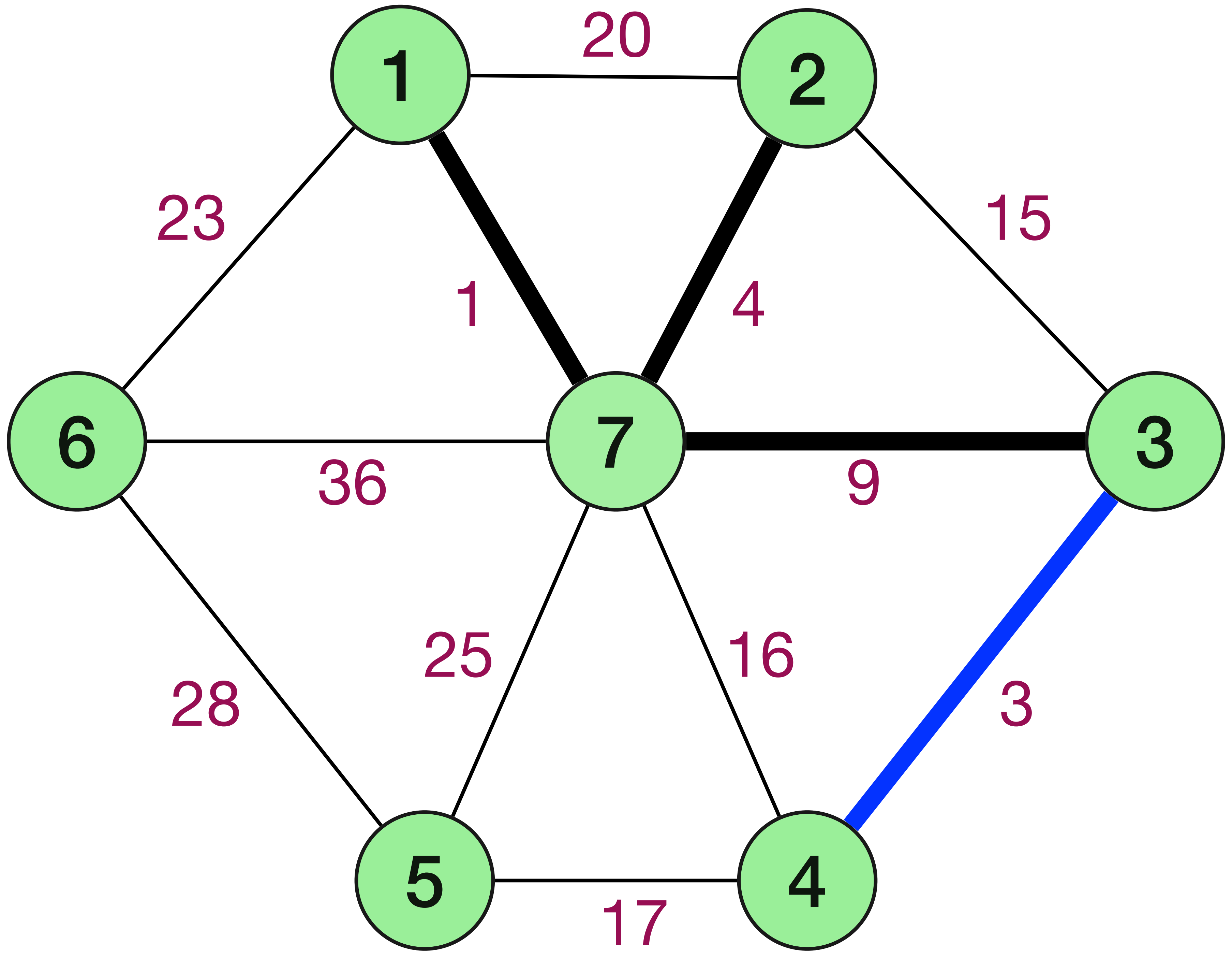

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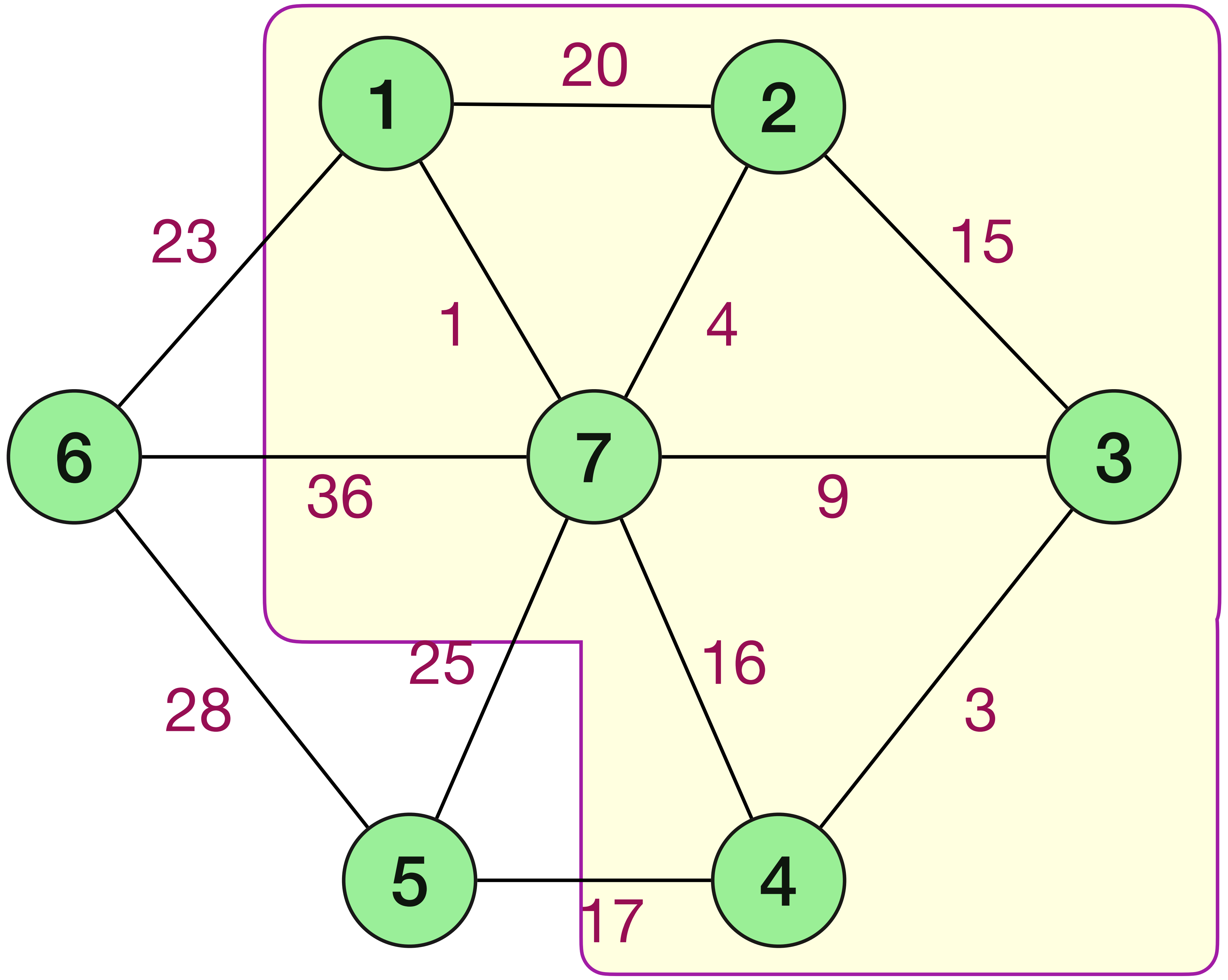

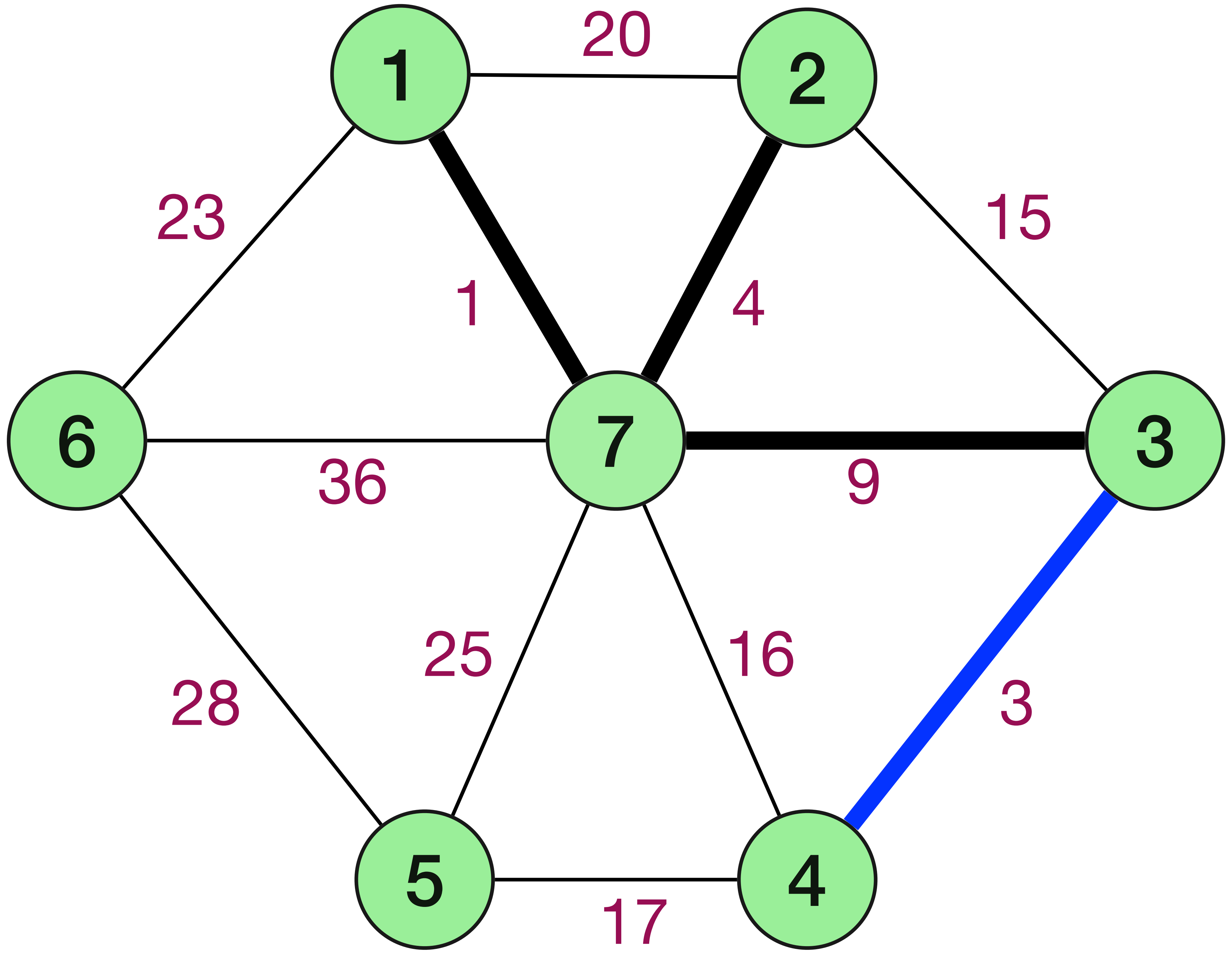

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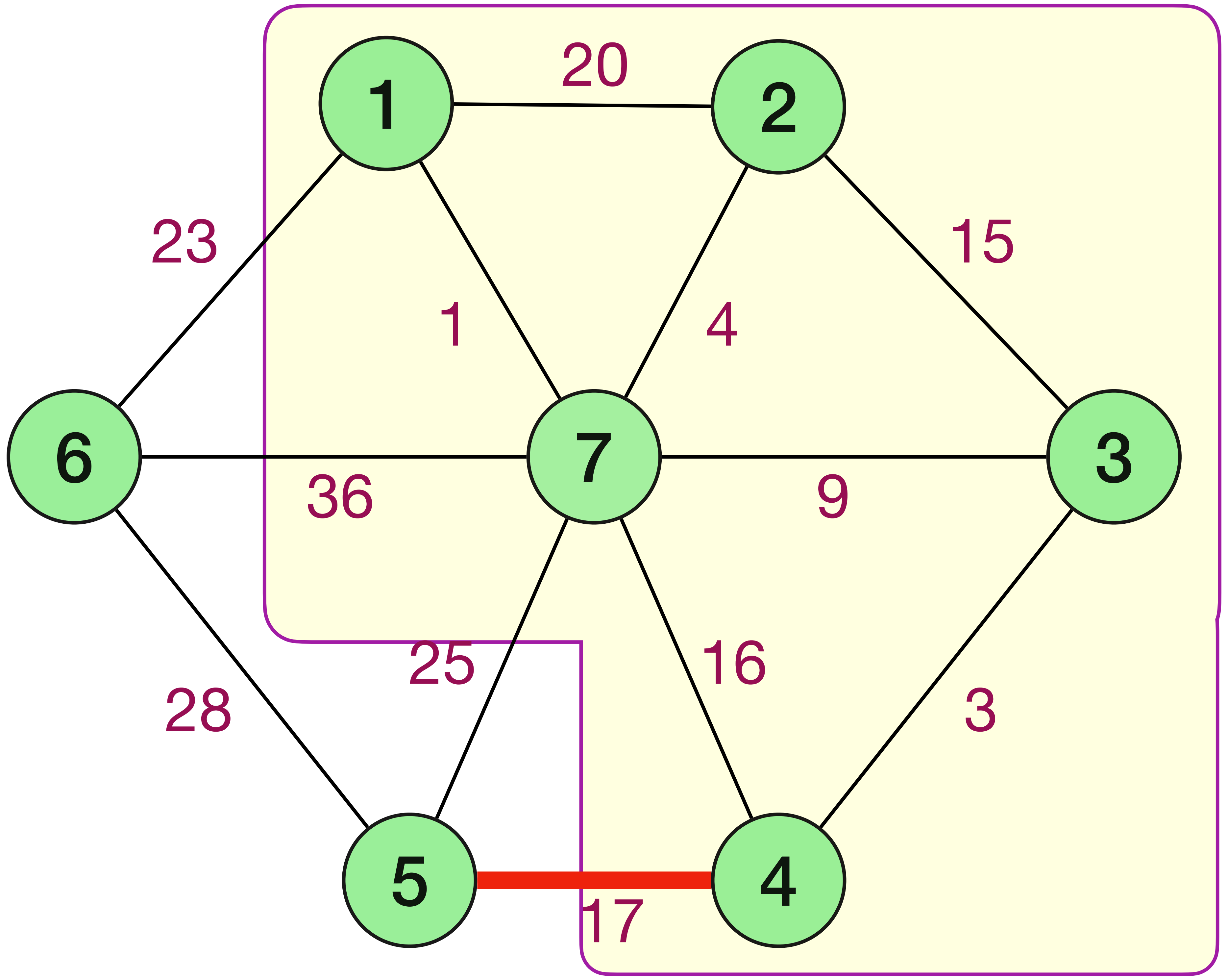

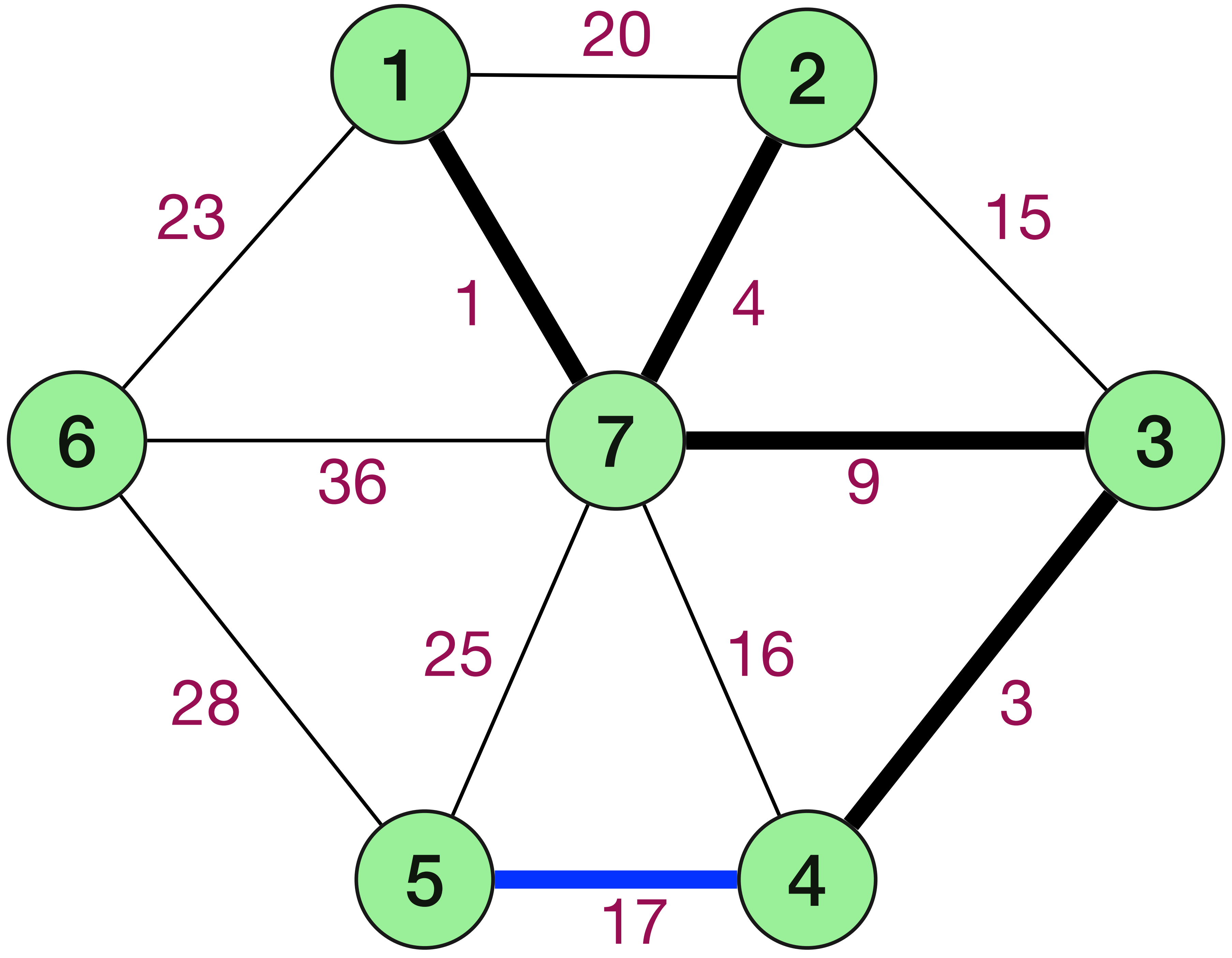

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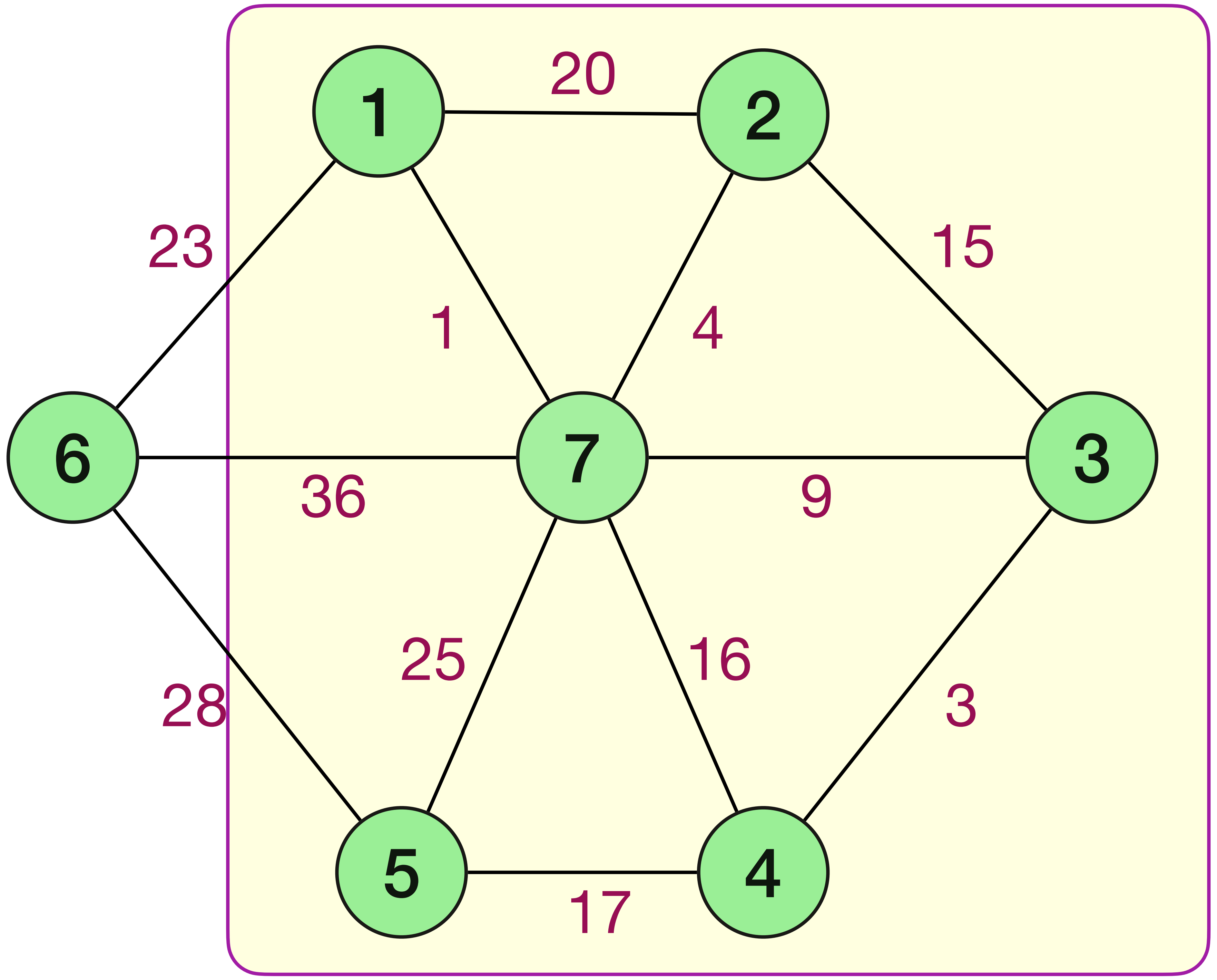

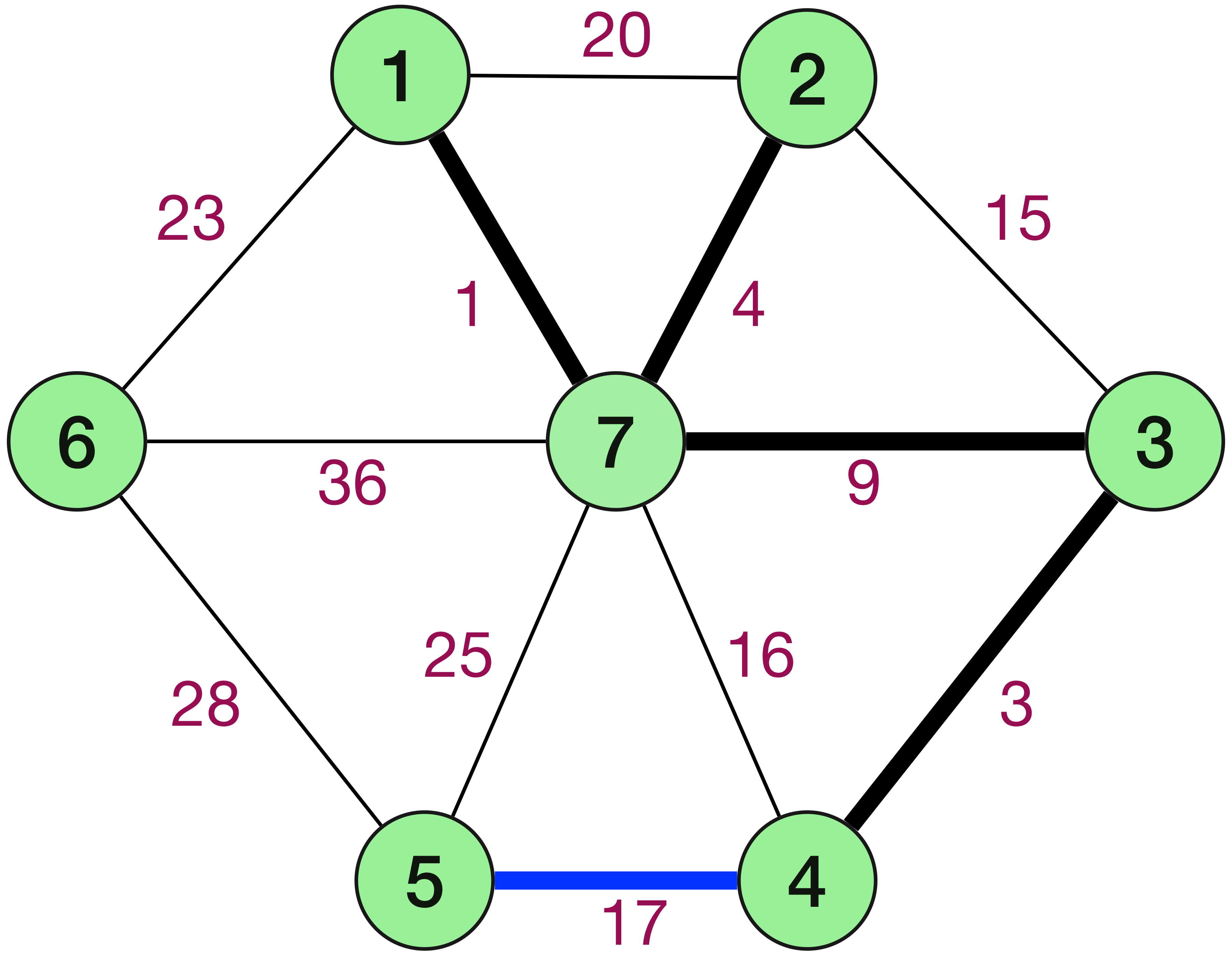

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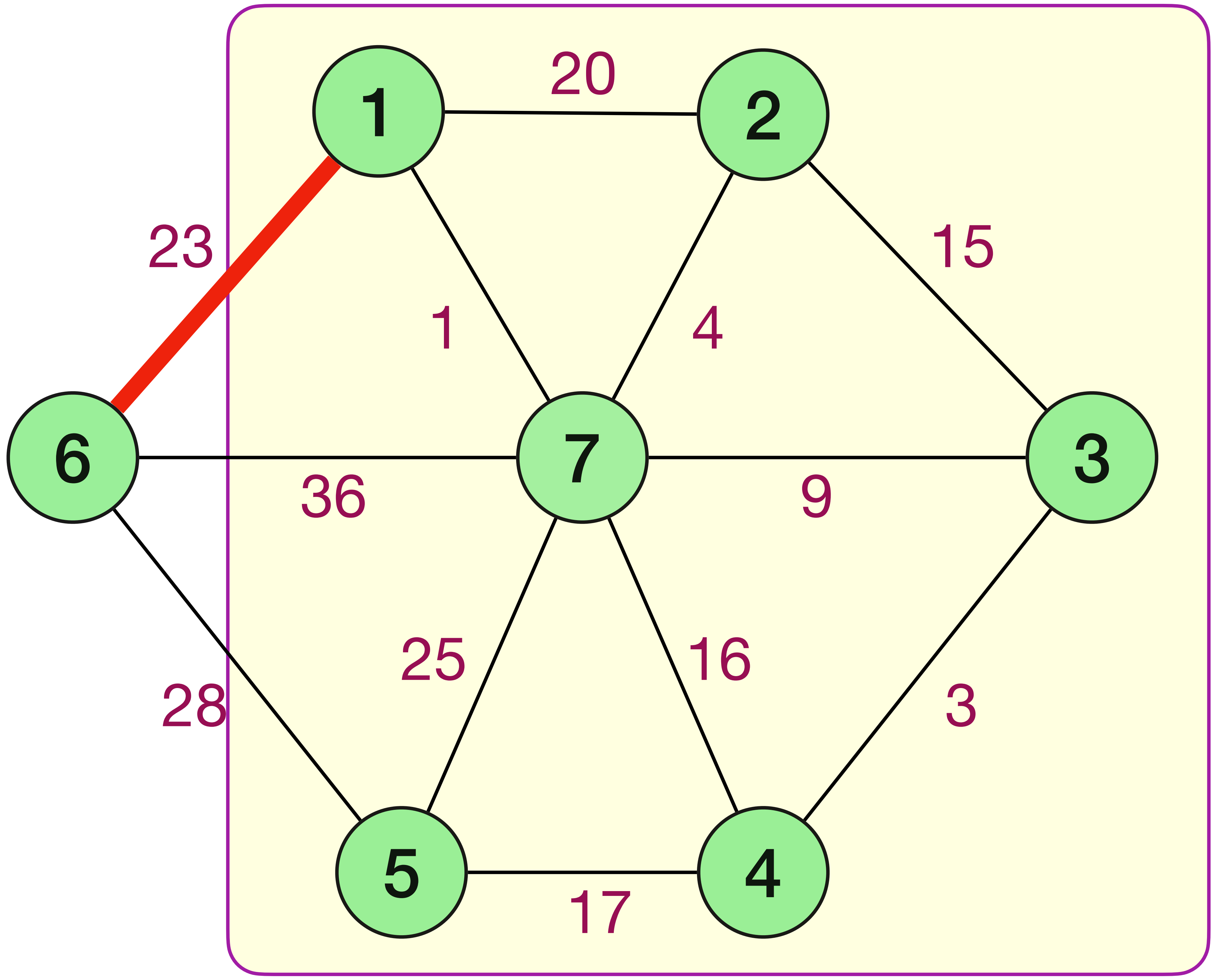

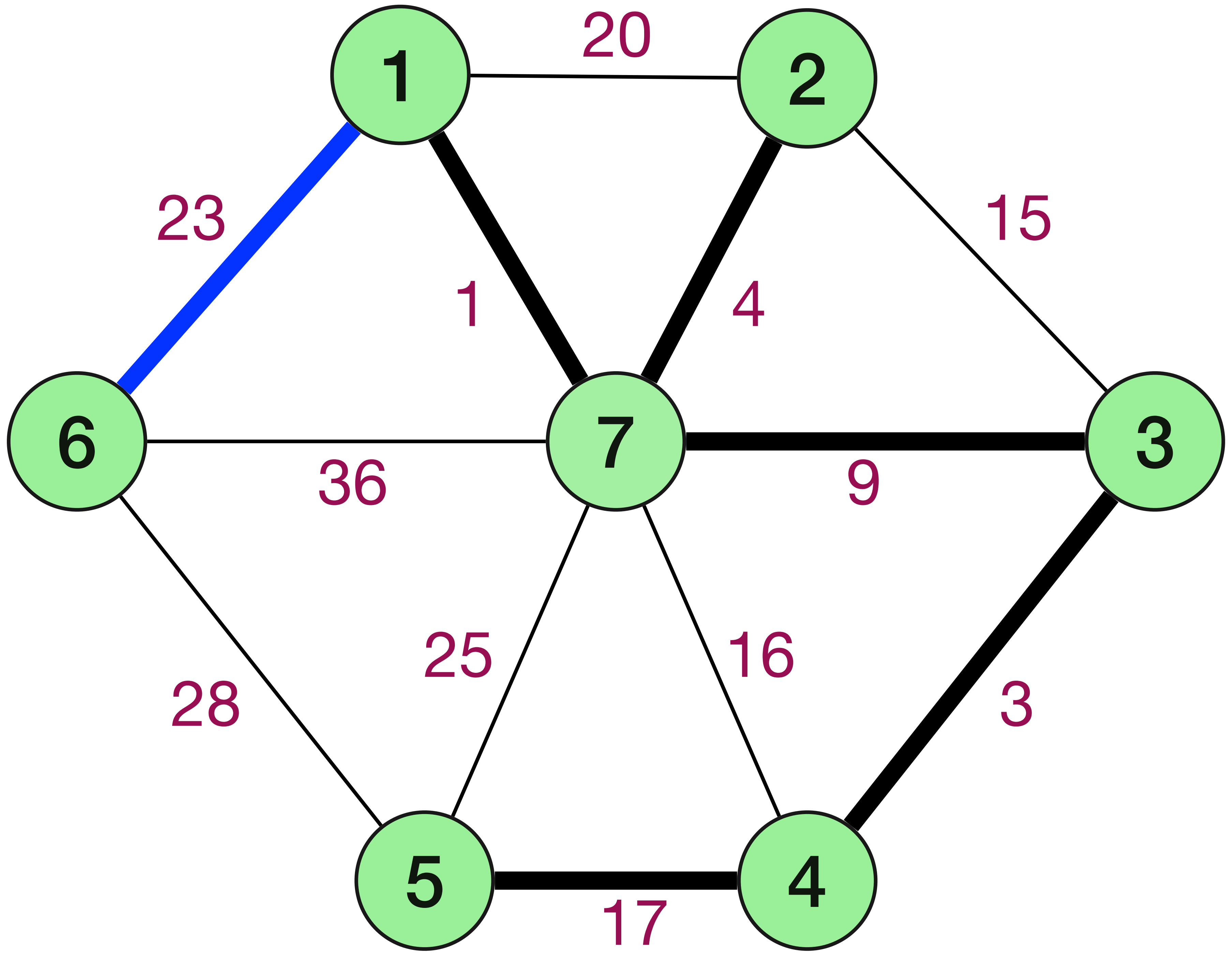

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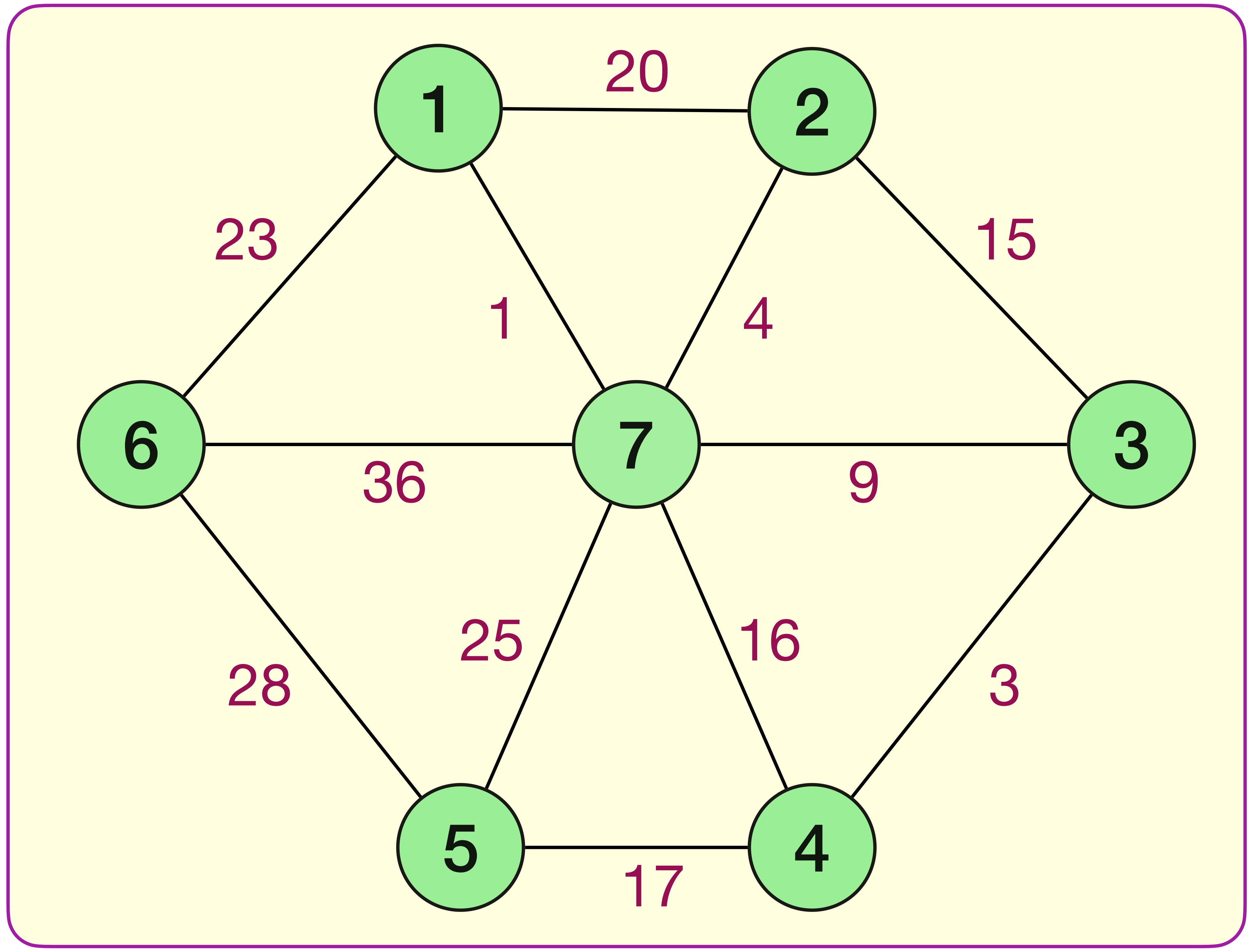

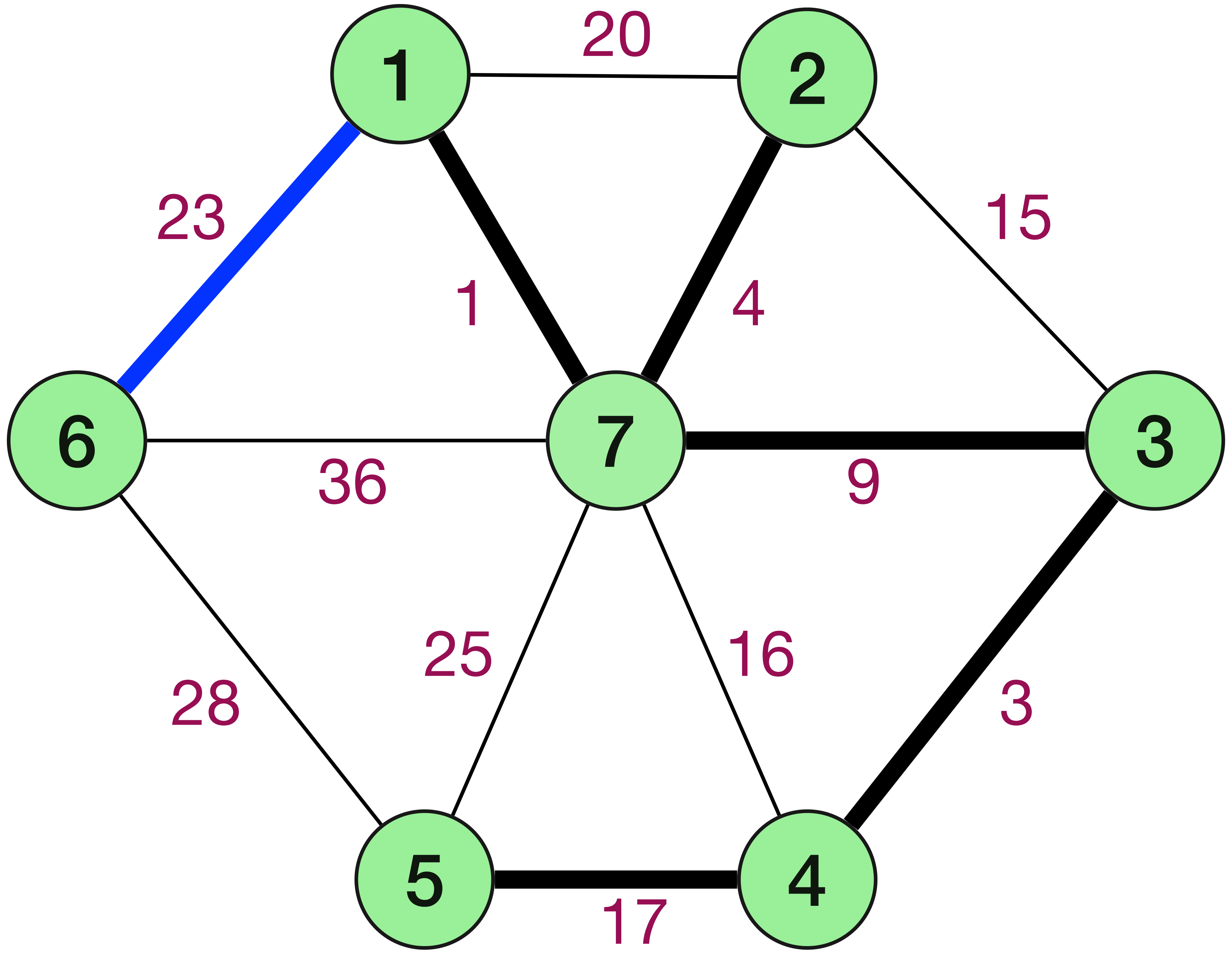

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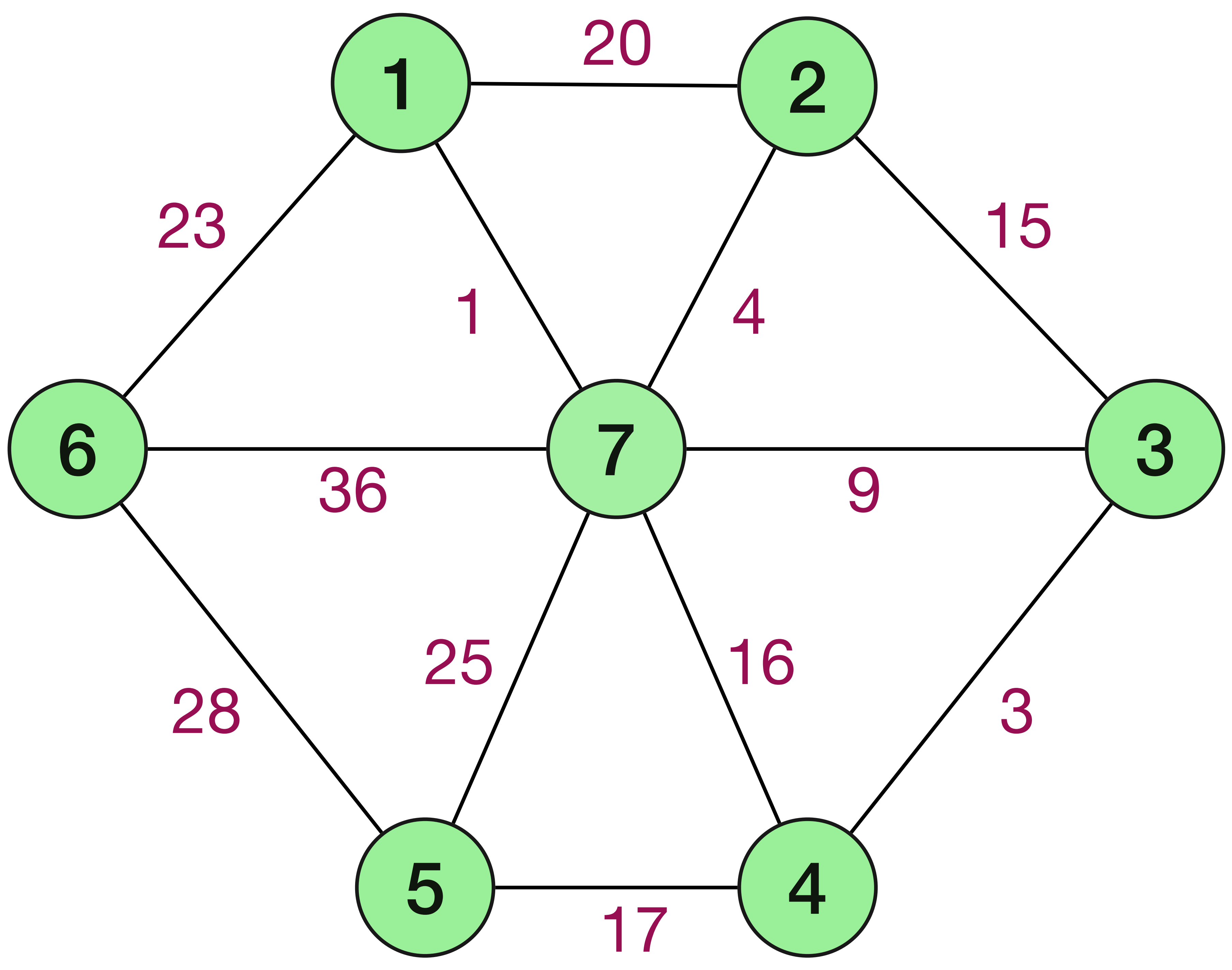

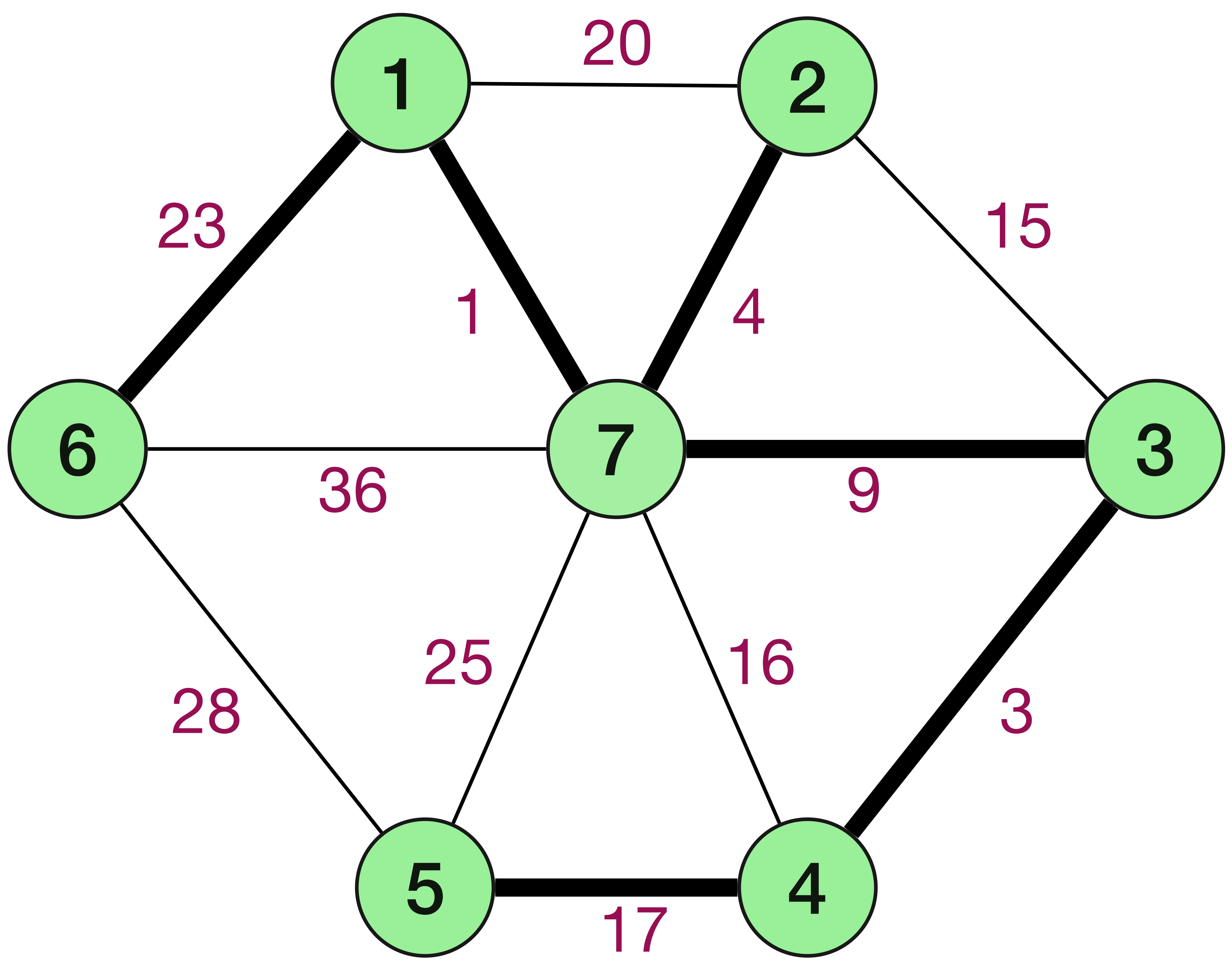

Correctness of Prim's Algorithm

Prim's Algorithm: Picking edge with minimum attachment cost to current tree, and adding to current tree generates a MST.

Proof: If e is added to tree, then e is safe and belongs to every MST.

- Let S be the vertices connected by edges in T when e is added.
- e is edge of lowest cost with one end in S and the other in $V \backslash S$ and hence e is safe.

- Set of edges output is a spanning tree
    - Set of edges output forms a connected graph: by induction, S is connected in each iteration and eventually $S = V$.
    - Only safe edges added and they do not have a cycle

Implementing Prim's Algorithm

```
Prim_ComputeMST
    E is the set of all edges in G
    S = {1}
    T is empty (* T will store edges of a MST *)
    while  S ≠ V do
        pick e =(v,w) ∈ E such that
            v ∈ S and w ∈ V\S
            e has minimum cost
        T = T ∪ e
        S = S ∪ w
    return the set T
```

Analysis

- Number of iterations = $O(n)$, where n is number of vertices
- Picking e is $O(m)$ where m is the number of edges
- Total time $O(nm)$

Prim's relation to Djikstra

```
Prim_ComputeMSTv1
    E is the set of all edges in G
    S ← {1}
    T is empty
    (* T will store edges of a MST *)
    for v ∉ S, d(v) = min_{x∈S} c(xv)
    for v ∉ S, p(v) = arg min_{x∈S} c(xv)
    while S ≠ V do
        pick v ∈ V\S with minimum d(v)
        e ← vp(v)
        T ← T ∪ {e}
        S ← S ∪ {v}
        update arrays d and p
    return the set T
```

```
Prim_ComputeMSTv2

    T ← ∅, S ← ∅, s = 1
    ∀v ∈ V (G) : d(v) ← ∞
    ∀v ∈ V (G) : p(v) ← Nil
    d(s) ← 0

    while S ≠ V do
        pick v ∈ V\S with minimum d(v)
        e ← vp(v)
        T ← T ∪ {e}
        S ← S ∪ {v}
        update arrays d and p
    return T
```

Prim's relation to Djikstra

```
Prim_ComputeMSTv2

    T ← ∅, S ← ∅, s = 1
    ∀v ∈ V (G) : d(v) ← ∞
    ∀v ∈ V (G) : p(v) ← Nil
    d(s) ← 0

    while S ≠ V do
        pick v ∈ V\S with minimum d(v)
        e ← vp(v)
        T ← T ∪ {e}
        S ← S ∪ {v}
        update arrays d and p
    return T
```

```
Prim_ComputeMSTv3
    T ← ∅, S ← ∅, s = 1
    ∀v ∈ V (G) : d(v) ← ∞, p(v) ← Nil
    d(s) ← 0
    while S ≠ V do
        v ← arg min_{u∈V\S} d(u)
        T ← T ∪ {vp(v)}
        S ← S ∪ {v}
        for each u in Adj(v) do
            d(u) ← min { d(u), c(vu) }
            if d(u) = c(vu) then
                p(u) ← v
    return T
```

Maintain vertices in $V \backslash S$ in a priority queue with key $d(v)$

Prim's relation to Djikstra

```
Prim_ComputeMSTv3
    T ← ∅, S ← ∅, s = 1
    ∀v ∈ V (G) : d(v) ← ∞, p(v) ← Nil
    d(s) ← 0
    while S ≠ V do
        v ← arg min_{u∈V\S} d(u)
        T ← T U {vp(v)}
        S ← S U {v}
        for each u in Adj(v) do
            d(u) ← min { d(u)
                       { c(vu)
            if d(u) = c(vu) then
                p(u) ← v
    return T
```

```
Dijkstra(G,s):
    ∀v ∈ V (G) : d(v) ← ∞, p(v) ← Nil
    S ← ∅, d(s) ← 0

    while S ≠ V do
        v ← arg min_{u∈V\S} d(u)
        S ← S U {v}
        for each u in Adj(v) do
            d(u) ← min { d(u)
                       { d(v) + l(v, u)
            if d(u) = d(v) + l(v,u) then
                p(u) ← v

    return d(v)
```

Prim's algorithm is essentially Dijkstra's algorithm!

Implementing Prim's algorithm with priority queues

Prim's using priority queues

```
Prim_ComputeMSTv3
    T ← ∅, S ← ∅, s = 1
    ∀v ∈ V (G) : d(v) ← ∞, p(v) ← Nil
    d(s) ← 0
    while S ≠ V do
        v ← arg min_{u∈V\S} d(u)
        T ← T ∪ {vp(v)}
        S ← S ∪ {v}
        for each u in Adj(v) do
            d(u) ← min { d(u), c(vu) }
            if d(u) = c(vu) then
                p(u) ← v
    return T
```

Maintain vertices in $V \backslash S$ in a priority queue with key $d(v)$

- Requires $O(n)$ extractMin operations
- Requires $O(m)$ decreaseKey operations

Running time of Prim's Algorithm

$O(n)$ extractMin operations and $O(m)$ decreaseKey operations

- Using standard Heaps, extractMin and decreaseKey take $O(\log n)$ time. Total: $O((m+n)\log n)$
- Using Fibonacci Heaps, $O(\log n)$ for extractMin and $O(1)$ (amortized) for decreaseKey. Total: $O(n\log n + m)$.
- Prim's algorithm and Dijkstra's algorithms are similar. Where is the difference?
- Prim's algorithm = Dijkstra where length of a path π is the weight of the heaviest edge in π. (Bottleneck shortest path.)

MST algorithm for negative weights, and non-distinct costs

When edge costs are not distinct

Heuristic argument: Make edge costs distinct by adding a small tiny and different cost to each edge

Formal argument: Order edges lexicographically to break ties

- $e_i \prec e_j$ if either $c(e_i) < c(e_j)$ or ($c(e_i) = c(e_j)$ and $i < j$)
- Lexicographic ordering extends to sets of edges. If $A, B \subseteq E$, $A \neq B$ then $A \prec B$ if either $c(A) < c(B)$ or ($c(A) = c(B)$ and $A \backslash B$ has a lower indexed edge than $B \backslash A$).
- Can order all spanning trees according to lexicographic order of their edge sets. Hence there is a unique MST.

Prim's and Kruskal's Algorithms are optimal with respect to lexicographic ordering.

Edge Costs: Positive and Negative

- Algorithms and proofs don't assume that edge costs are non-negative! MST algorithms work for arbitrary edge costs.

- Another way to see this: make edge costs non-negative by adding to each edge a large enough positive number. Why does this work for MSTs but not for shortest paths?

- Can compute maximum weight spanning tree by negating edge costs and then computing an MST.

Question: Why does this not work for shortest paths?

MST: An epilogue

Best Known Asymptotic Running Times for MST

Prim's algorithm using Fibonacci heaps: $O(n \log n + m)$.

If m is $O(n)$ then running time is $\Omega(n \log n)$.

Question: Is there a linear time ($O(m + n)$ time) algorithm for MST?

- $O(m \log^* m)$ time **[Fredman and Tarjan 1987]**
- $O(m + n)$ time using bit operations in RAM model **[Fredman, Willard 1994]**
- $O(m + n)$ expected time (randomized algorithm) **[Karger, Klein, Tarjan 1995]**
- $O((n + m)\alpha(m, n))$ time **[Chazelle 2000]**
- Still open: Is there an $O(n + m)$ time deterministic algorithm in the comparison model?