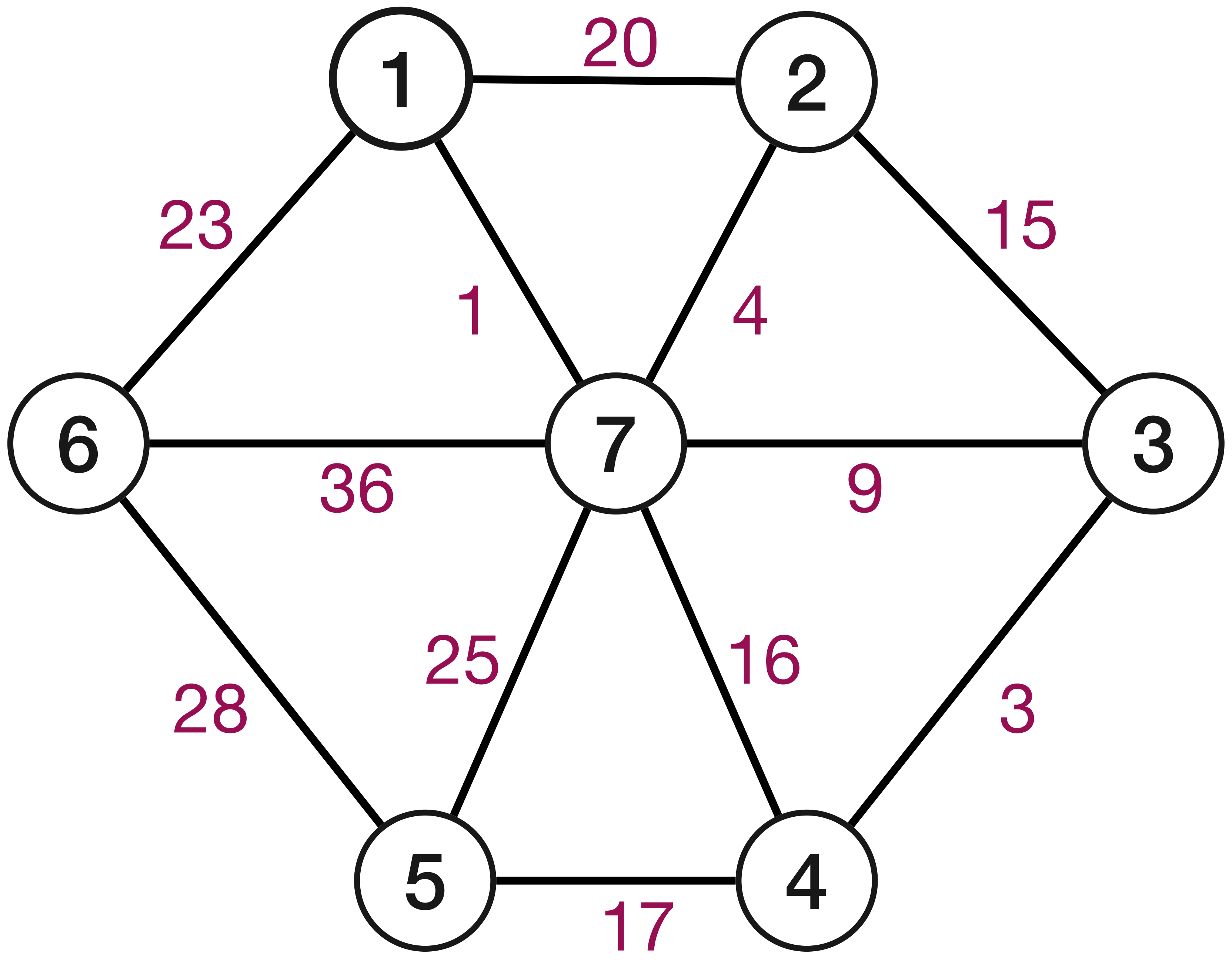
Minimum spanning trees (MSTs)

Sides based on material by Kani, Erickson, Chekuri, et. al.

All mistakes are my own! - Ivan Abraham (Fall 2024)

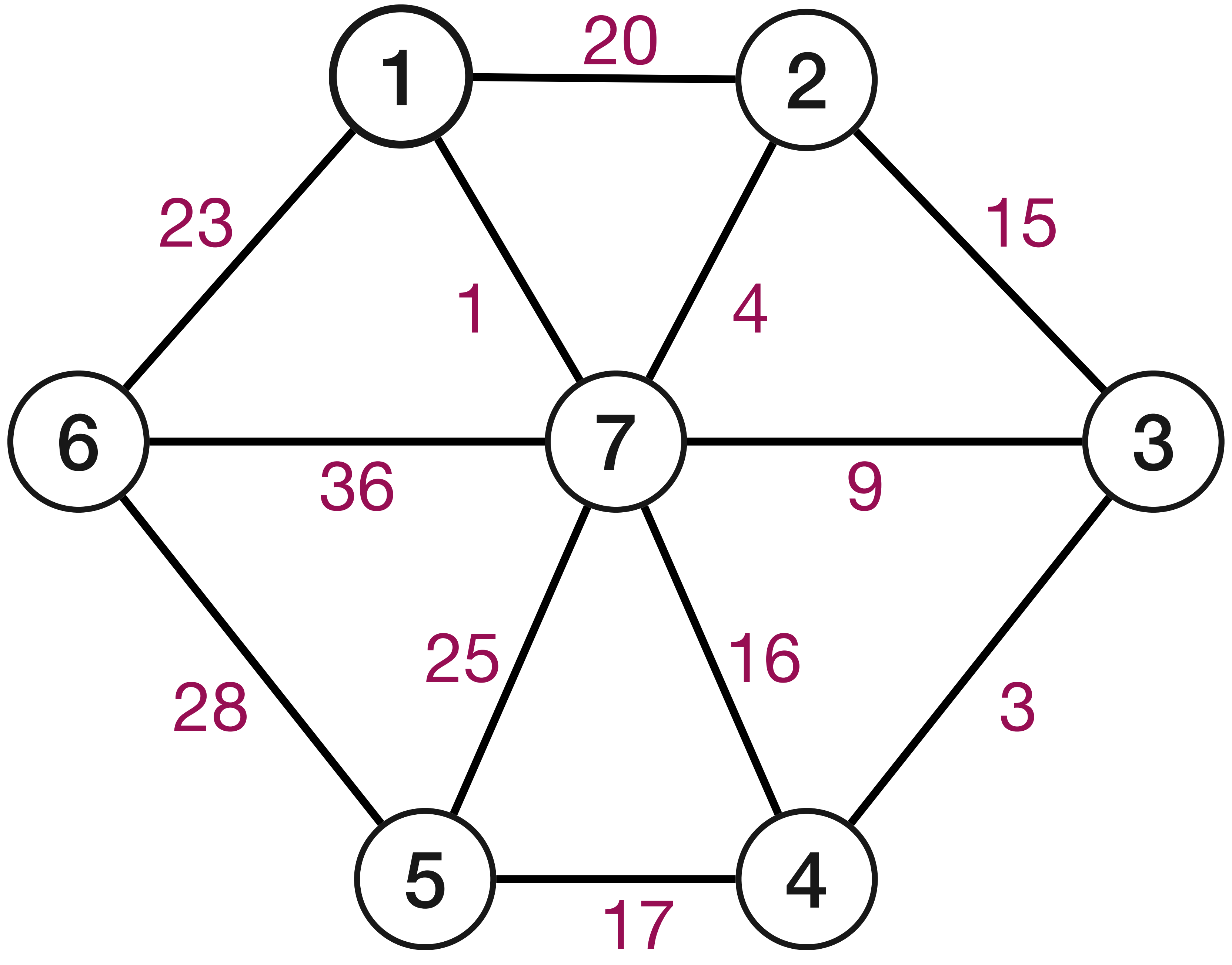
Image by ChatGPT (probably collaborated with DALL-E)

Minimum Spanning Tree



Minimum Spanning Tree

Input: Connected graph $G = (V, E)$ with edge costs



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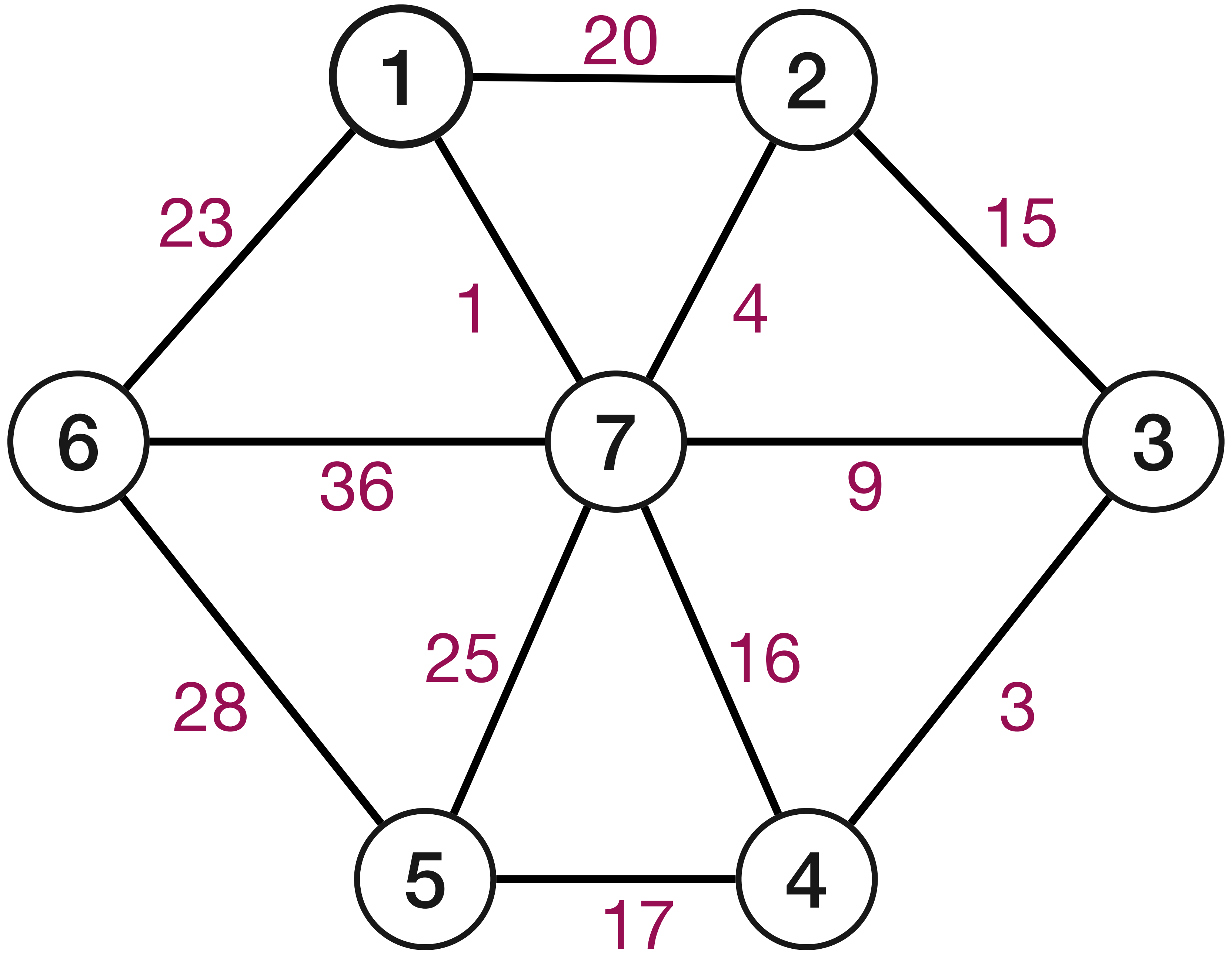

Minimum Spanning Tree

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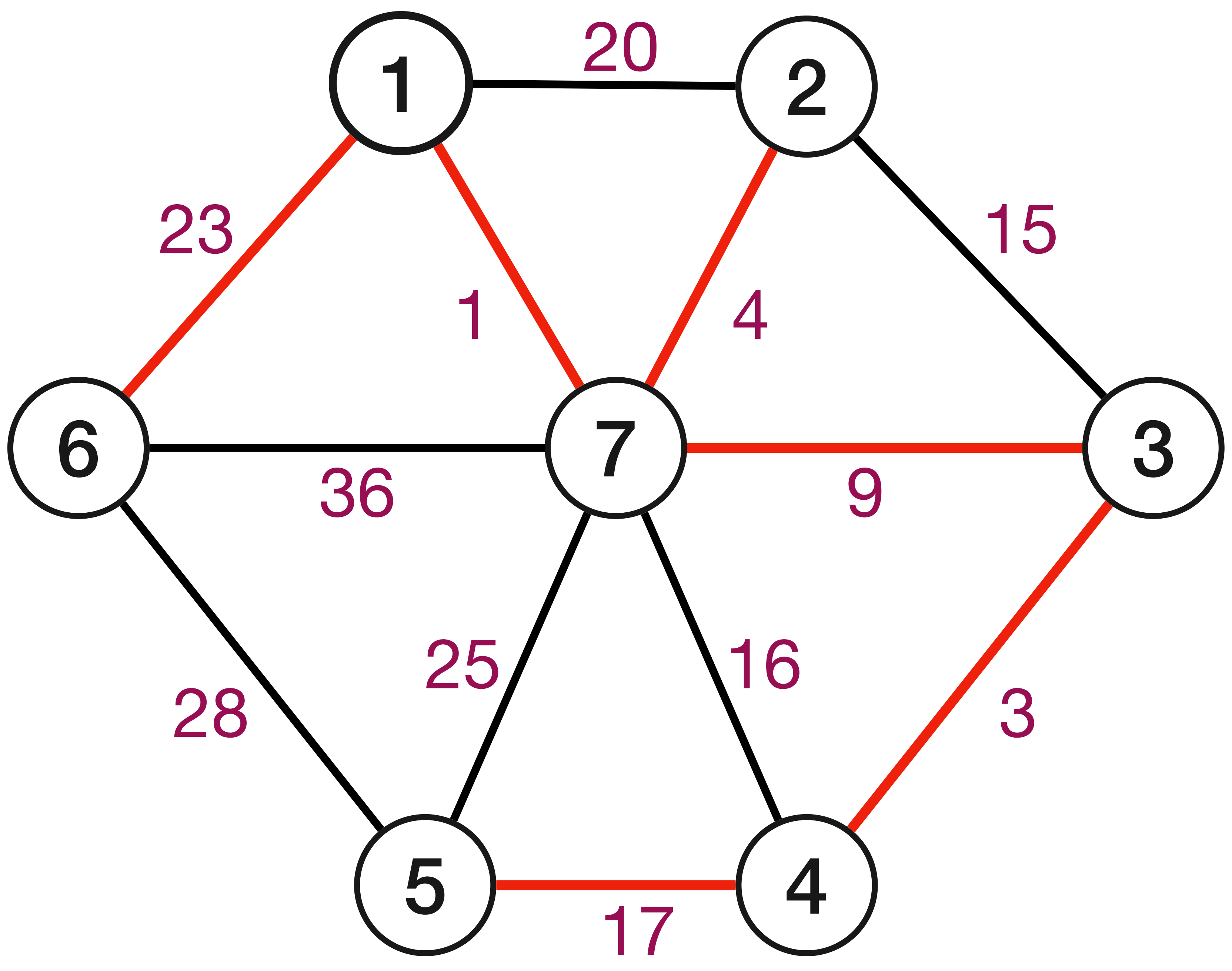
T is the **minimum spanning tree (MST)** of G.



Minimum Spanning Tree

Applications

- Network design

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- Network design
    - Designing networks with minimum cost but maximum connectivity

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- Network design
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- Cluster analysis
    - Social networks, epidemiological networks, etc.
- Approximation algorithms
    - Can be used to bound the optimality of algorithms to approximate Traveling Salesman Problem, Steiner Trees, etc.

Spanning Trees
Basic properties

- Subgraph H of G is **spanning** for G, if G and H have same connected components.

Spanning Trees

Basic properties

MSTs as they relate to directed graphs

rooted directed trees

↳ arborescence.

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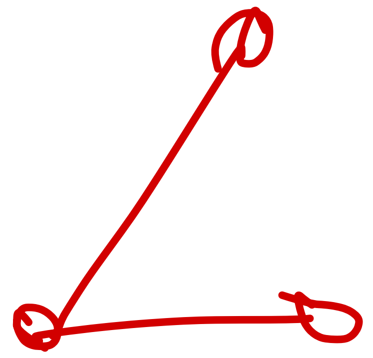

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- Every ***spanning tree*** of a graph on n nodes has $n - 1$ edges.
- A graph G is connected $\Longleftrightarrow$ it has a spanning tree.

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There is some work in 1909 by a Polish anthropologist Jan Czekanowski on clustering, which is a precursor to MST.

Exchanging an edge in a spanning tree

Useful lemma

Let $T = (V, E_T)$ be a spanning tree of $G = (V, E)$. Then,

Exchanging an edge in a spanning tree

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E setminus E_T

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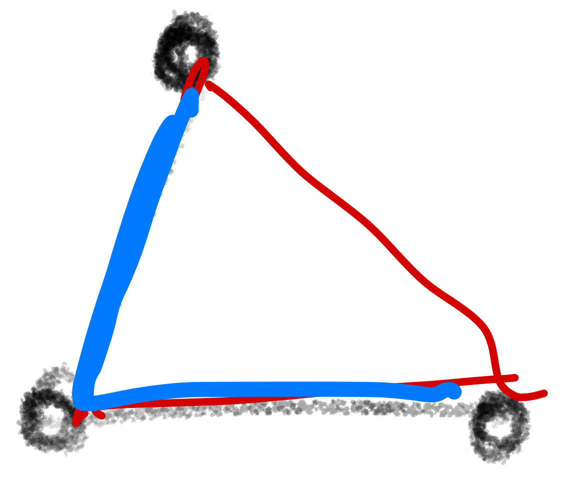
- For every non-tree edge $e \in E \setminus E_T$ there is a unique cycle C in $T + e$.

Exchanging an edge in a spanning tree

Useful lemma



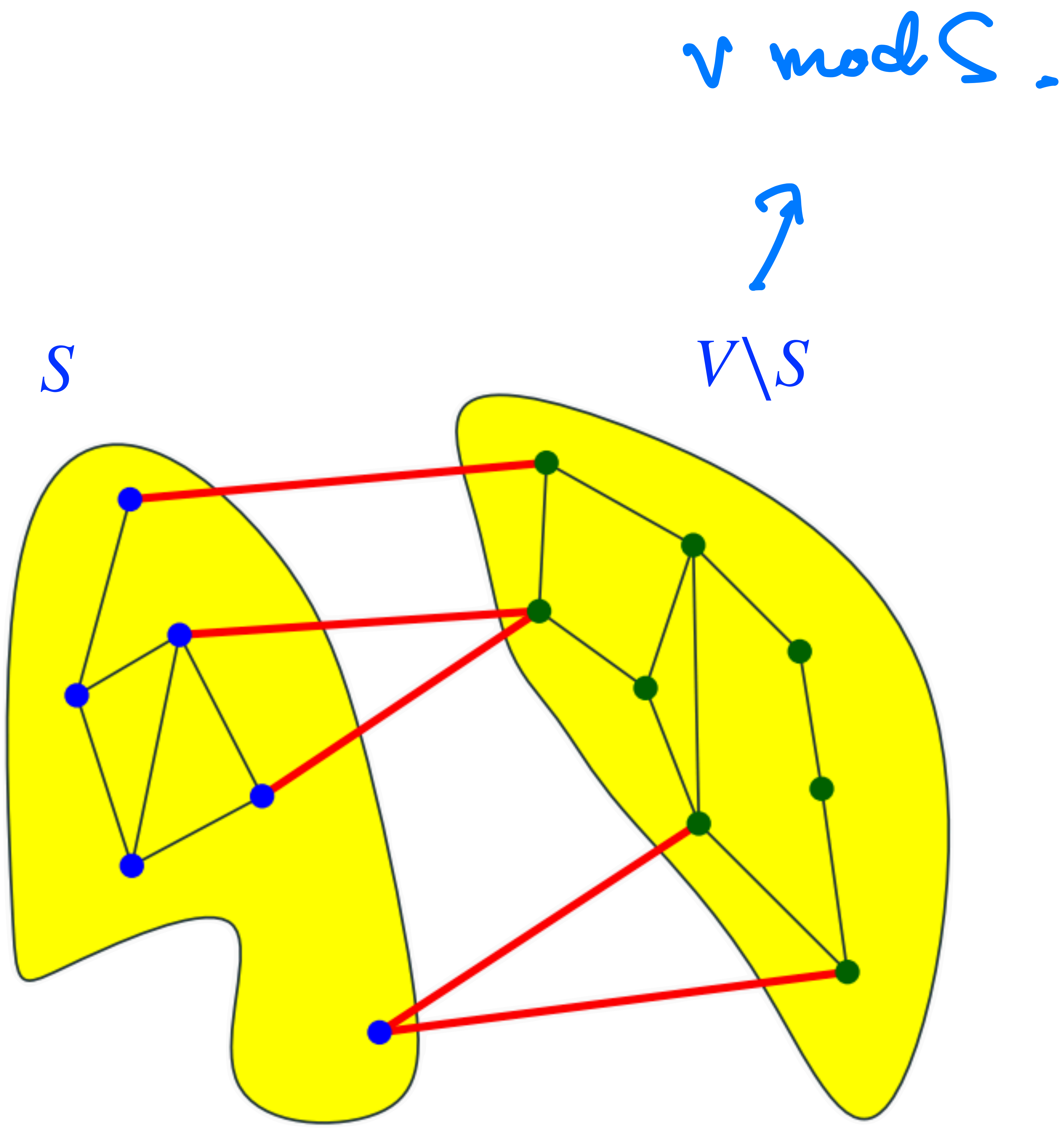
Let $T = (V, E_T)$ be a spanning tree of $G = (V, E)$. Then,

- For every non-tree edge $e \in E \backslash E_T$ there is a unique cycle C in $T + e$.
- For every edge $f \in C - \{e\}$, $T - f + e$ is another spanning tree of G.

Cuts

Definition

- Given a graph $G = (V, E)$, a ***cut*** is a partition of the vertices of the graph into two sets $(S, V \backslash S)$.



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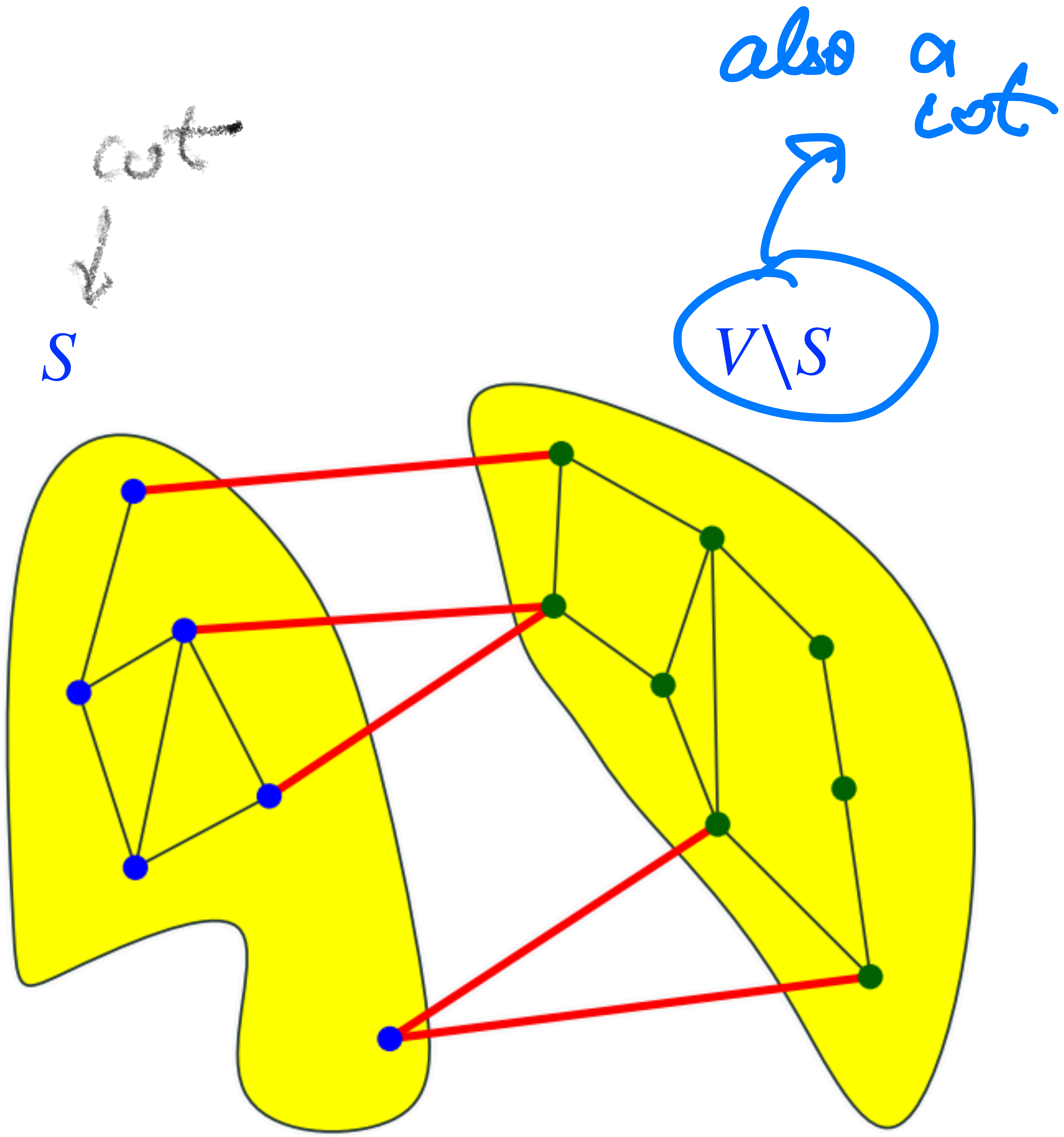

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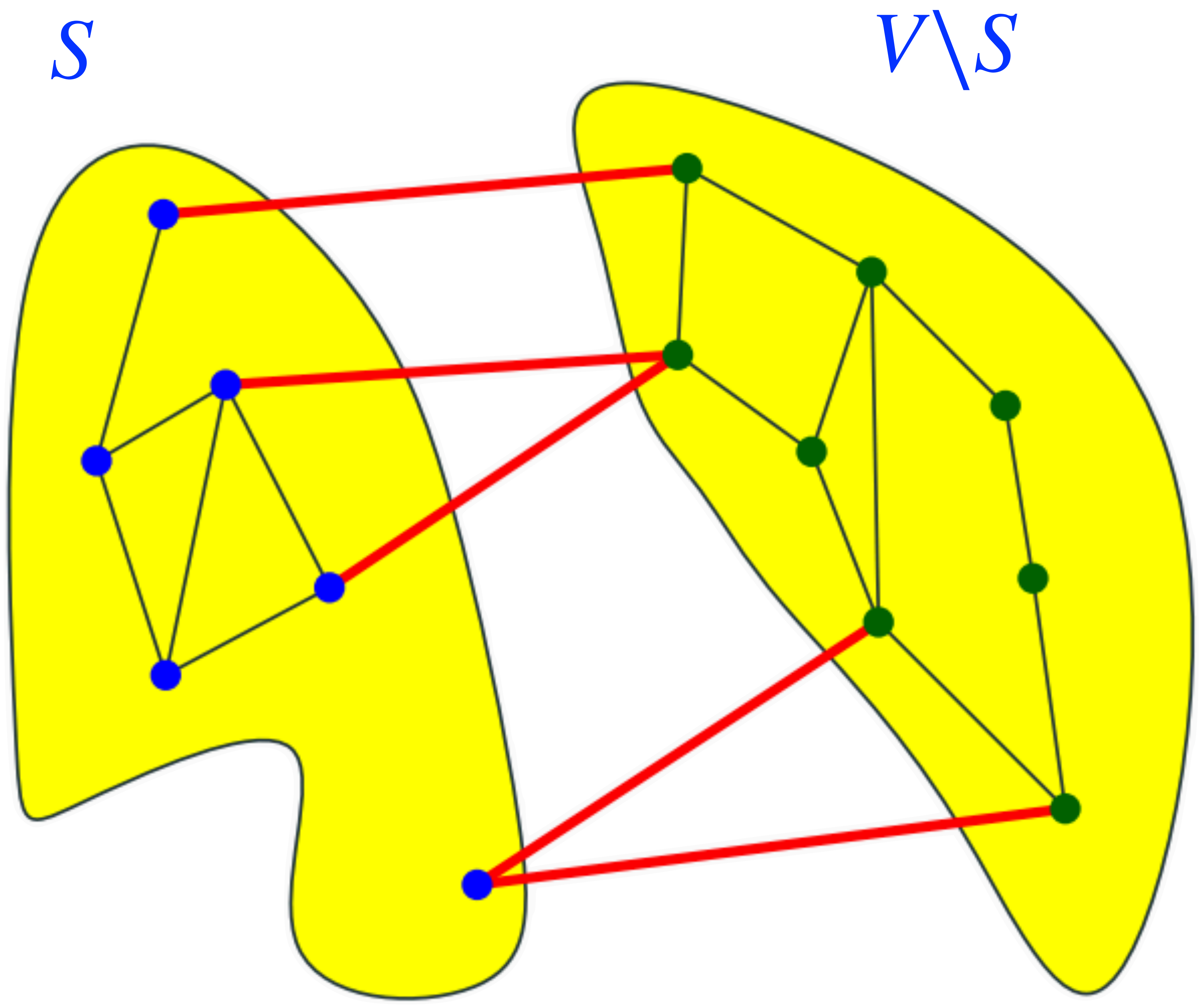
- A cut edge is *crossing* the cut.



Safe and unsafe edges

Assumption: Edge costs are distinct, that is no two edge costs are equal.

Safe edge:

An edge $e = (u, v)$ is a *safe edge* if there is some partition of V into S and $V \backslash S$ and e is the unique minimum cost edge crossing S (one end in S and the other in $V \backslash S$).

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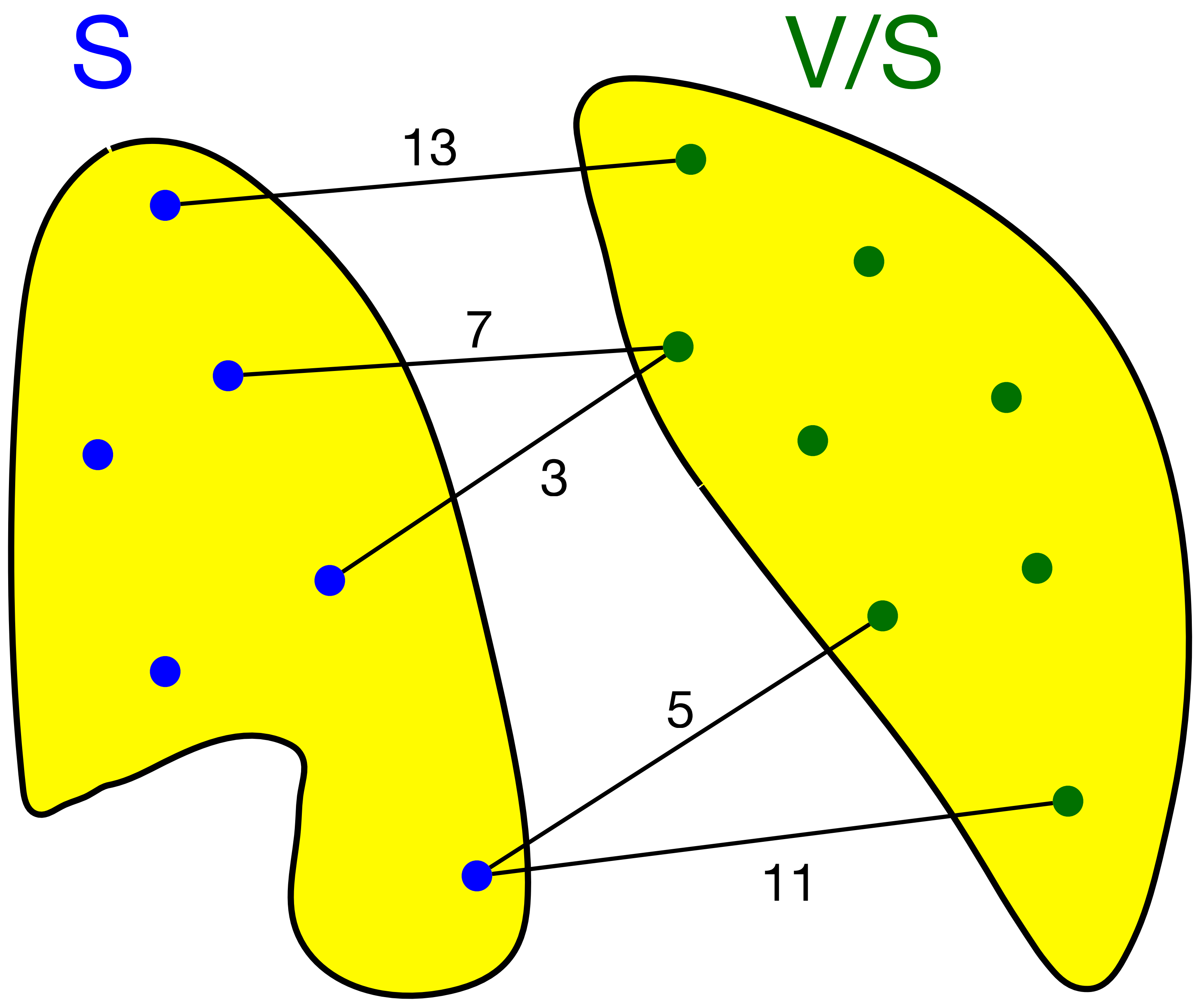
Safe edge

Example



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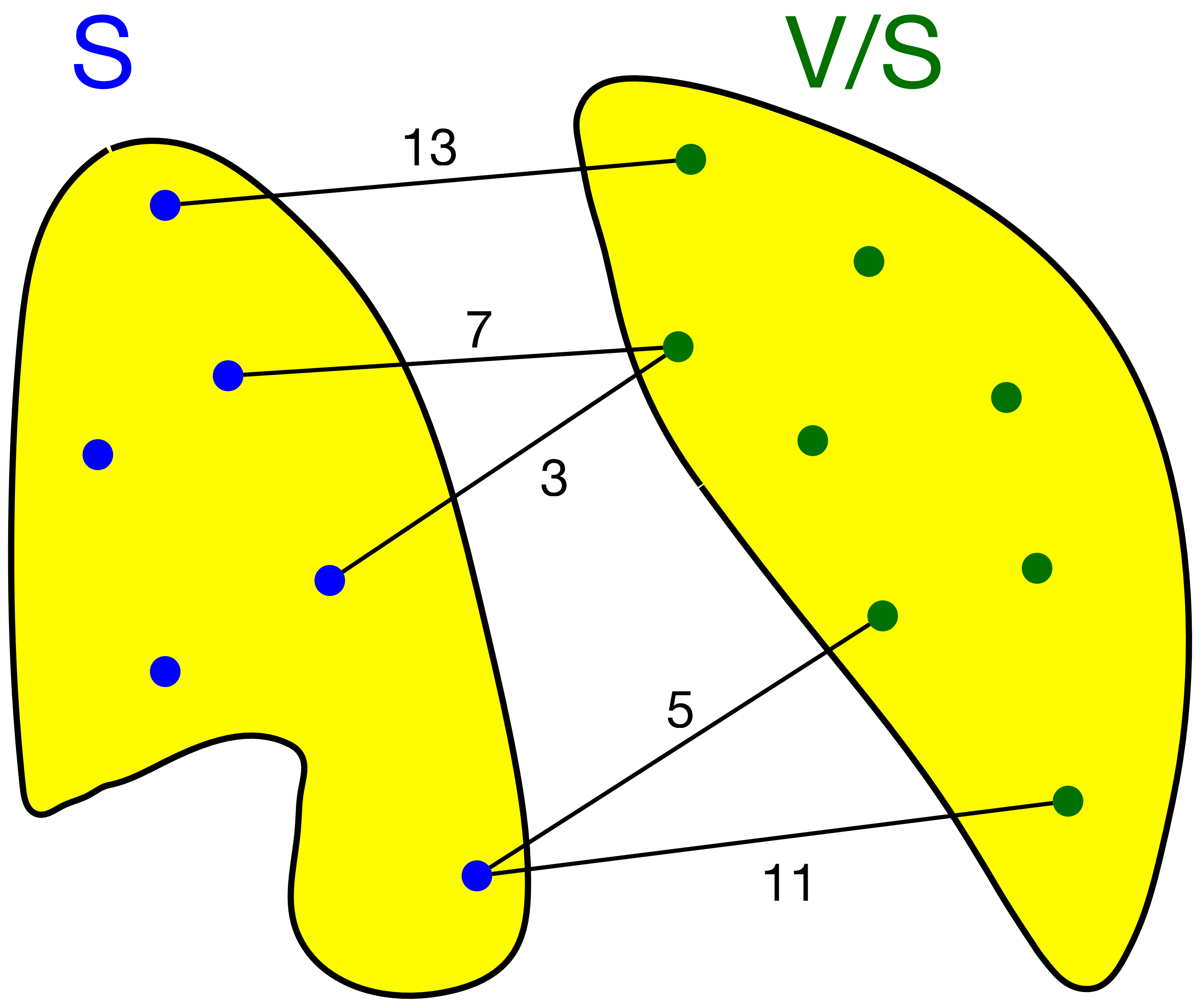
- Every cut identifies one safe edge …



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- Every cut identifies one safe edge …
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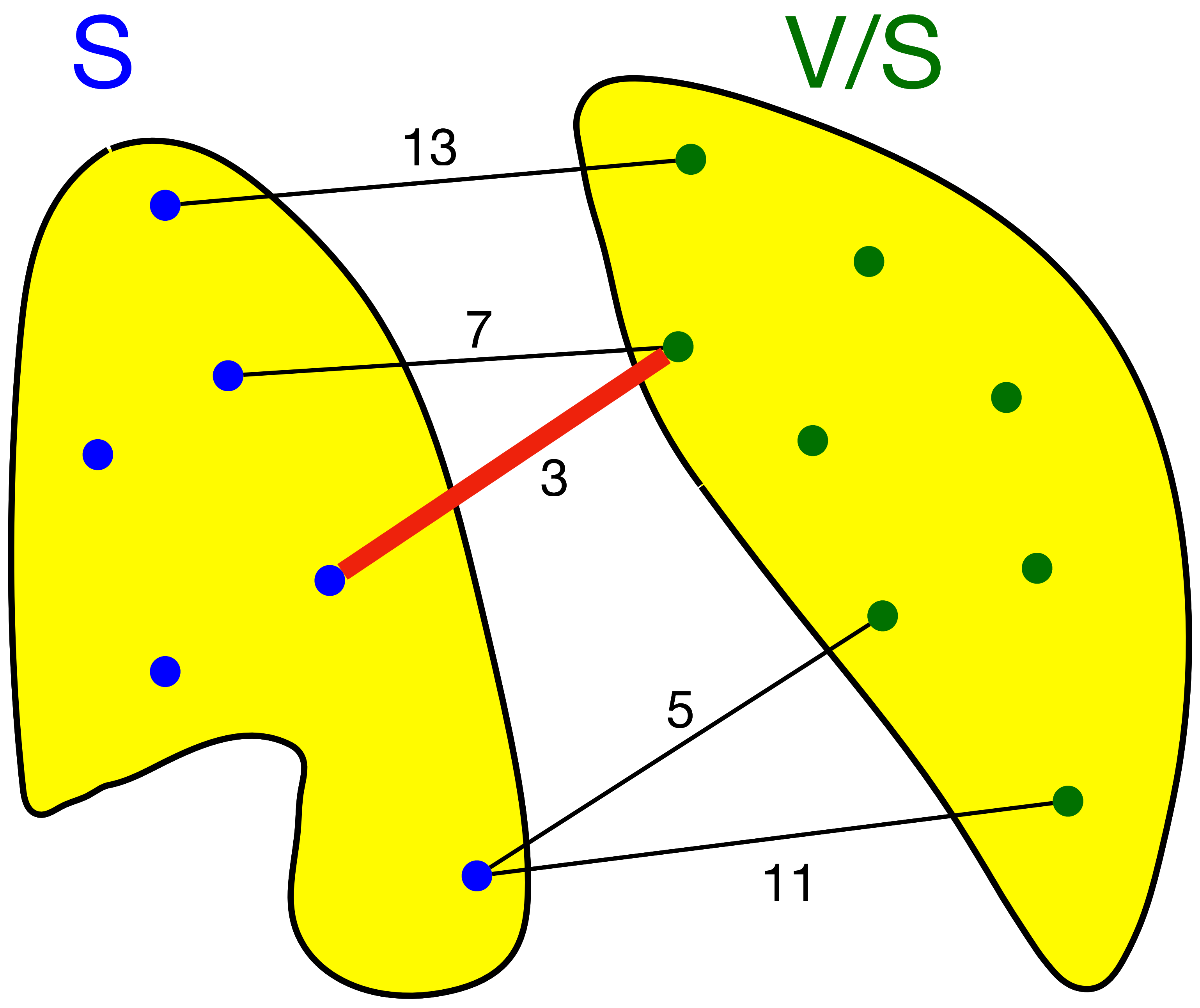
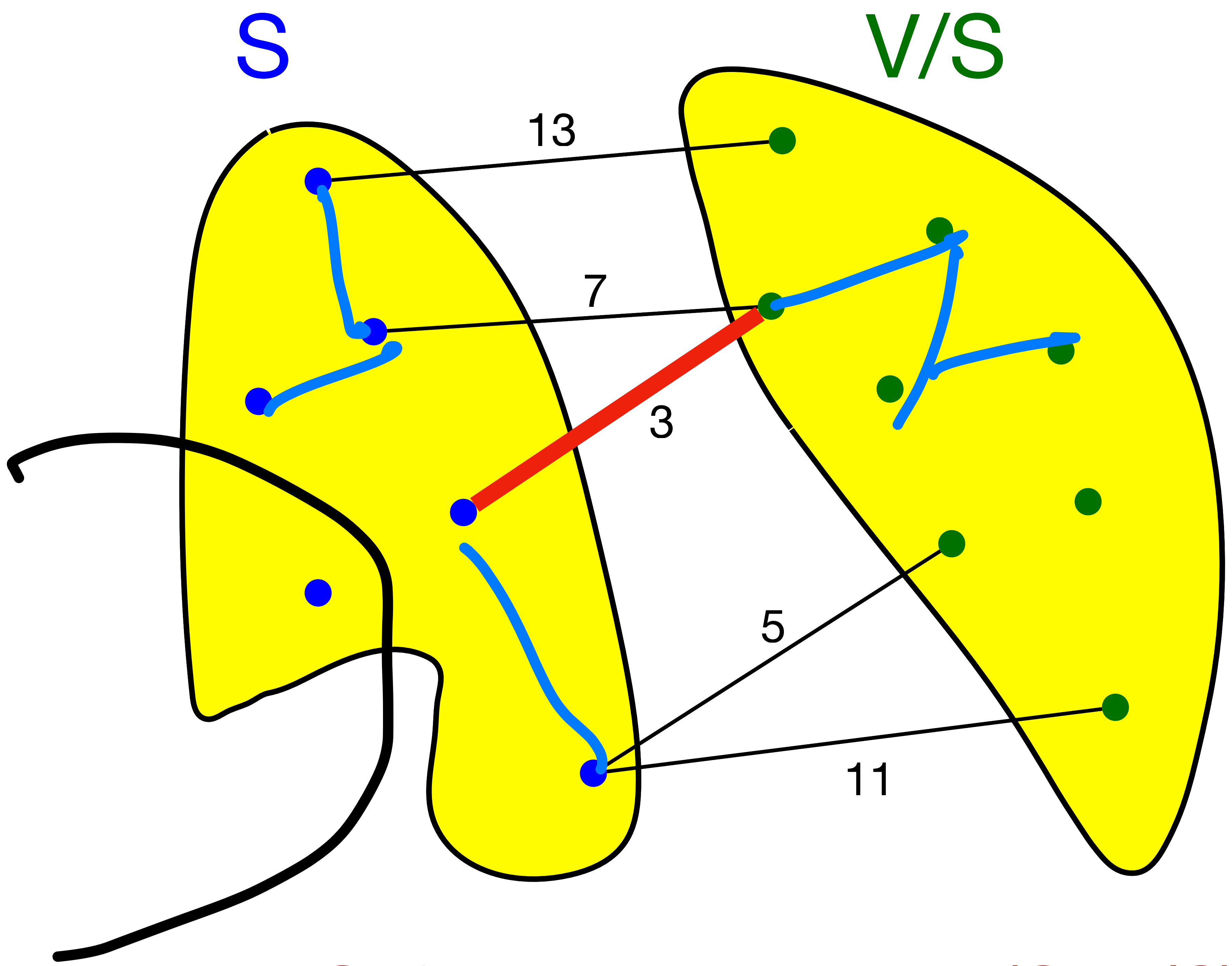


Safe edge in the cut (*S*, *V*/*S*)

Safe edge

Example

- Every cut identifies one safe edge …
- … the cheapest edge in the cut.
- Note: An edge e may be a safe edge for many cuts!



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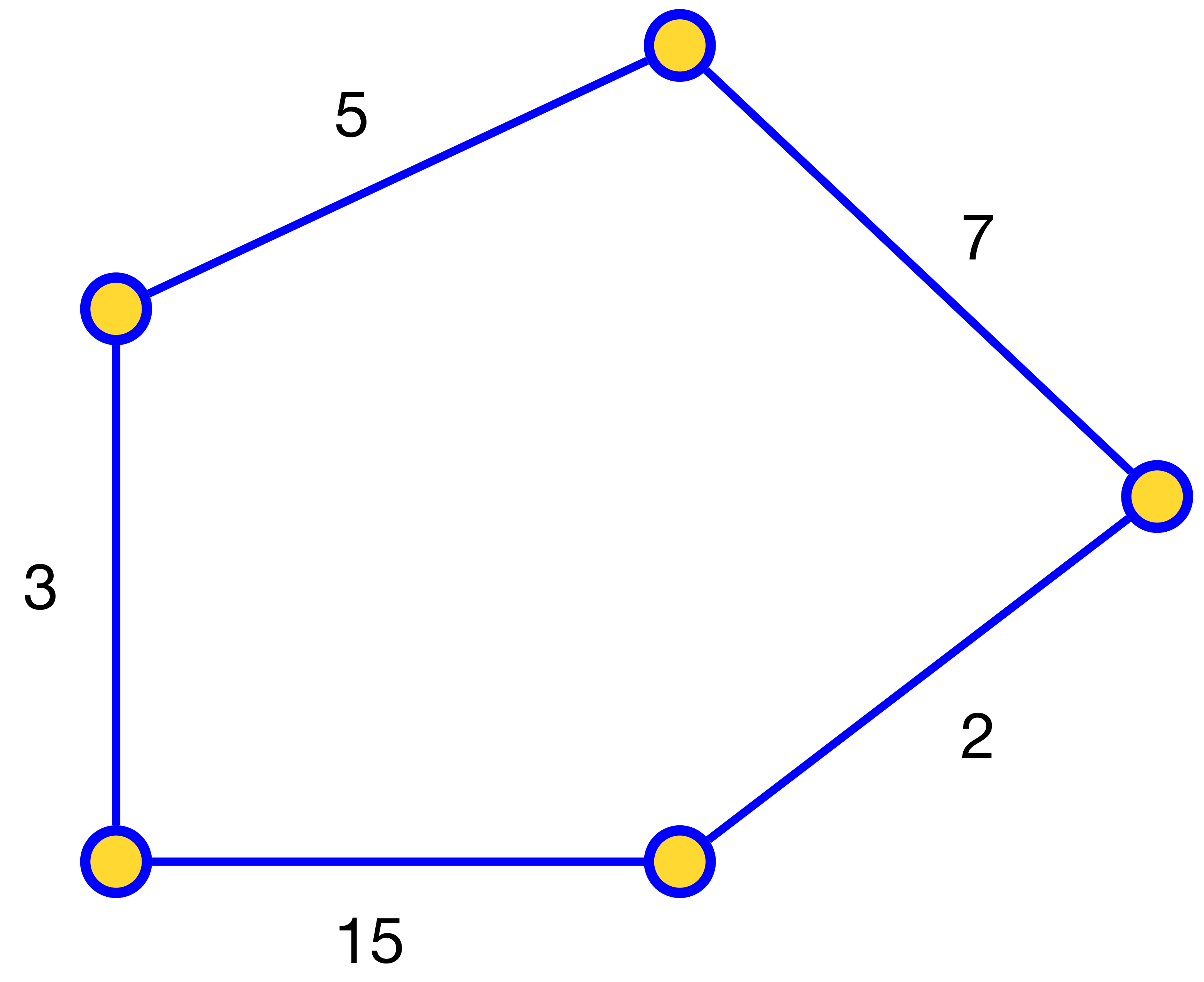
Unsafe edge

An edge $e = (u, v)$ is an *unsafe edge* if there is some cycle C such that e is the unique maximum cost edge in C.

Unsafe edge

Example



Unsafe edge

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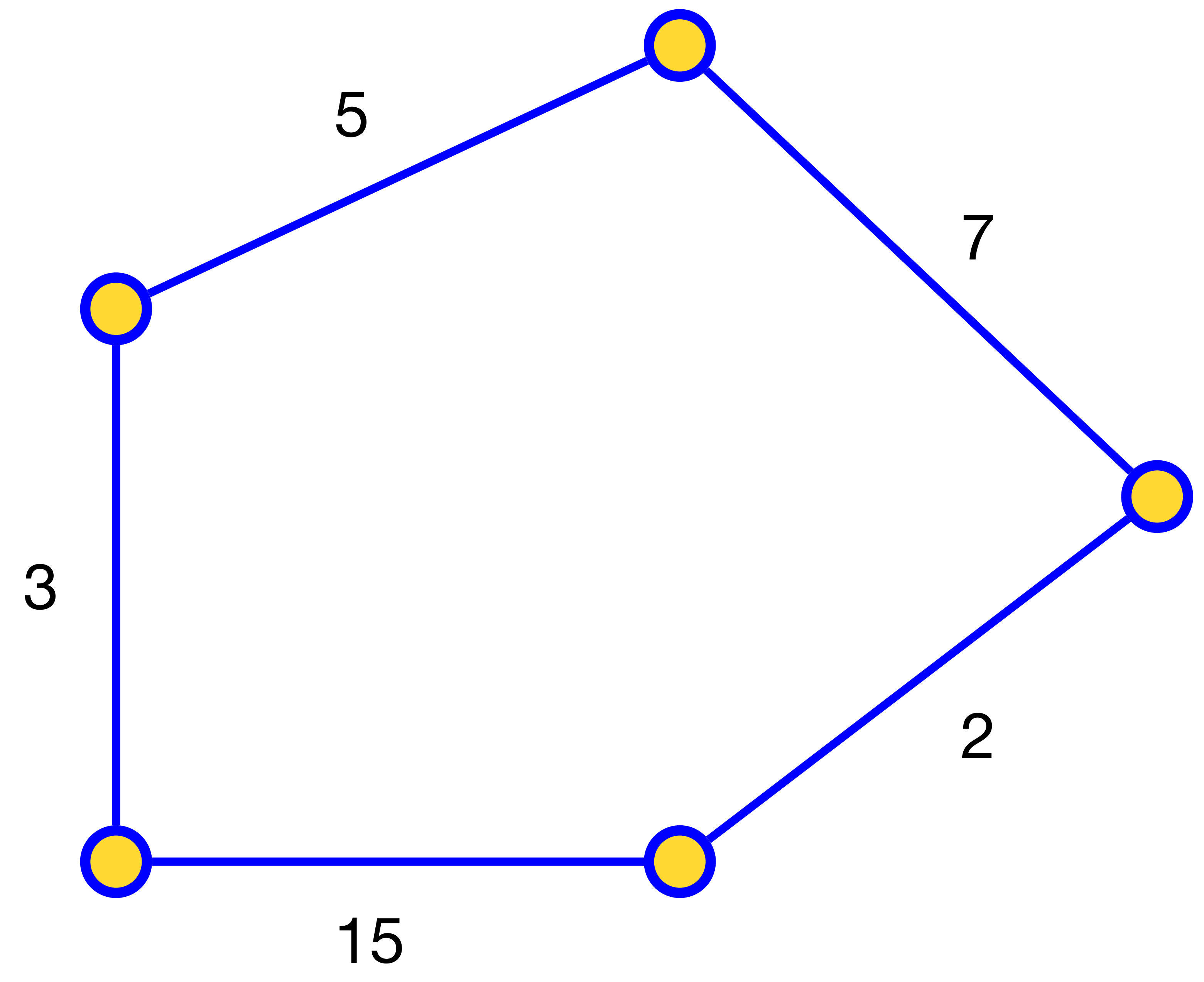
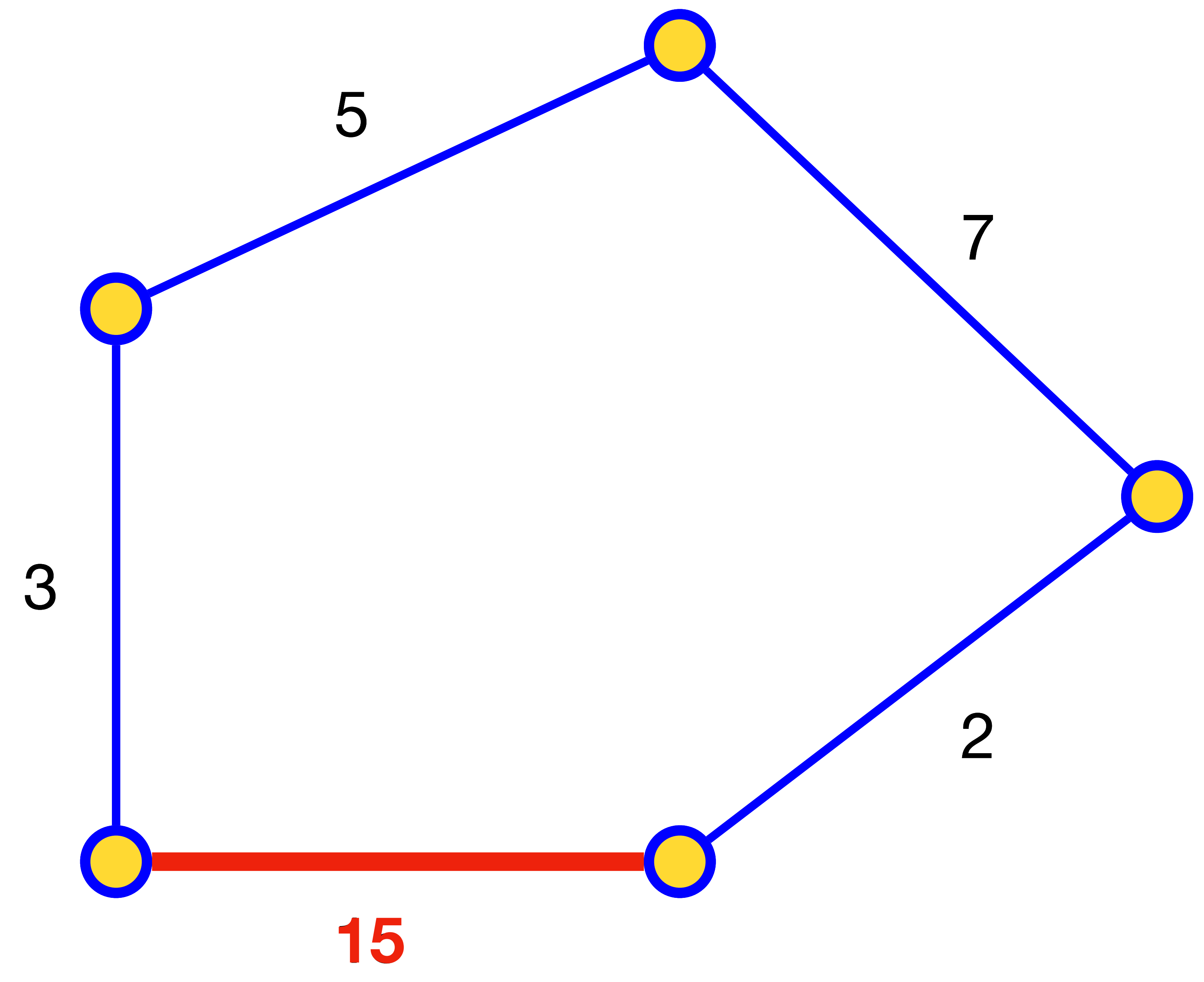


Unsafe edge
Example

- Every cycle identifies one unsafe edge …
- … the most expensive edge in the cycle.



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Proof: Consider any edge $e = uv$. Let
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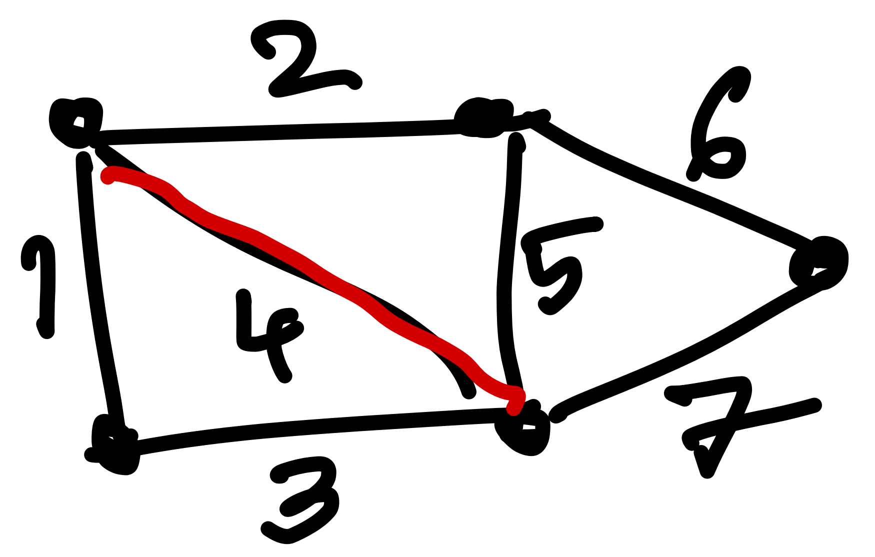

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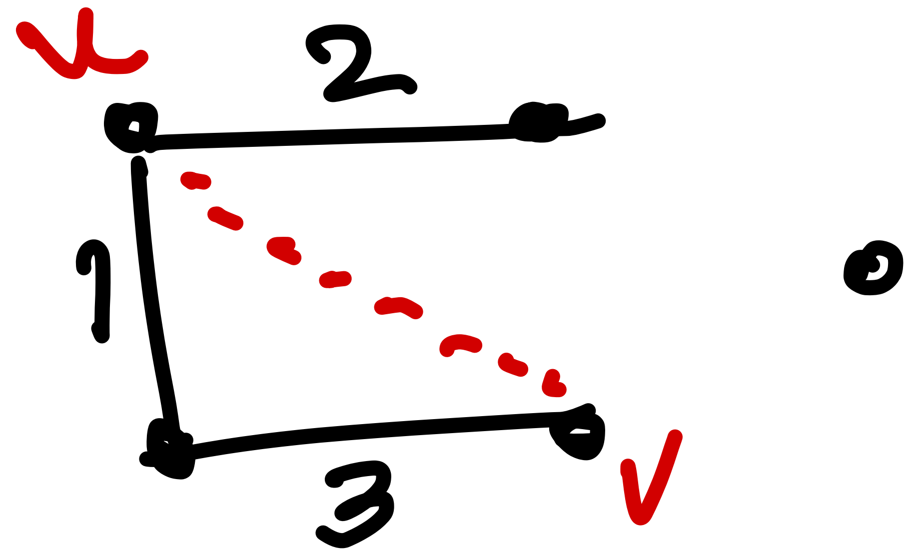

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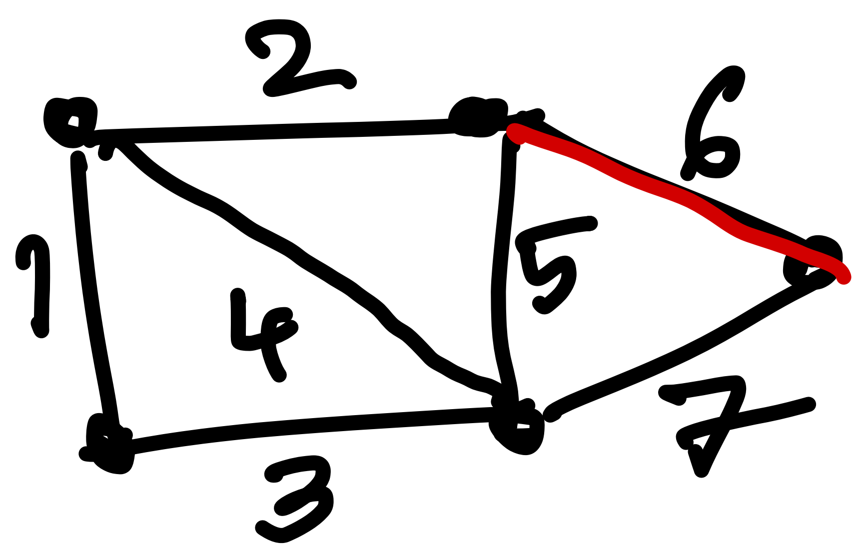

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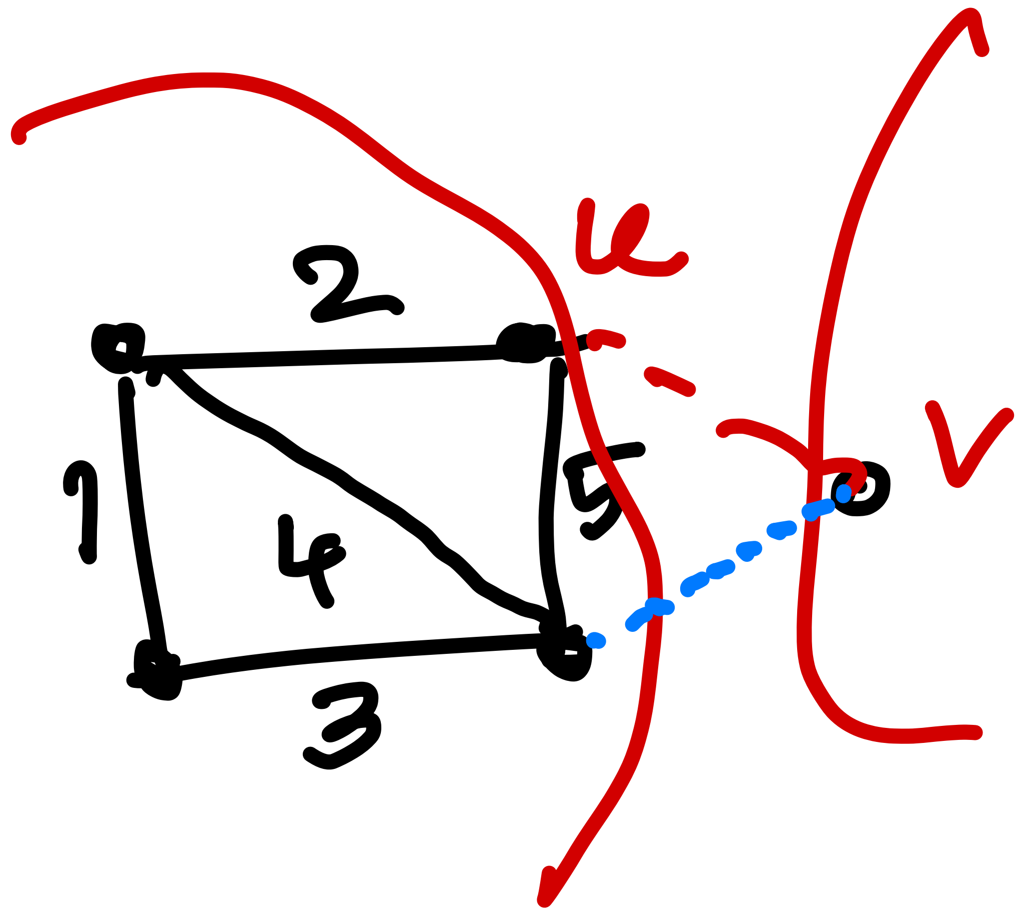

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- If u, v in same connected component of $G_{<w(e)}$, then $G_{<w(e)} + e$ contains a cycle where e is most expensive $\implies$ e is unsafe.
- If u and v are in different connected components of $G_{<w(e)}$, let S be the vertices of connected component of $G_{<w(e)}$ containing x . The edge e is cheapest edge in cut $(S, V \backslash S)$ $\implies$ e is safe.

Example

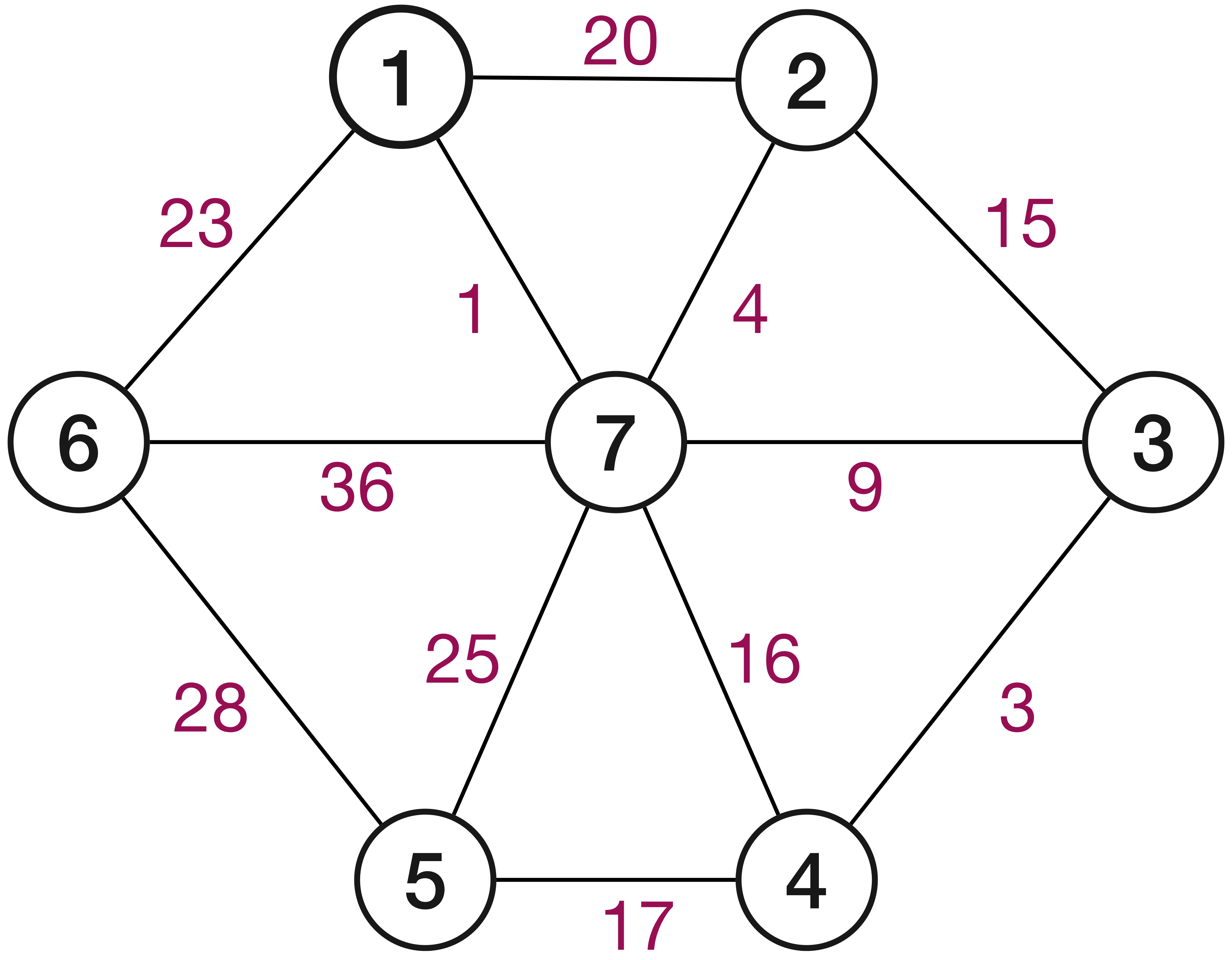
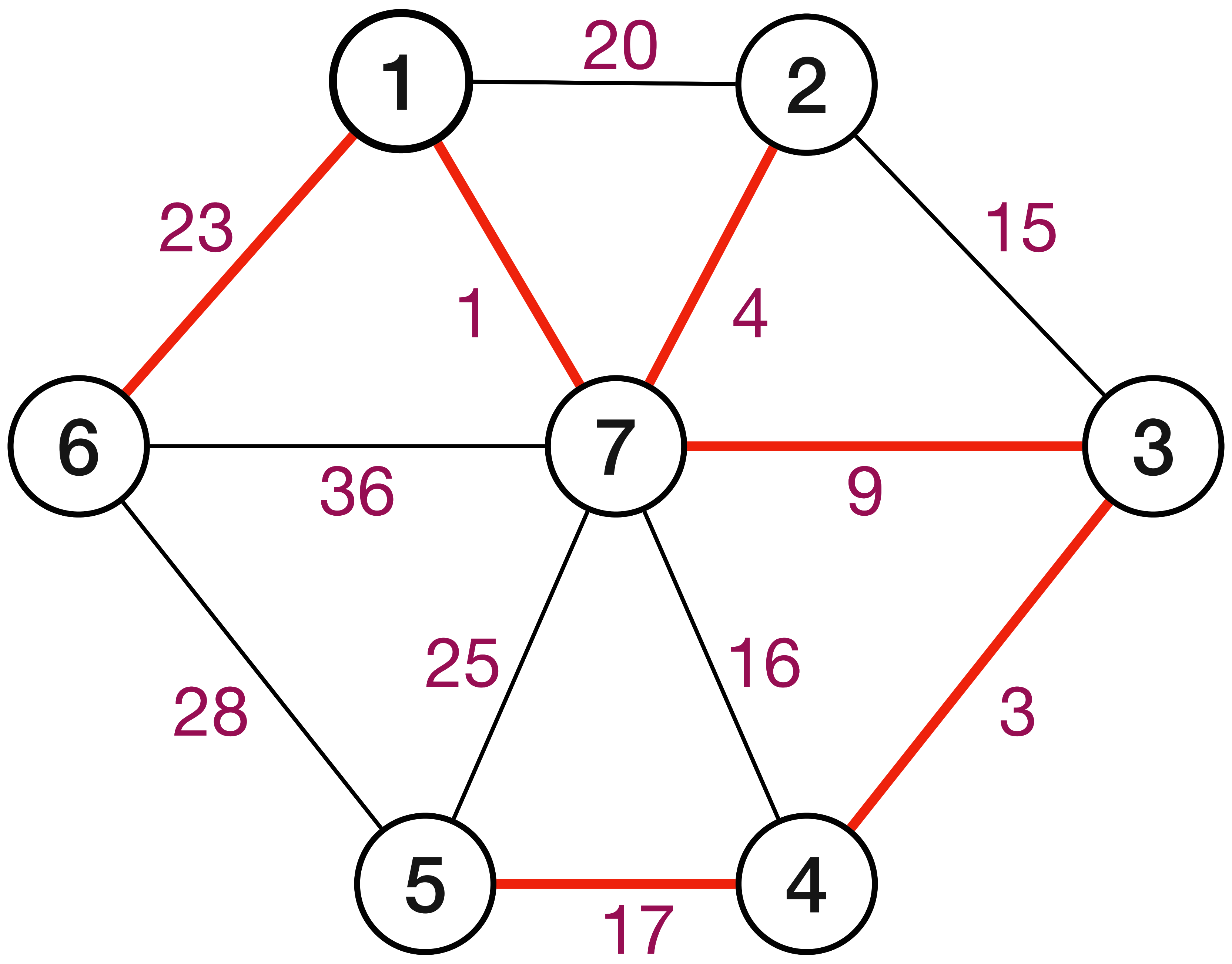


Figure 1: Graph with unique edge costs.

Example



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Safe edges are red, rest are unsafe.

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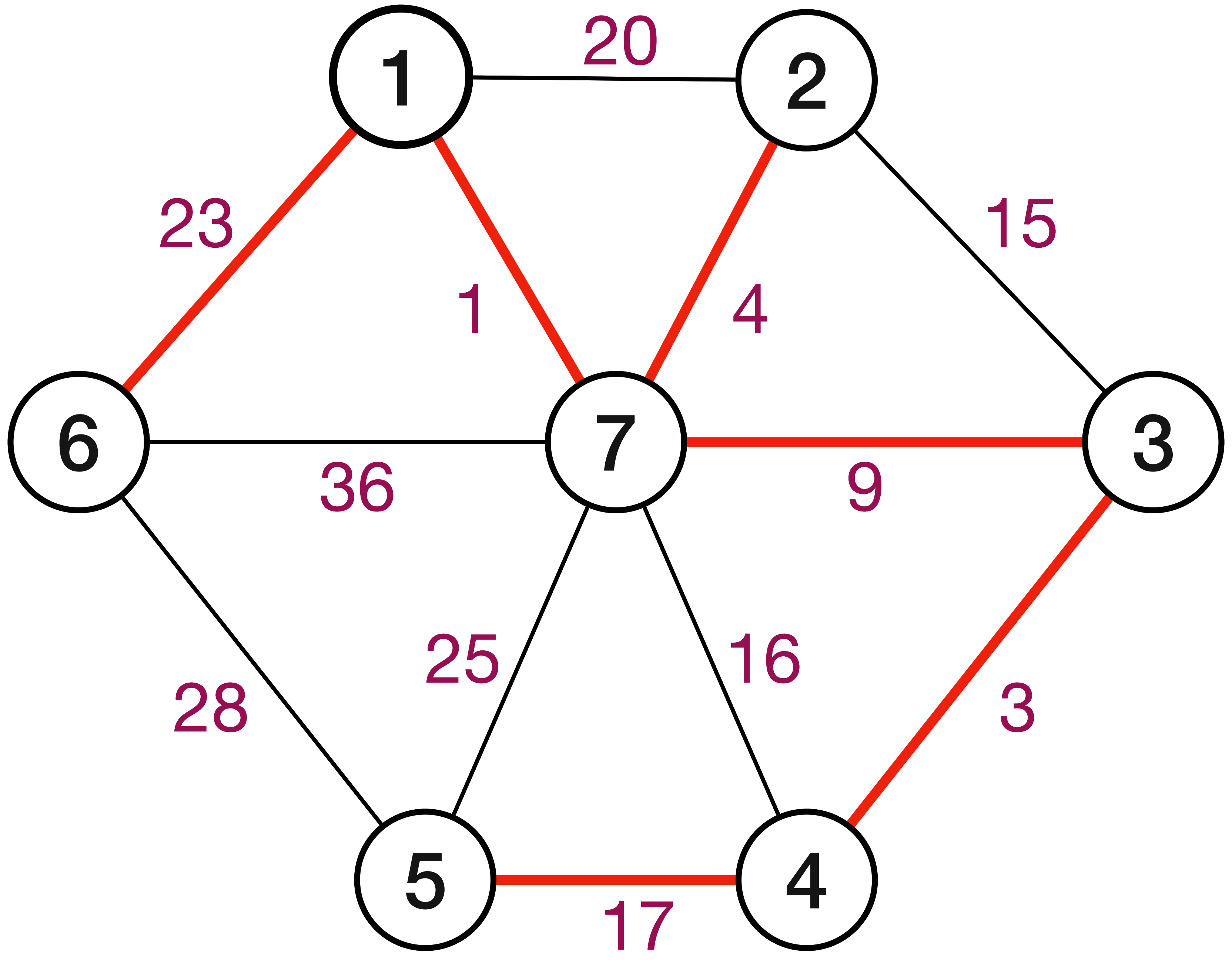

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And all safe edges are in the MST in this case …

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Lemma: **(a)** If e is a safe edge then every minimum spanning tree contains e and **(b)** if e is an unsafe edge then no MST of G contains e.

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- Part two of the lemma is called the ***Cycle Property*****.**

Key observation

Cut property

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Is the proof correct?

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problem: Didn't show T' is a spanning tree.

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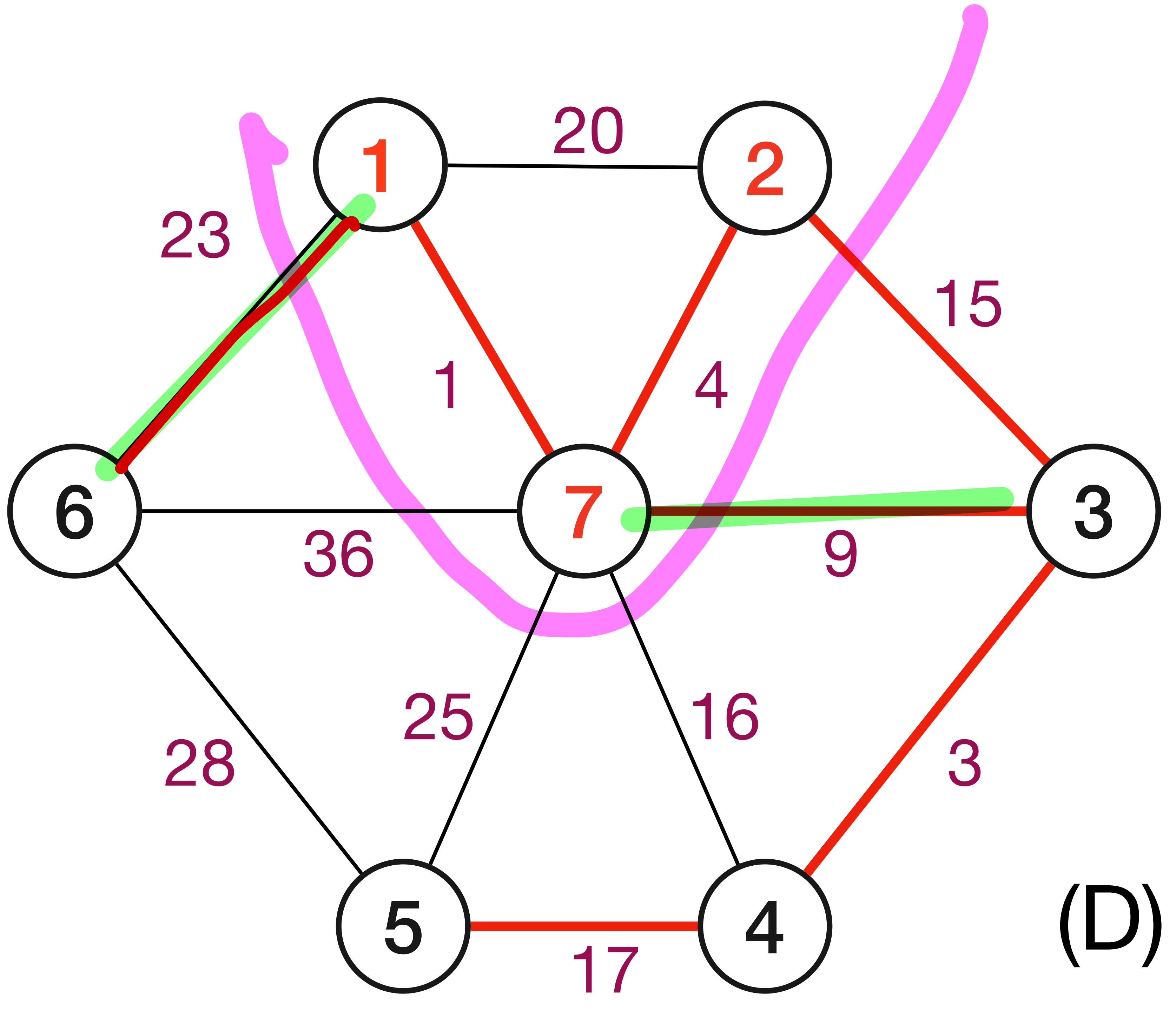

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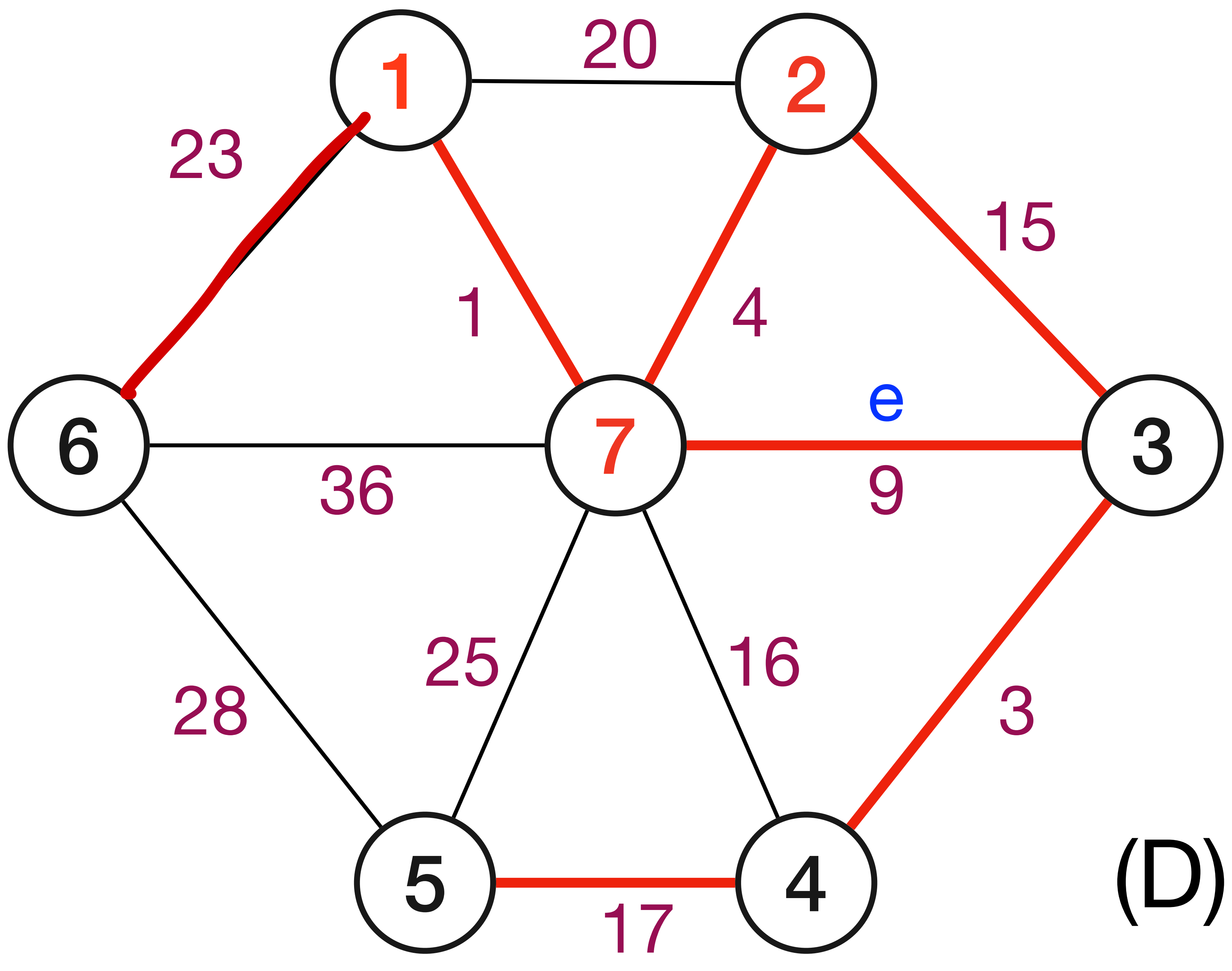

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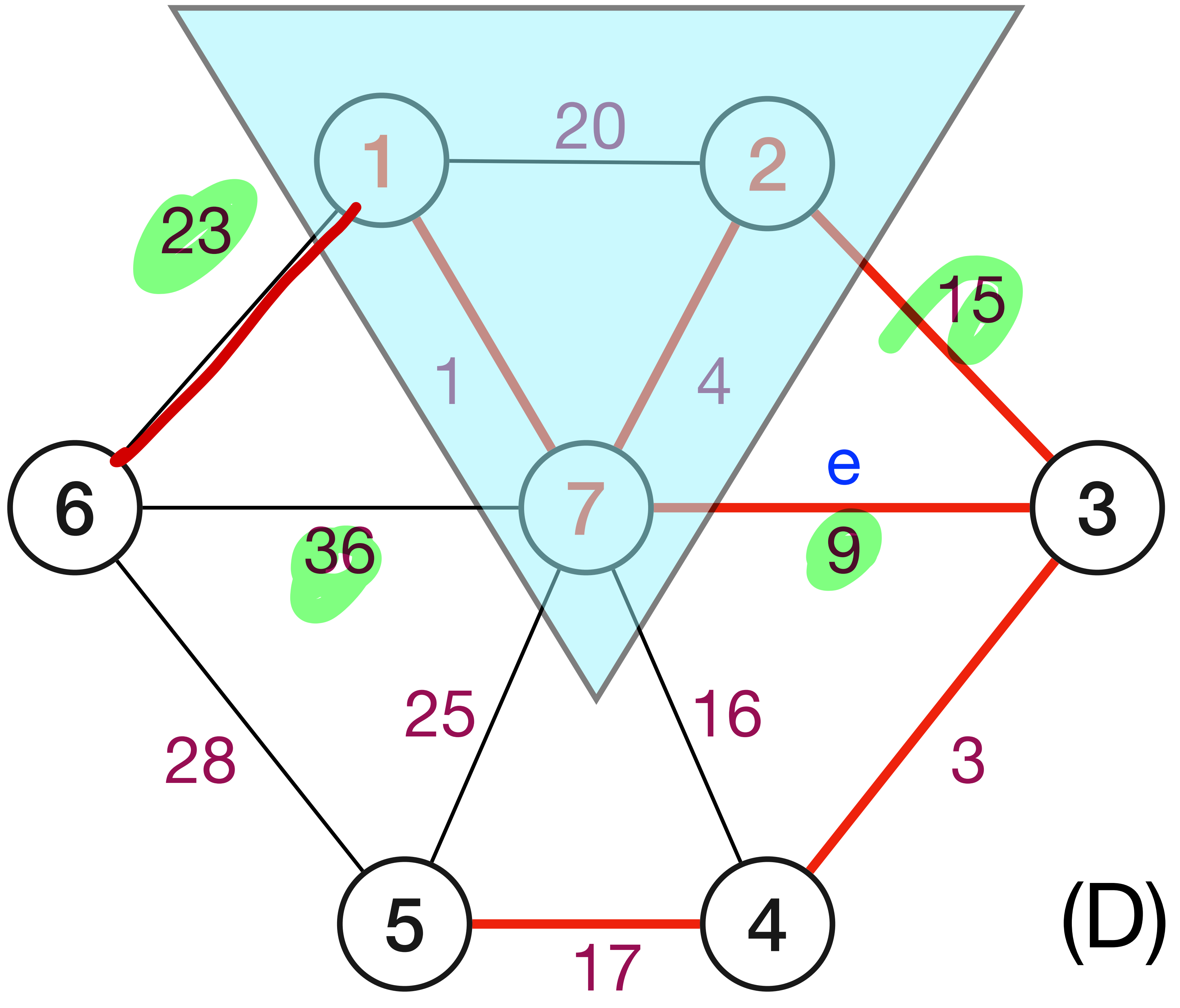

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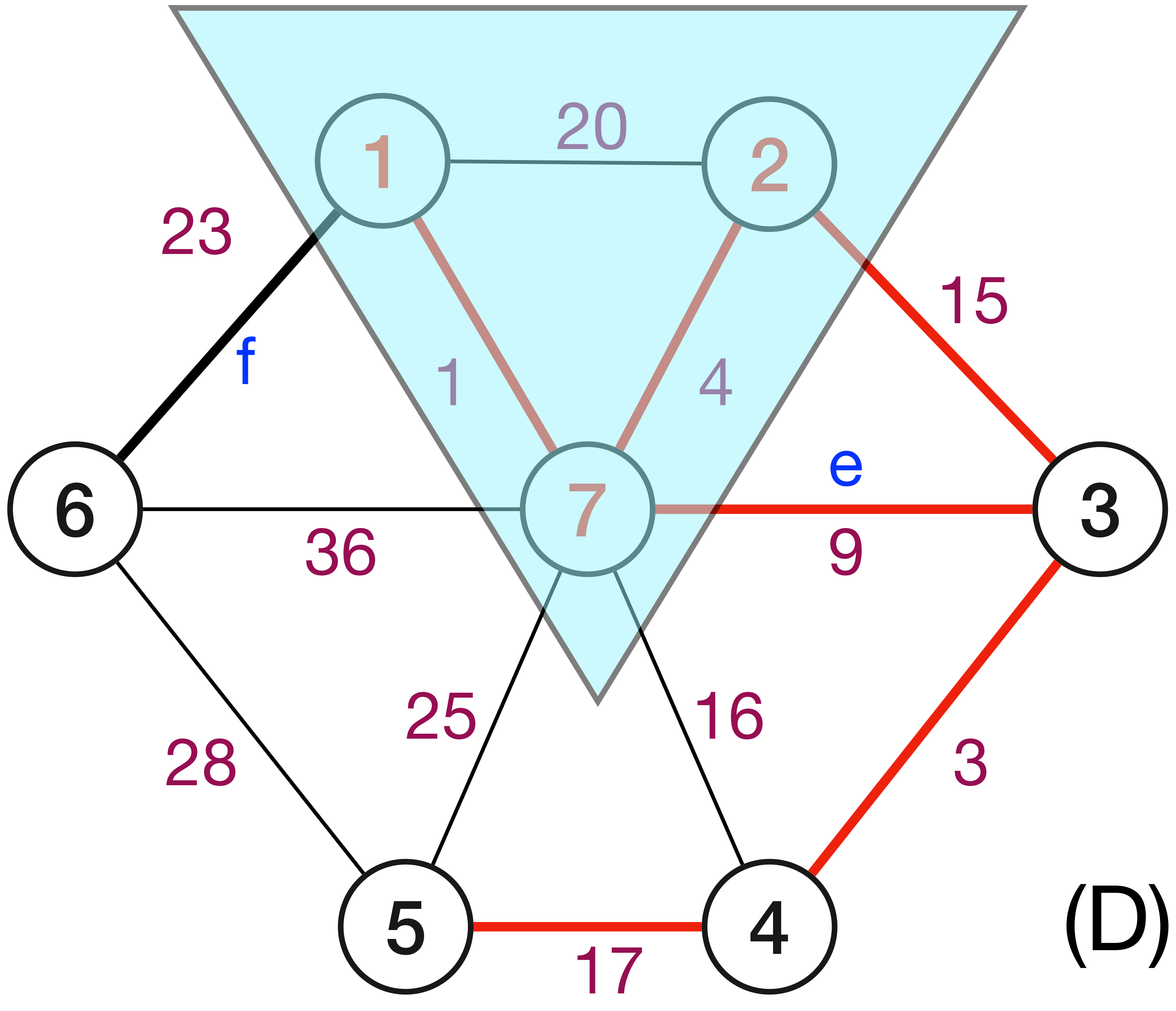

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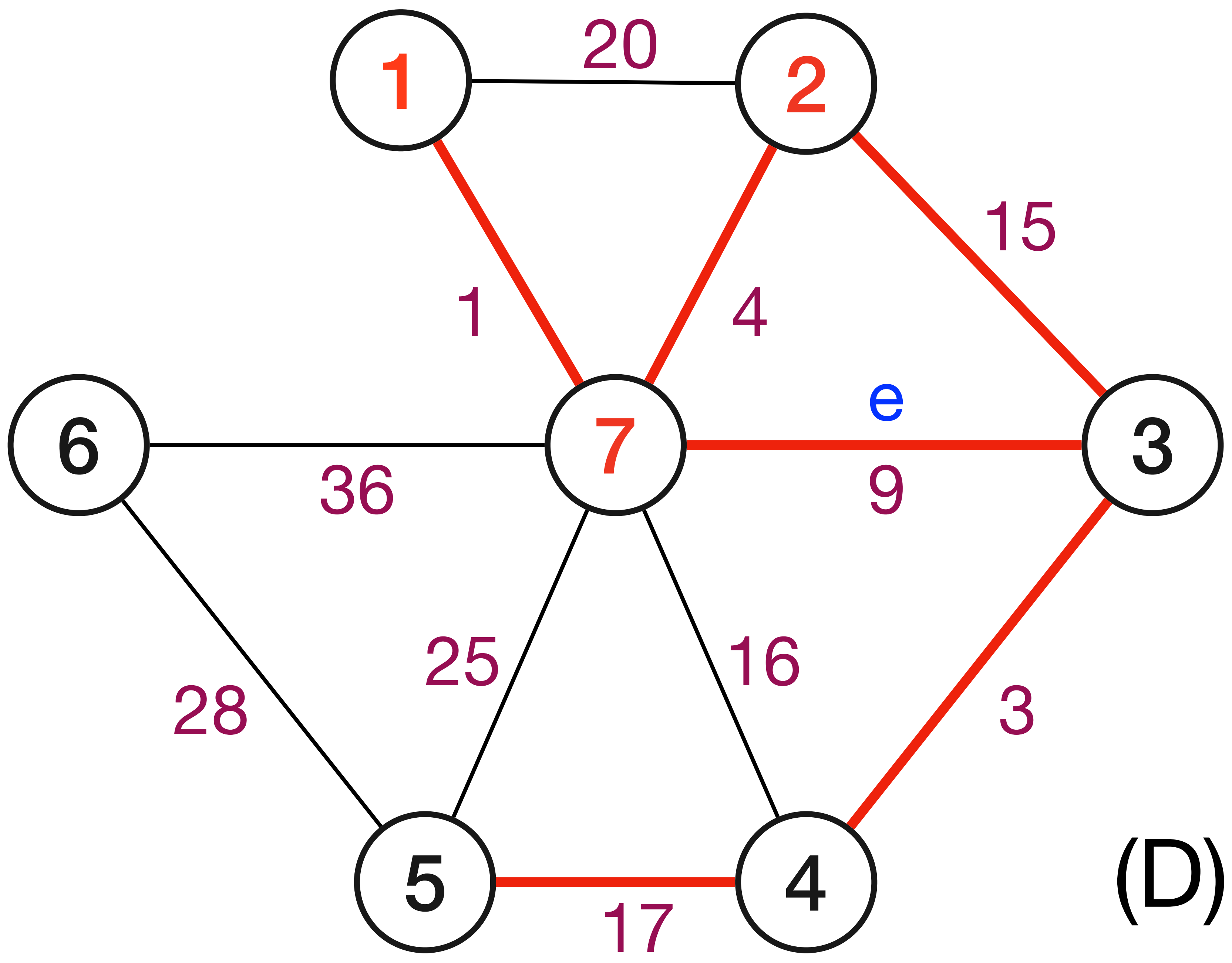

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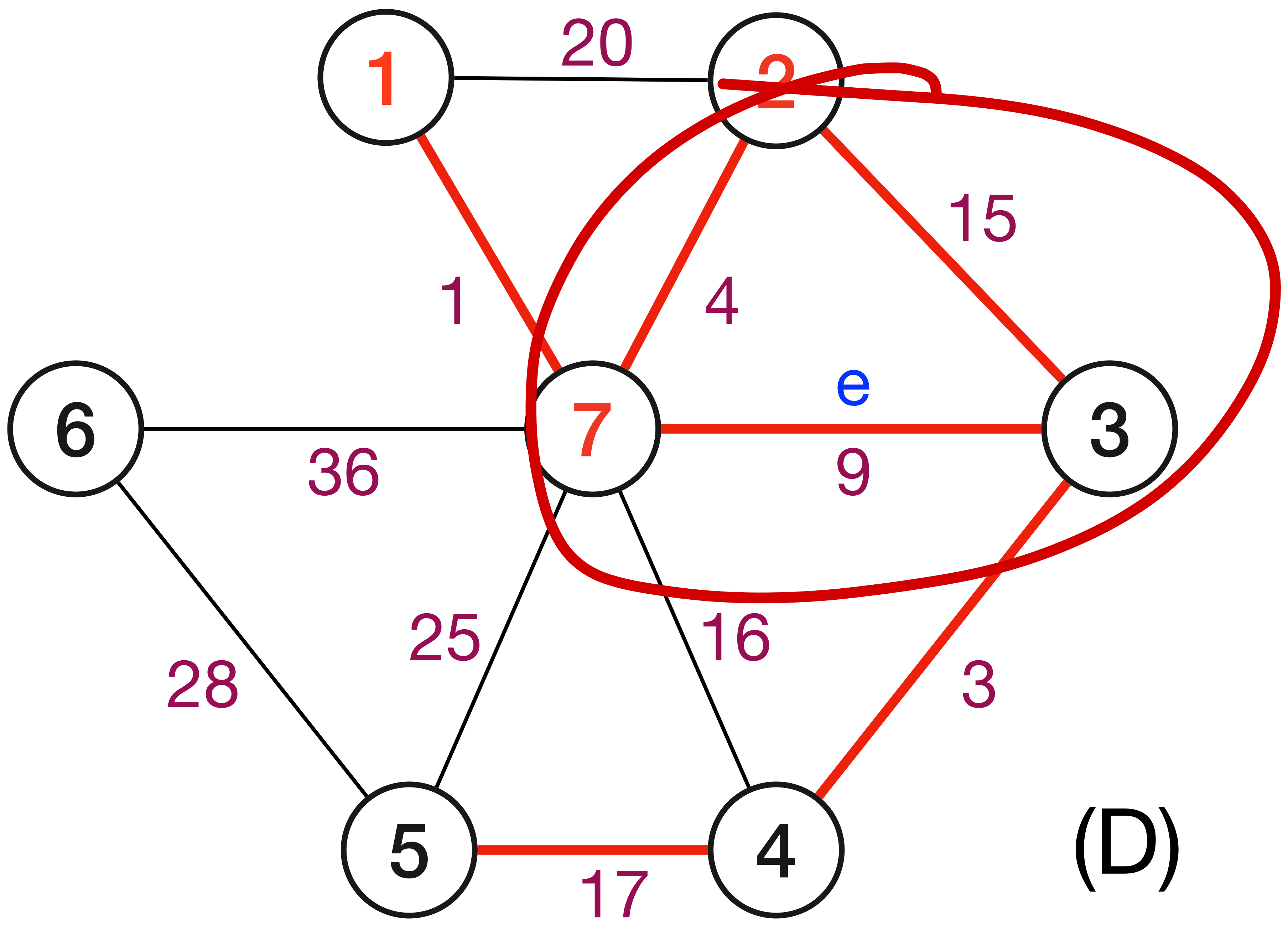

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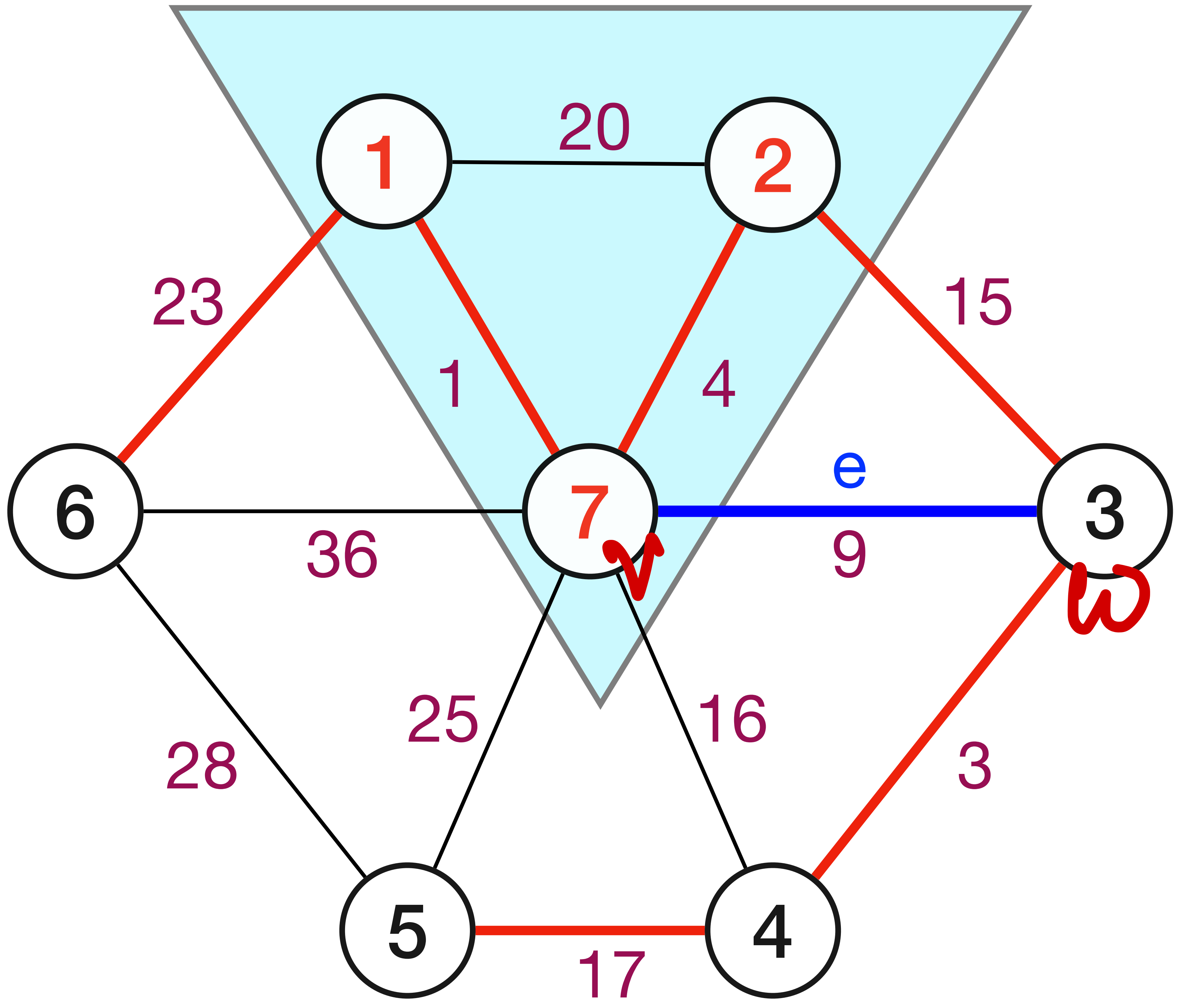
(D) New graph of selected edges is not a tree!

Proof of Cut Property



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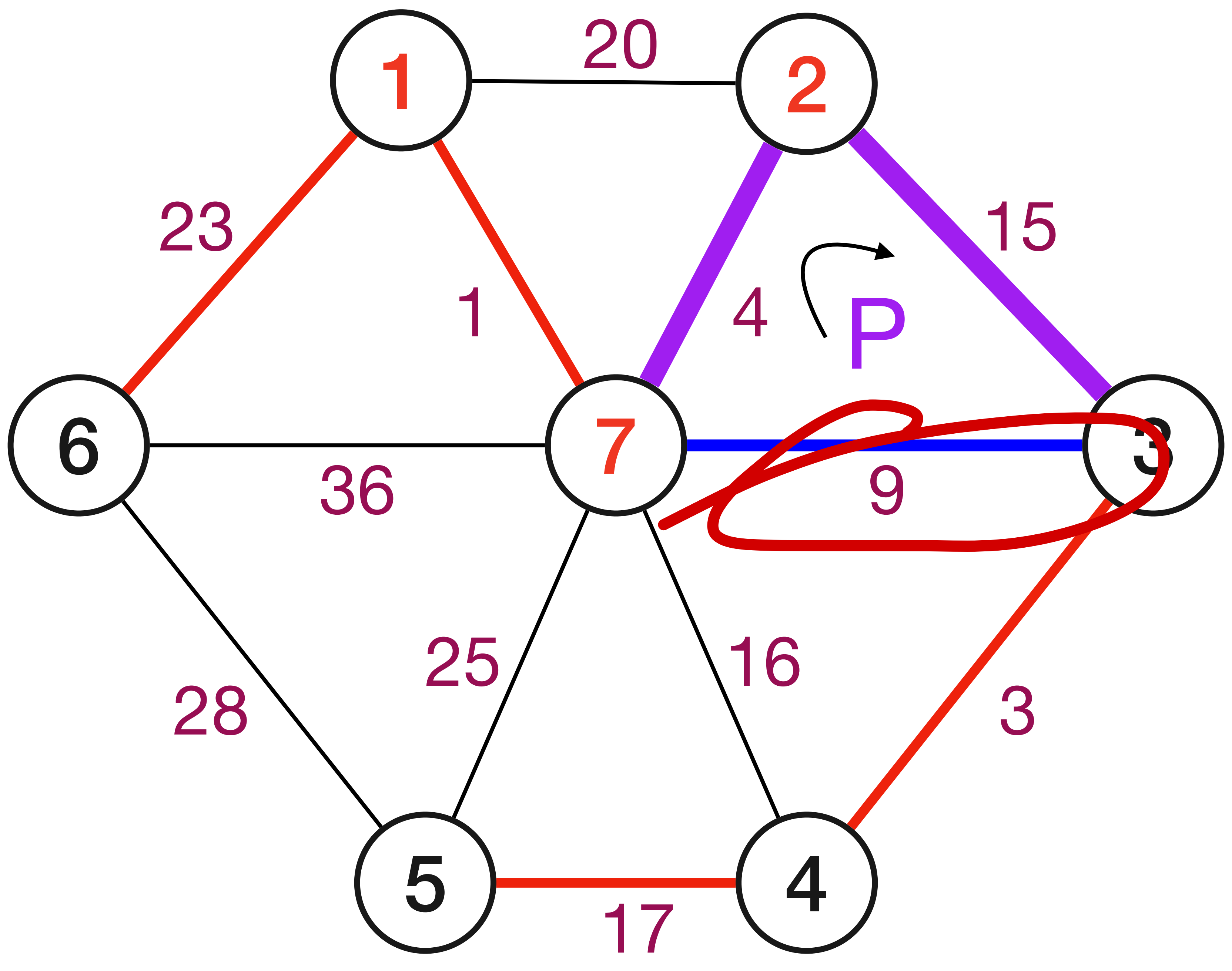

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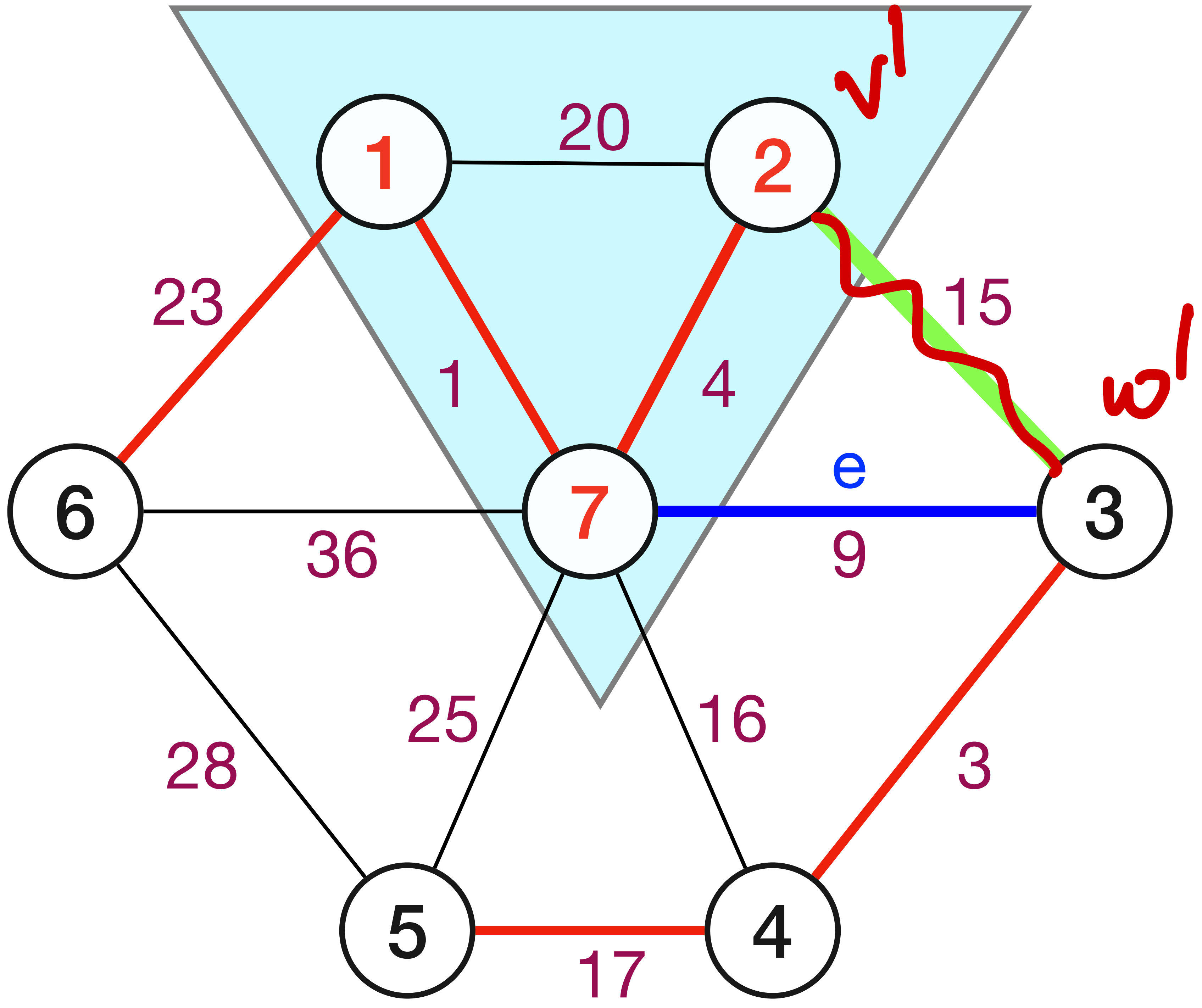

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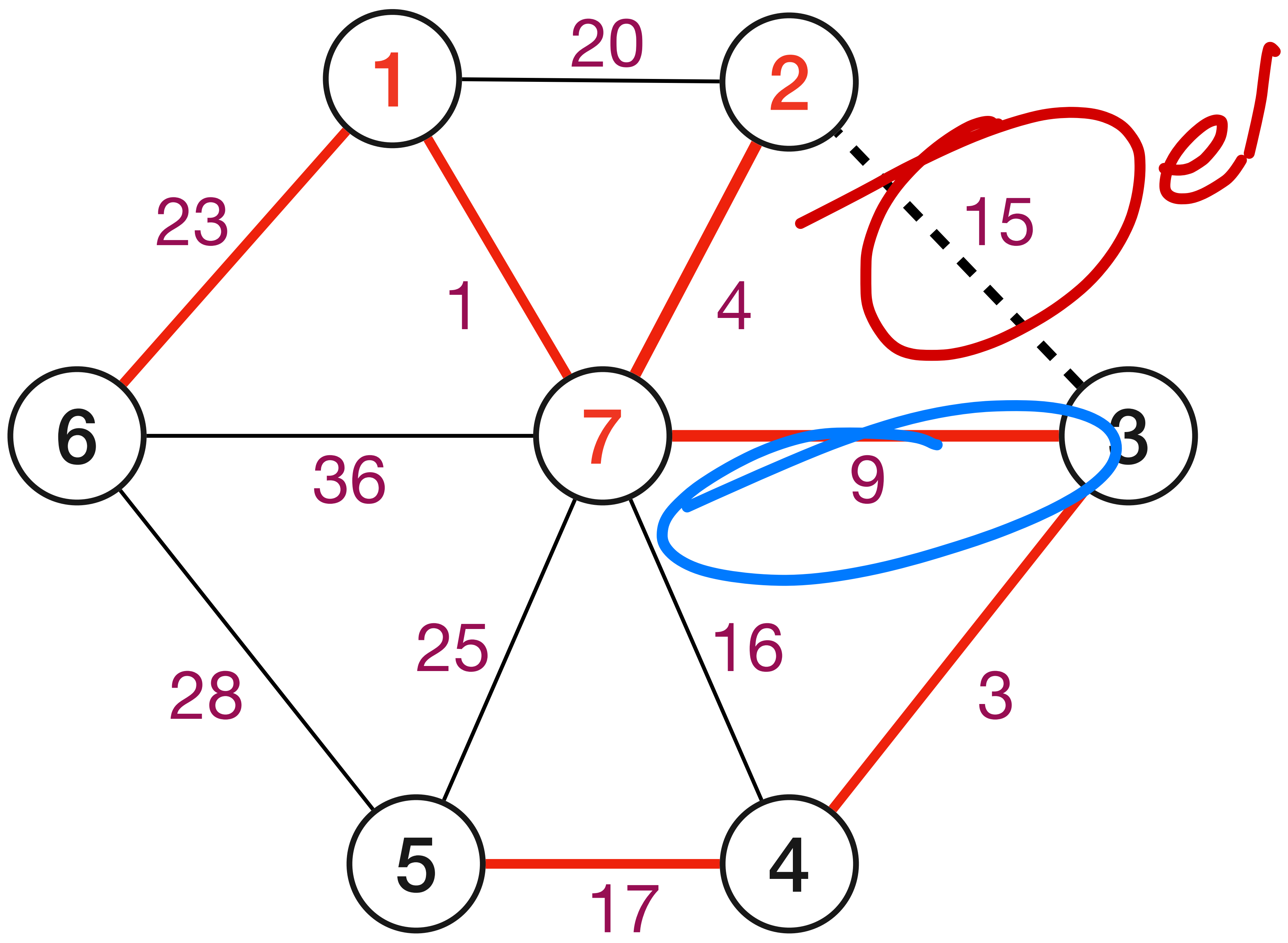
3. Let w' be the first vertex in P belonging to $V \backslash S$; let v' be the vertex just before it on P, and let $e' = (v', w')$

Proof of Cut Property



1. Suppose $e = (v, w)$ is not in MST T and e is min weight edge in cut $(S, V \backslash S)$. Assume $v \in S$. It is safe because it is the cheapest edge in the cut.

2. T is spanning tree: there is a unique path P from v to w in T.

3. Let w' be the first vertex in P belonging to $V \backslash S$; let v' be the vertex just before it on P, and let $e' = (v', w')$

4. $T' = (T \backslash \{e'\}) \cup \{e\}$ is spanning tree of lower cost. (Why?)

↳ Tree not minimum to begin w/ so contradiction

Proof of Cut Property

(contd)

Observation: $T' = (T \backslash \{e'\}) \cup \{e\}$ is a spanning tree.

Proof of Cut Property
(contd)

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Proof of Cut Property

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Removed $e' = (v', w')$ from T but v' and w' are connected by the path $P - f + e$ in T' . Hence T' is connected if T is.

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Proof: T' is a tree

T' is connected and has $n - 1$ edges (since T had $n - 1$ edges) and hence T' is a tree.

Safe edges form a connected graph

Lemma: Let G be a connected graph with distinct edge costs, then the set of safe edges form a connected graph.

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Proof:

- Suppose not. Let S be a connected component in the graph induced by the safe edges.

- Consider the edges crossing S, there must be a safe edge among them since edge costs are distinct and so we must have picked it.

Safe edges, cycles and MST

Lemma: Let G be a connected graph with distinct edge costs, then the set of safe edges does not contain a cycle.

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Corollary: Let G be a connected graph with distinct edge costs, then set of safe edges form the unique MST of G.

Safe edges, cycles and MST

Lemma: Let G be a connected graph with distinct edge costs, then the set of safe edges does not contain a cycle.

Corollary: Let G be a connected graph with distinct edge costs, then set of safe edges form the unique MST of G.

Consequence: Every correct MST algorithm when G has unique edge costs includes exactly the safe edges.

Borůvka's Algorithm

Simplest to implement. Assume G is a connected graph.

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```
T is ∅ (* T will store edges of a MST *)

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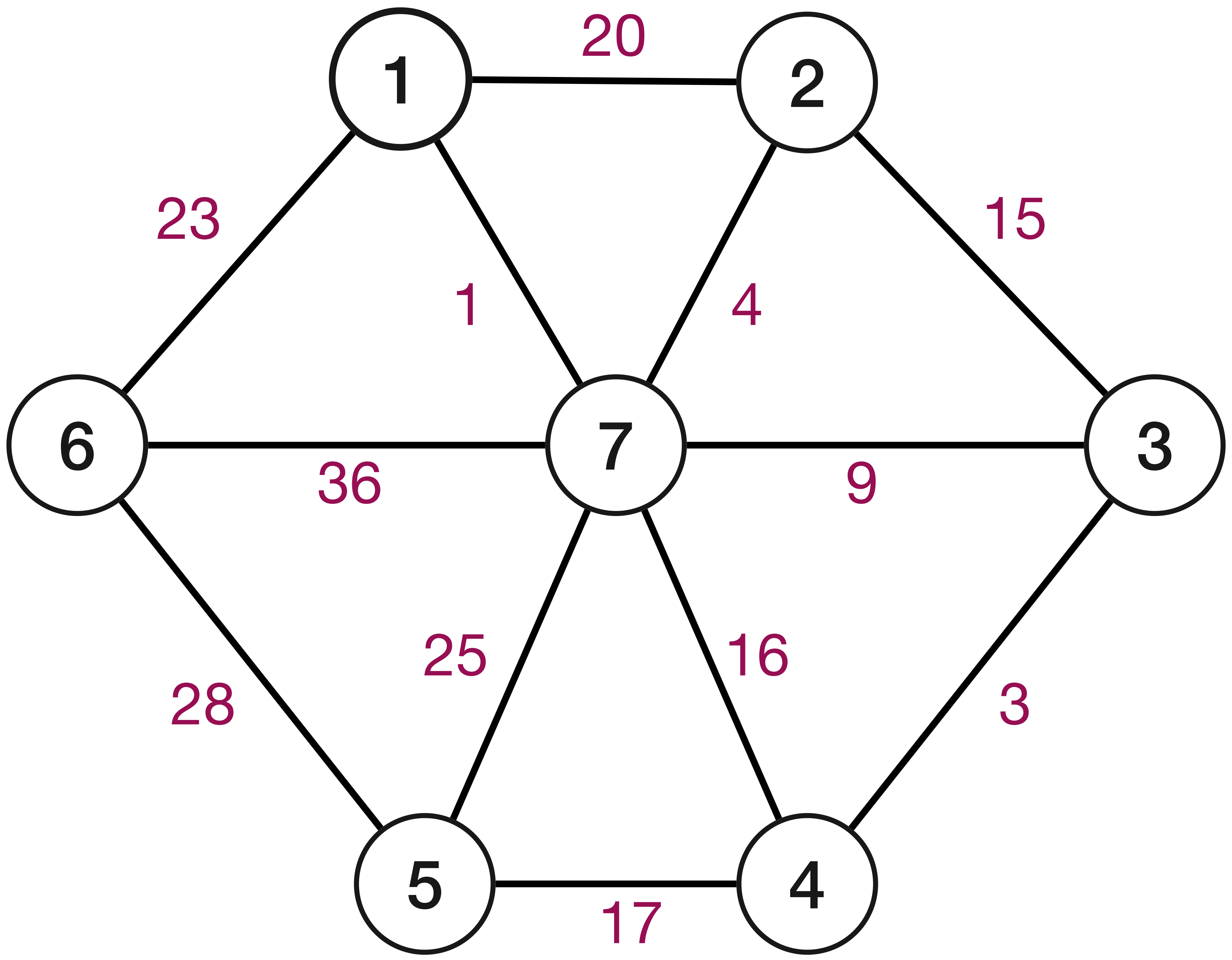
Borůvka's Algorithm



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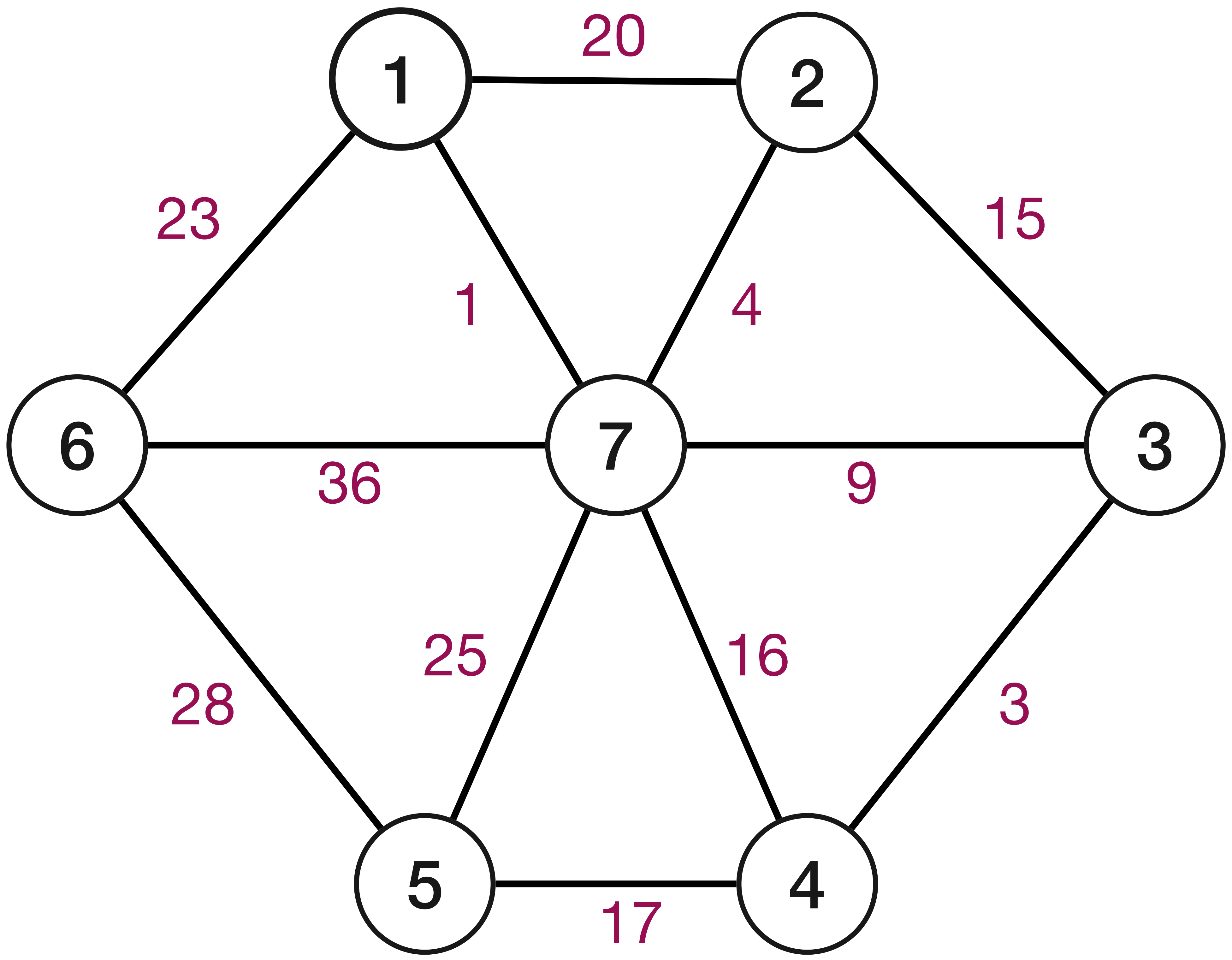
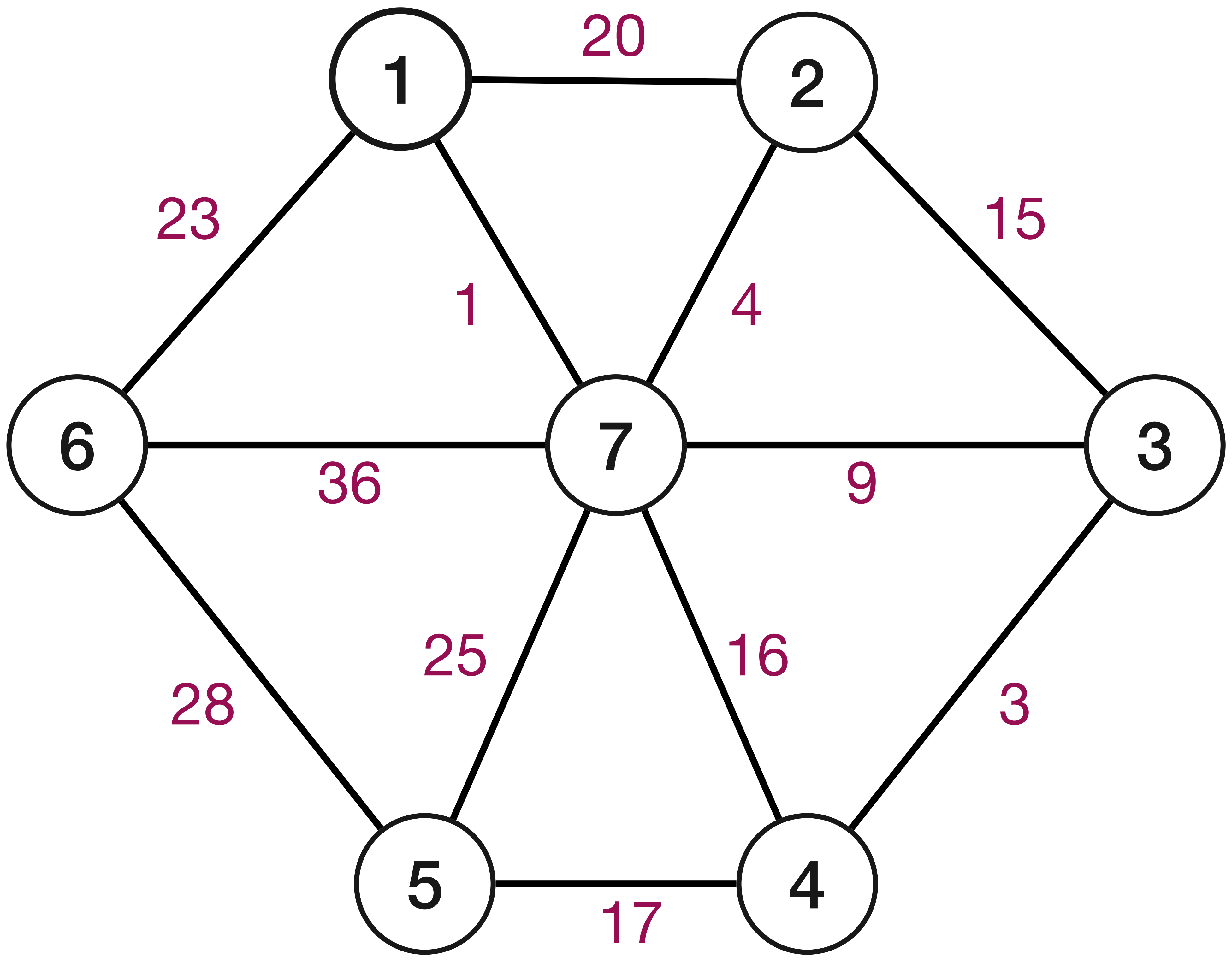


Initialize: All vertices are singleton connected components.

Borůvka's Algorithm



Initialize: All vertices are singleton connected components.

Heuristic: Each vertex tries to expand its "network" (connected component) by gaining the "least expensive friend."

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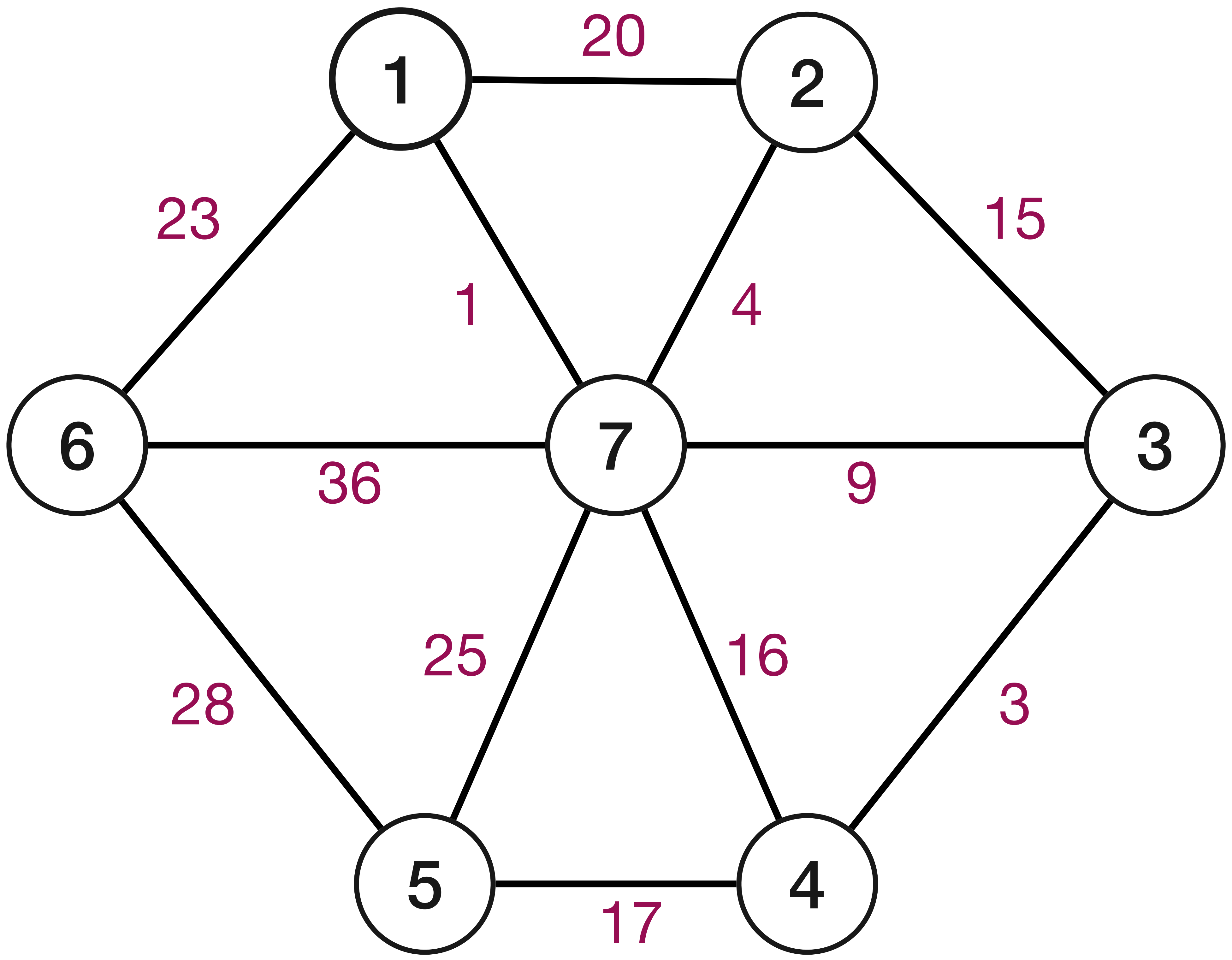

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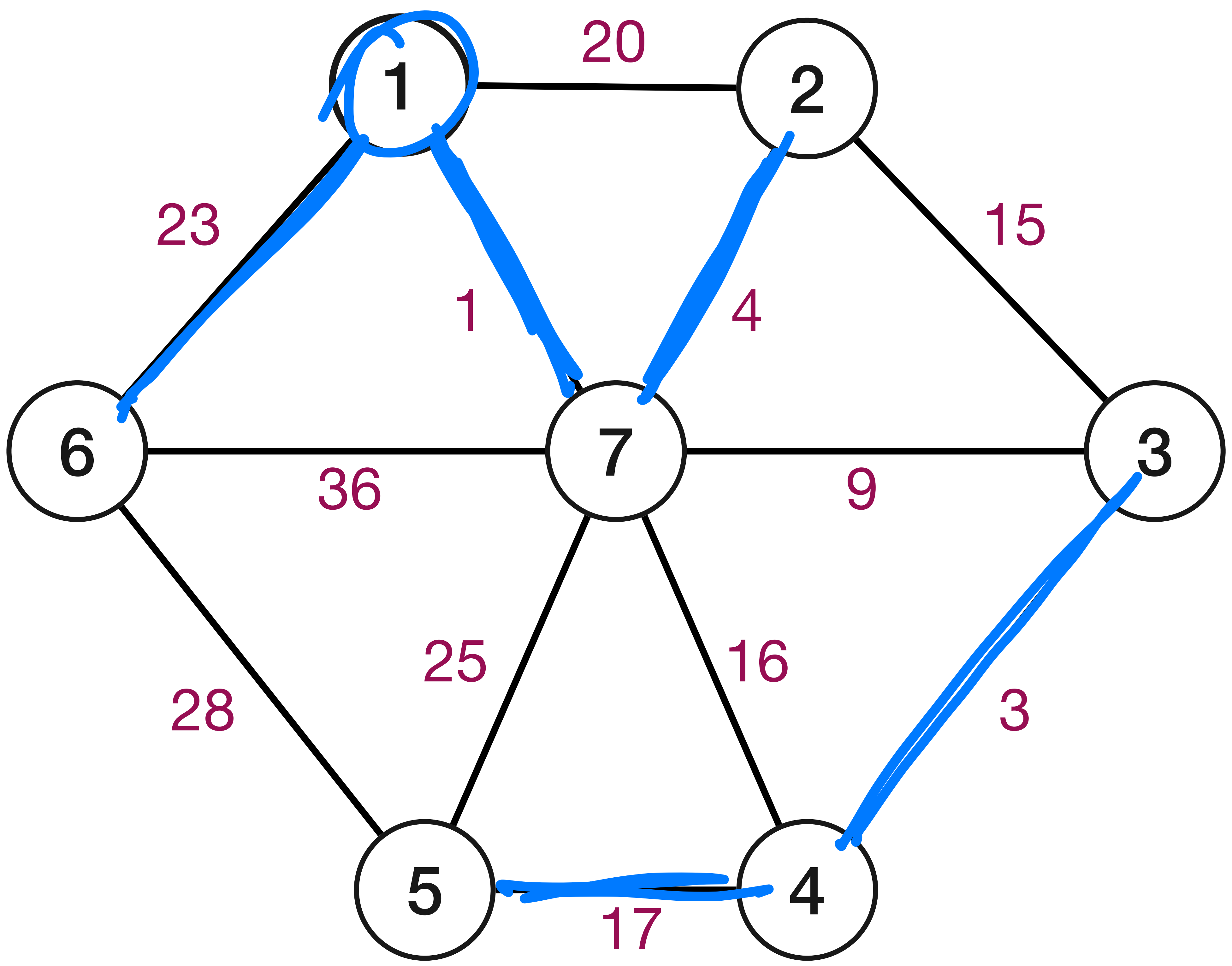

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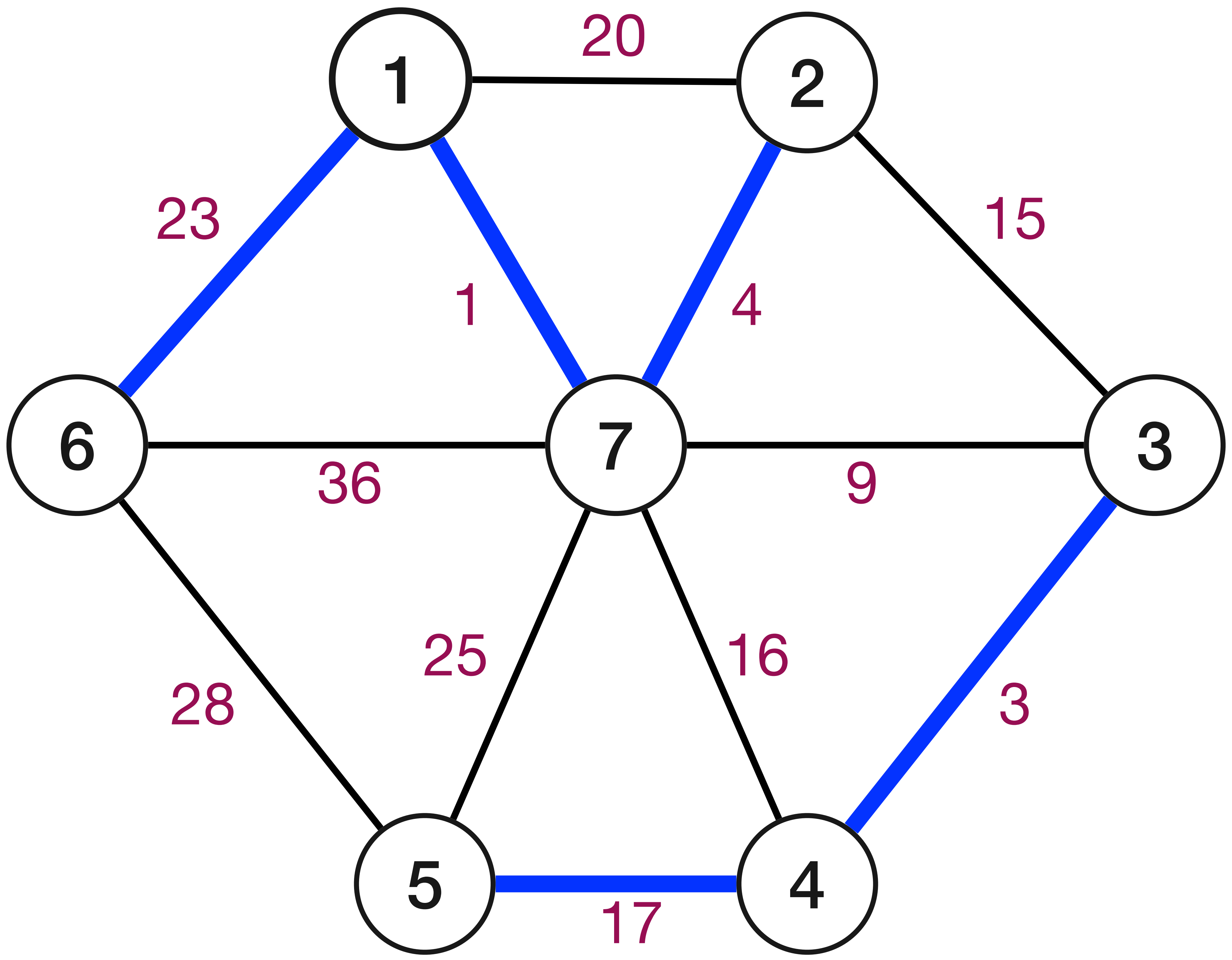

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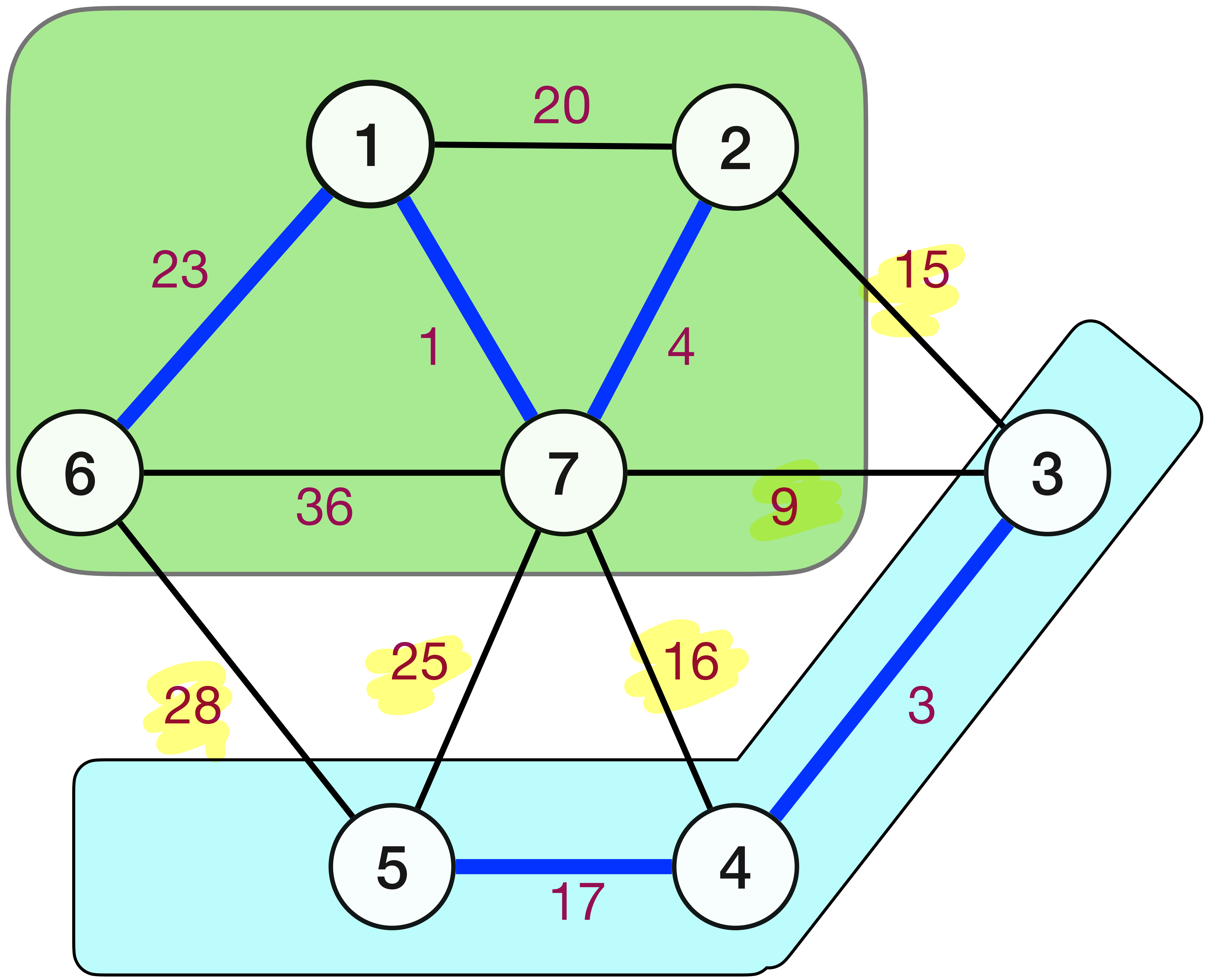

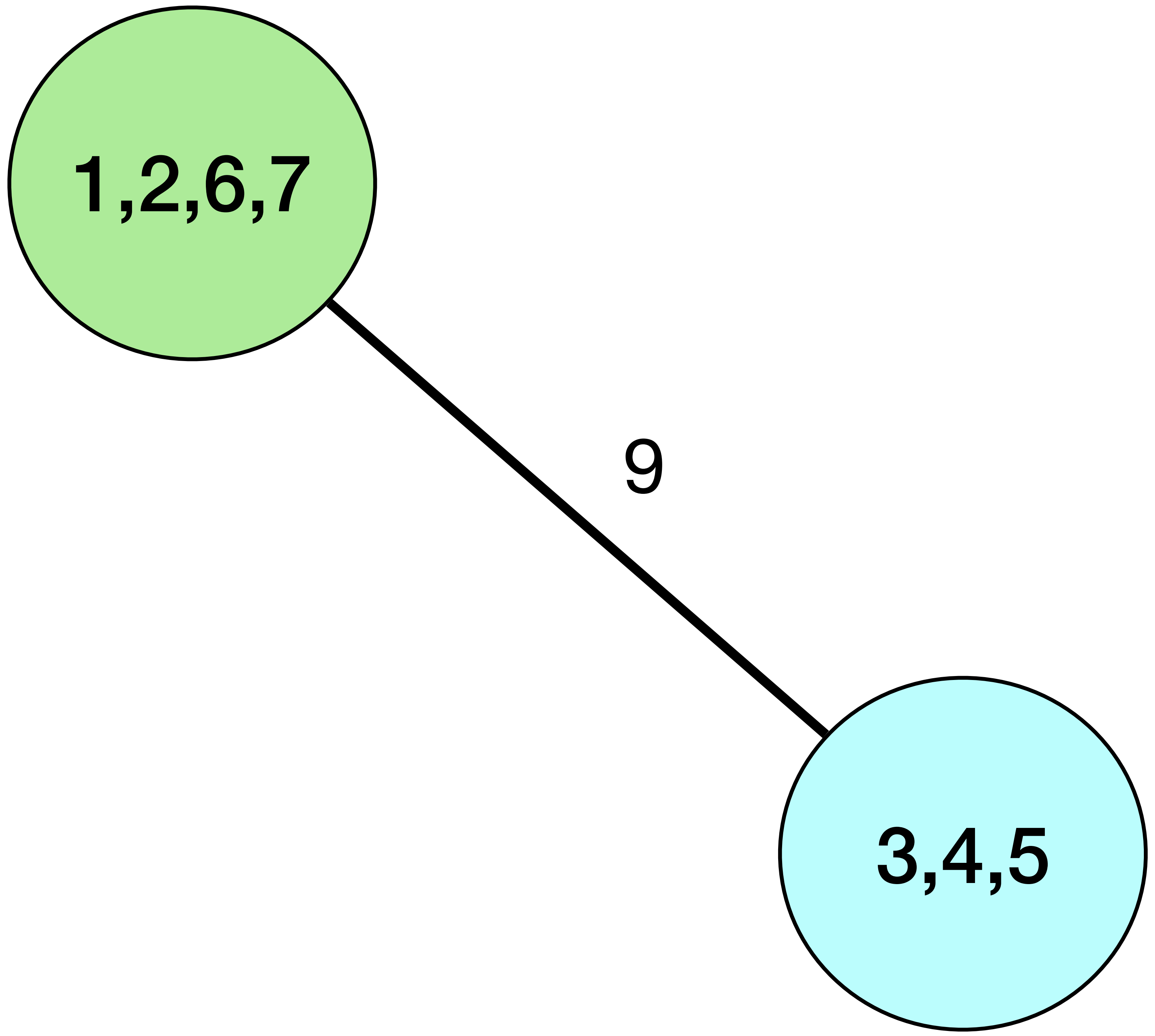

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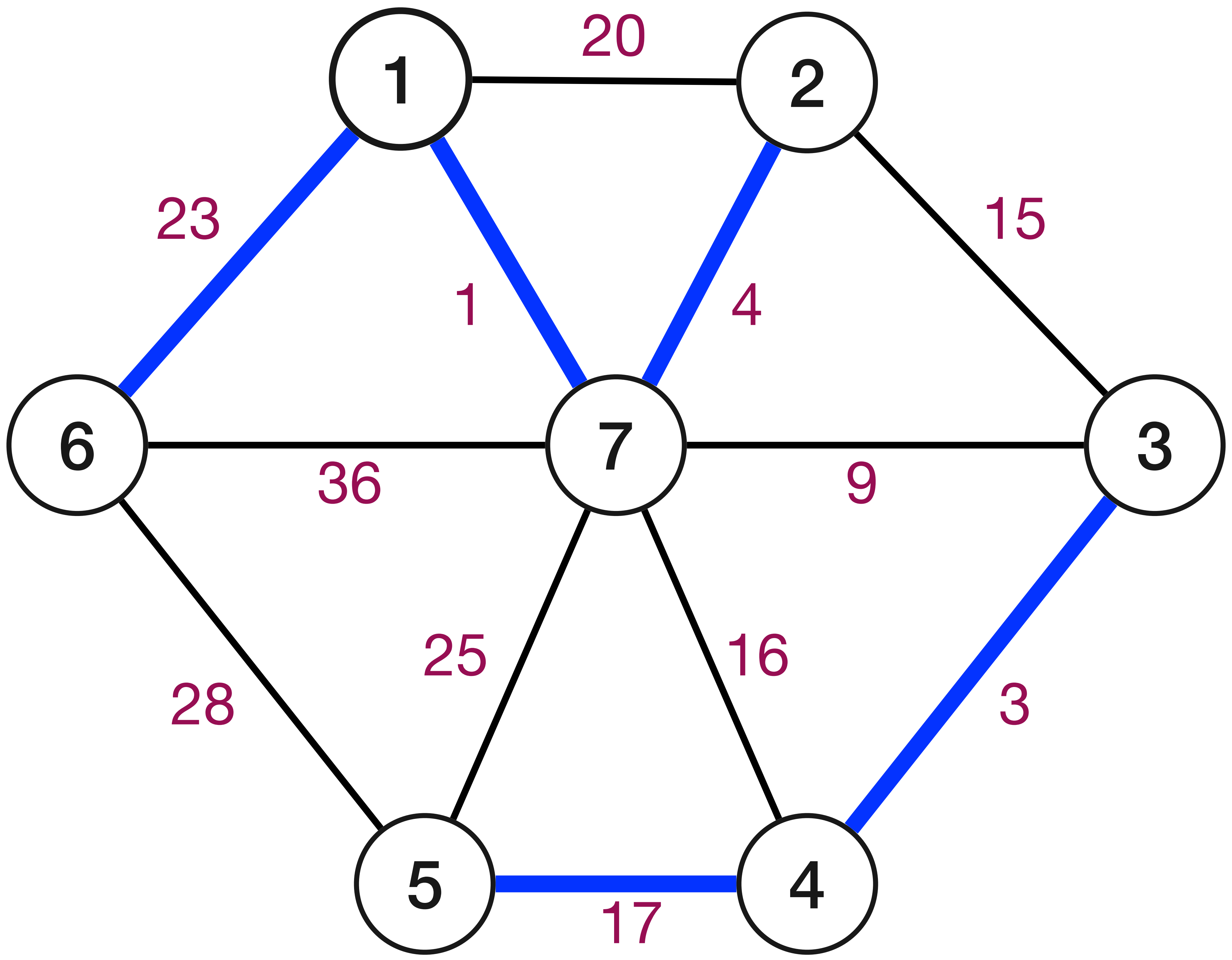

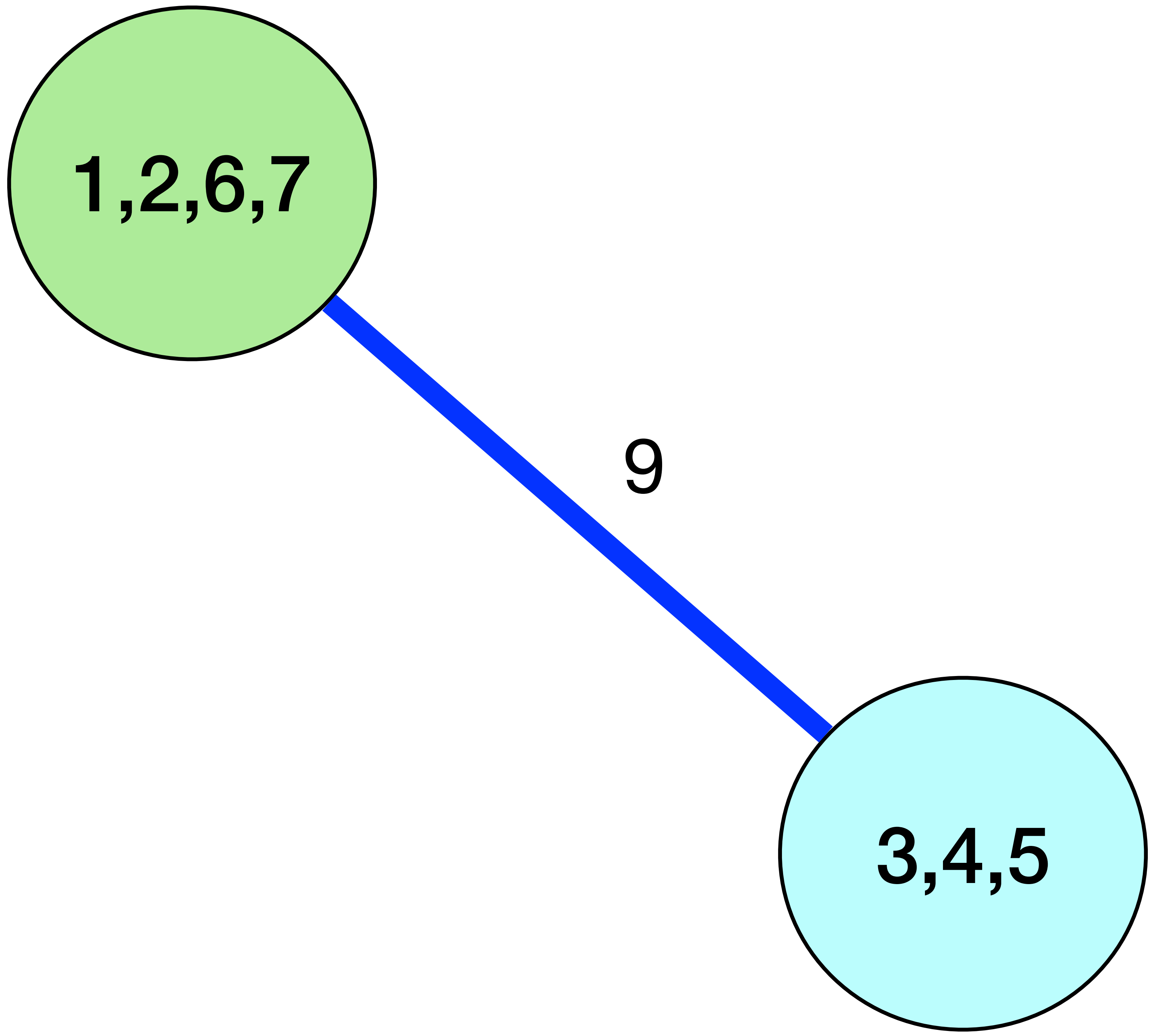

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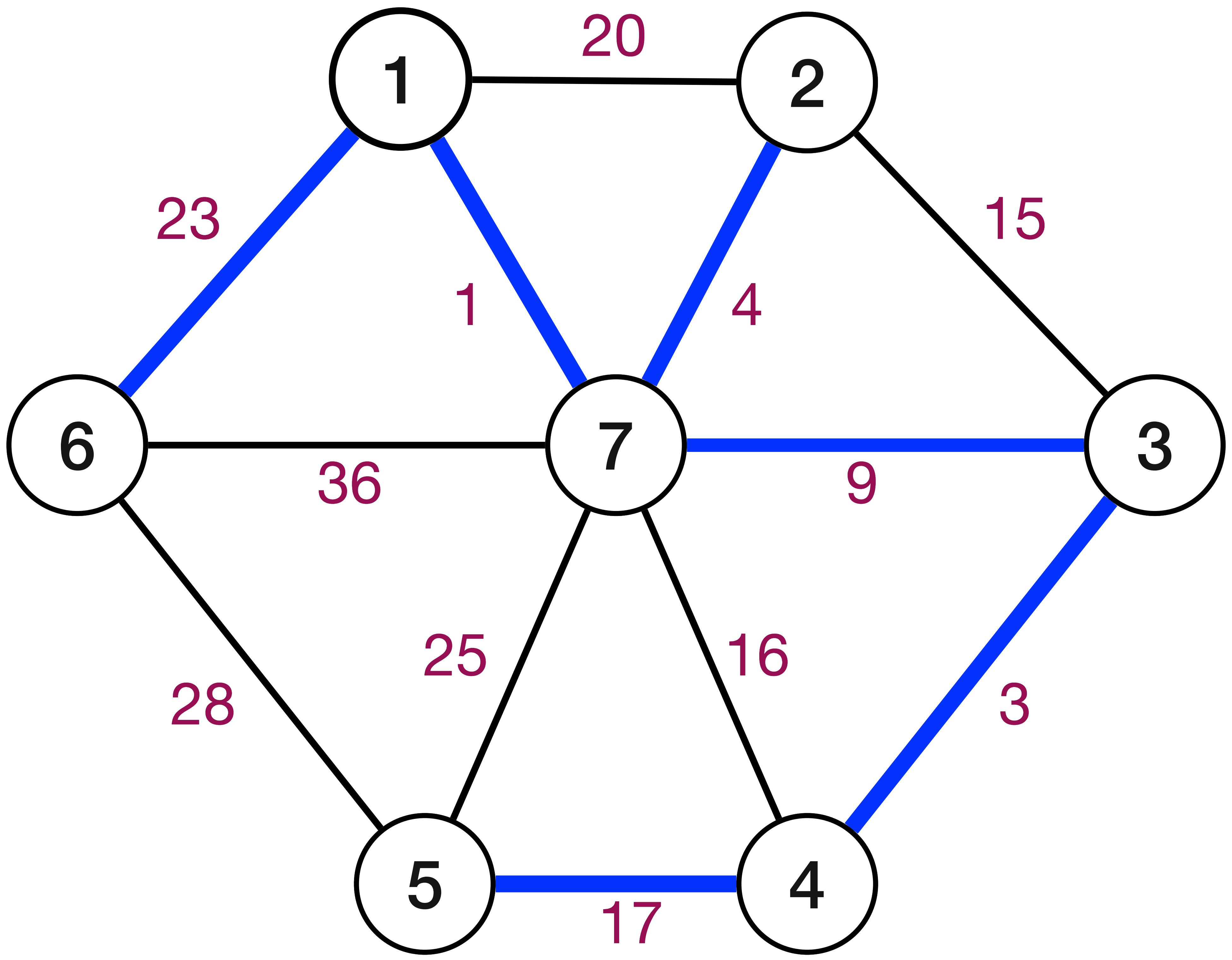


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Implementing Borůvka's Algorithm

- $O(\log n)$ iterations of while loop.
  Why?

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Mininimum Spanning Trees

Greedy template

```
Initially E is the set of all edges in G

T is empty (*T will store edges of a MST*)

while E is not empty do
  choose e ∈ E
  if (e satisfies condition)
    add e to T
                      → remove e from E
return the set T
```

Mininimum Spanning Trees

Greedy template

- In what order should the edges be processed?

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Mininimum Spanning Trees

Greedy template

- In what order should the edges be processed?
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- Leads to Kruskal’s and Prim’s algorithms.

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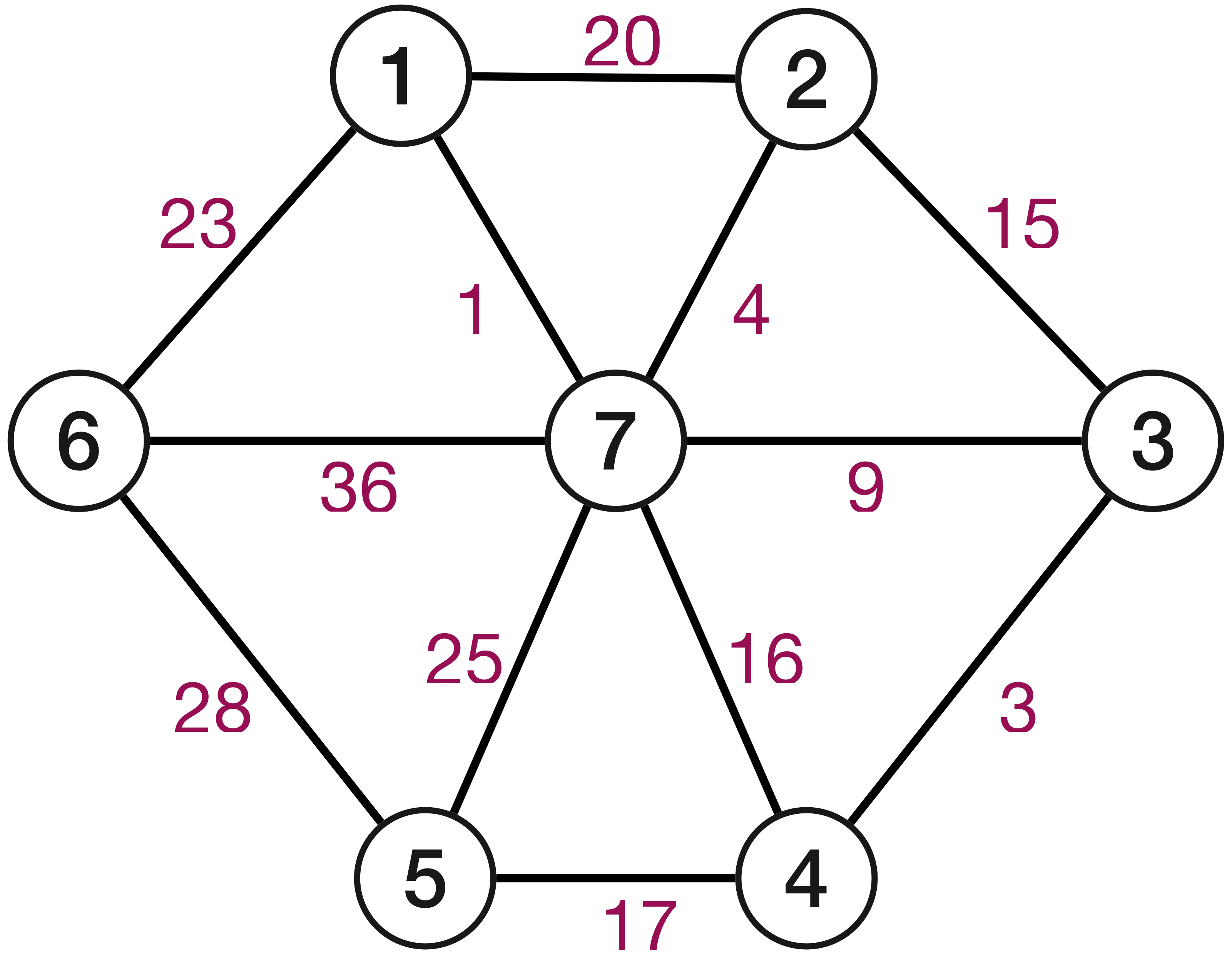
Kruskal's Algorithm

Process edges in the order of their costs (starting from the least) and add edges to T as long as they don't form a cycle.



Graph G

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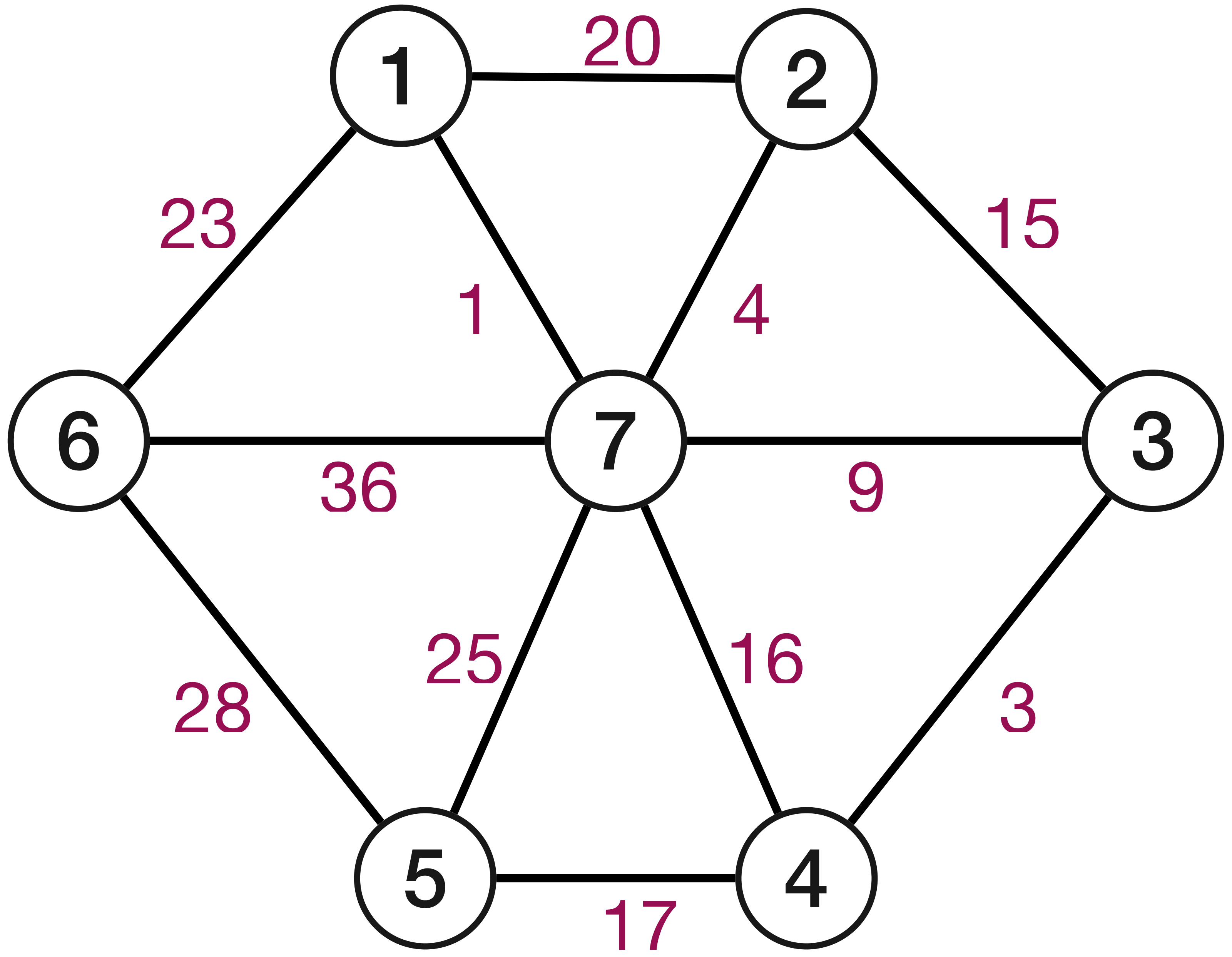

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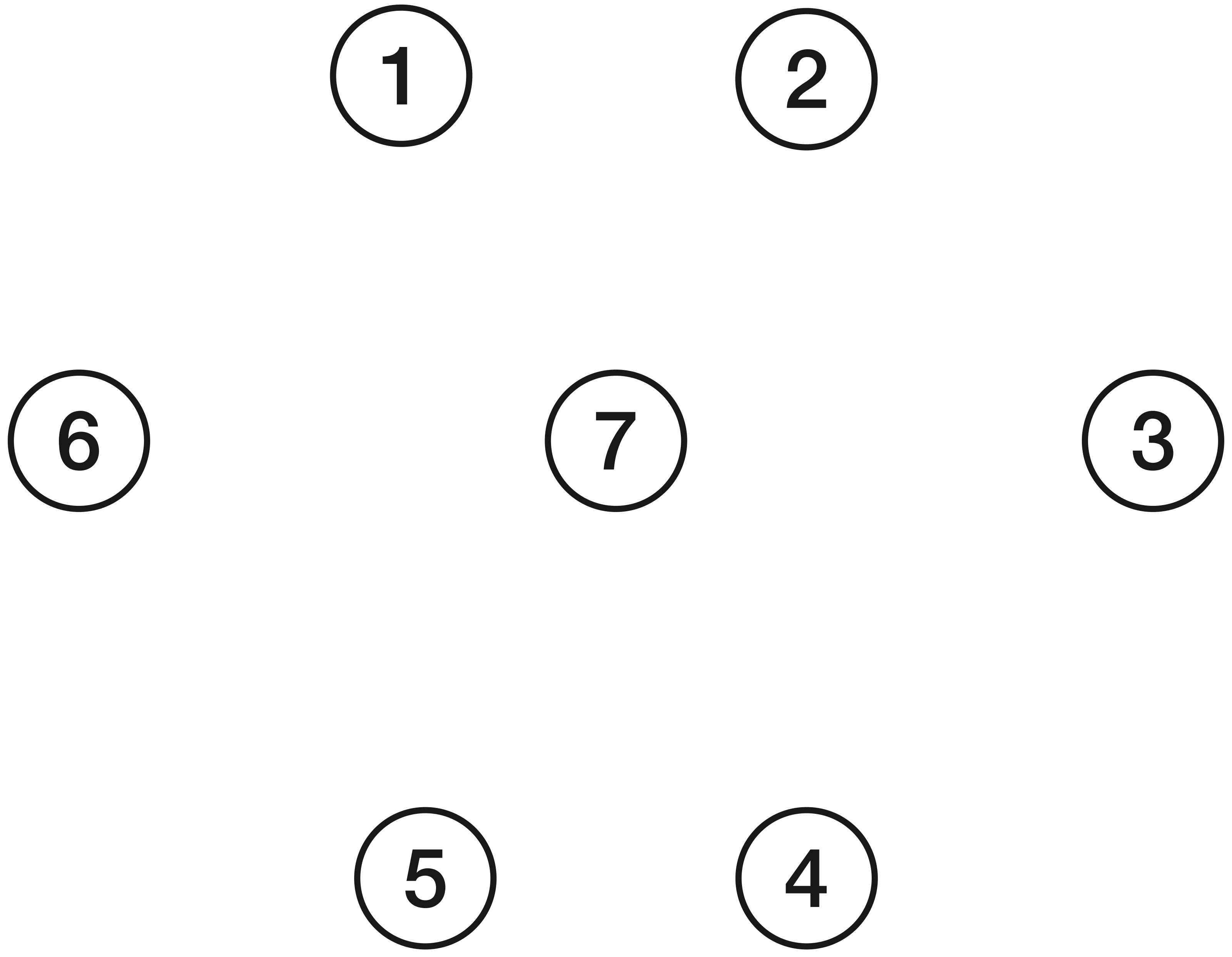

MST of G

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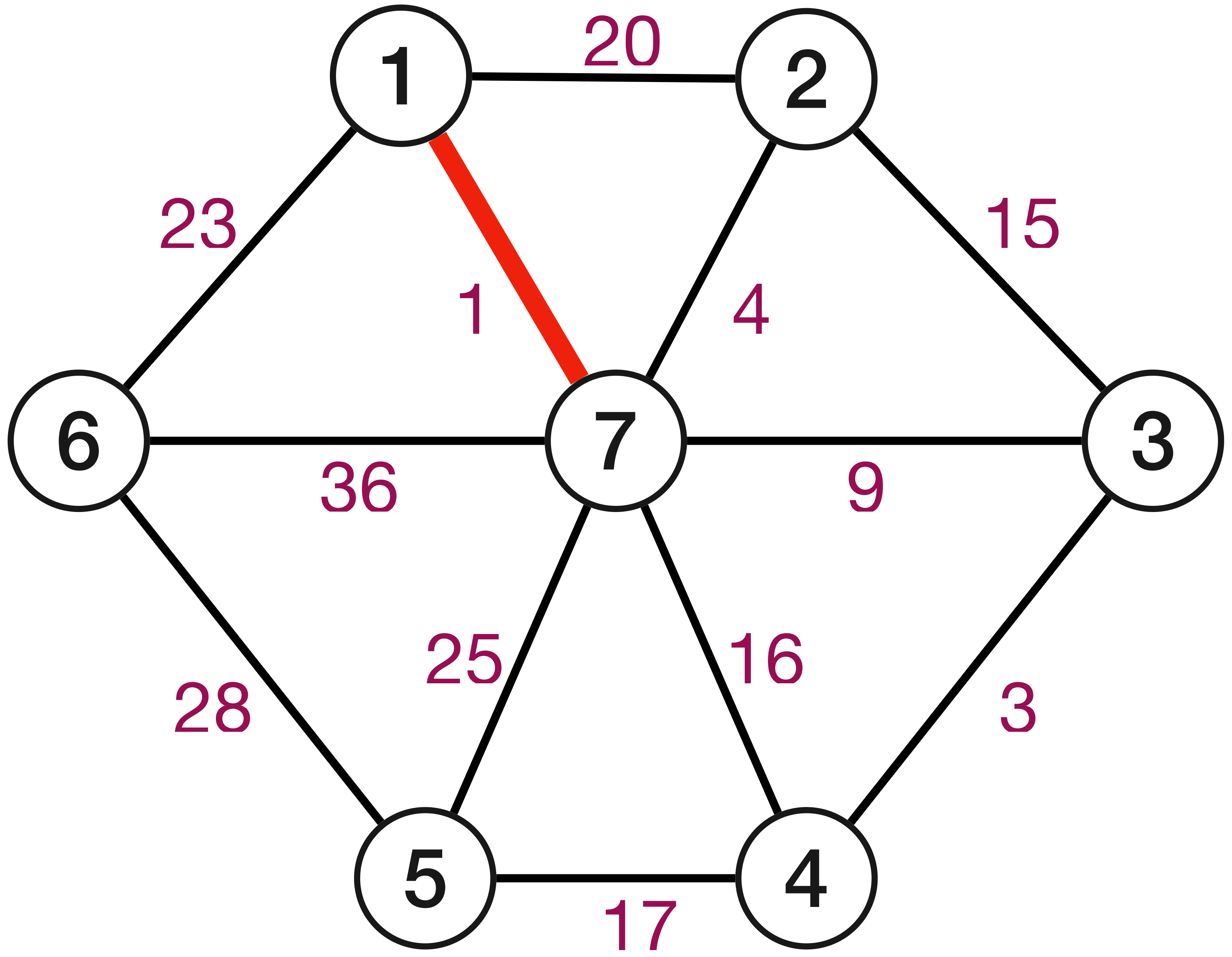

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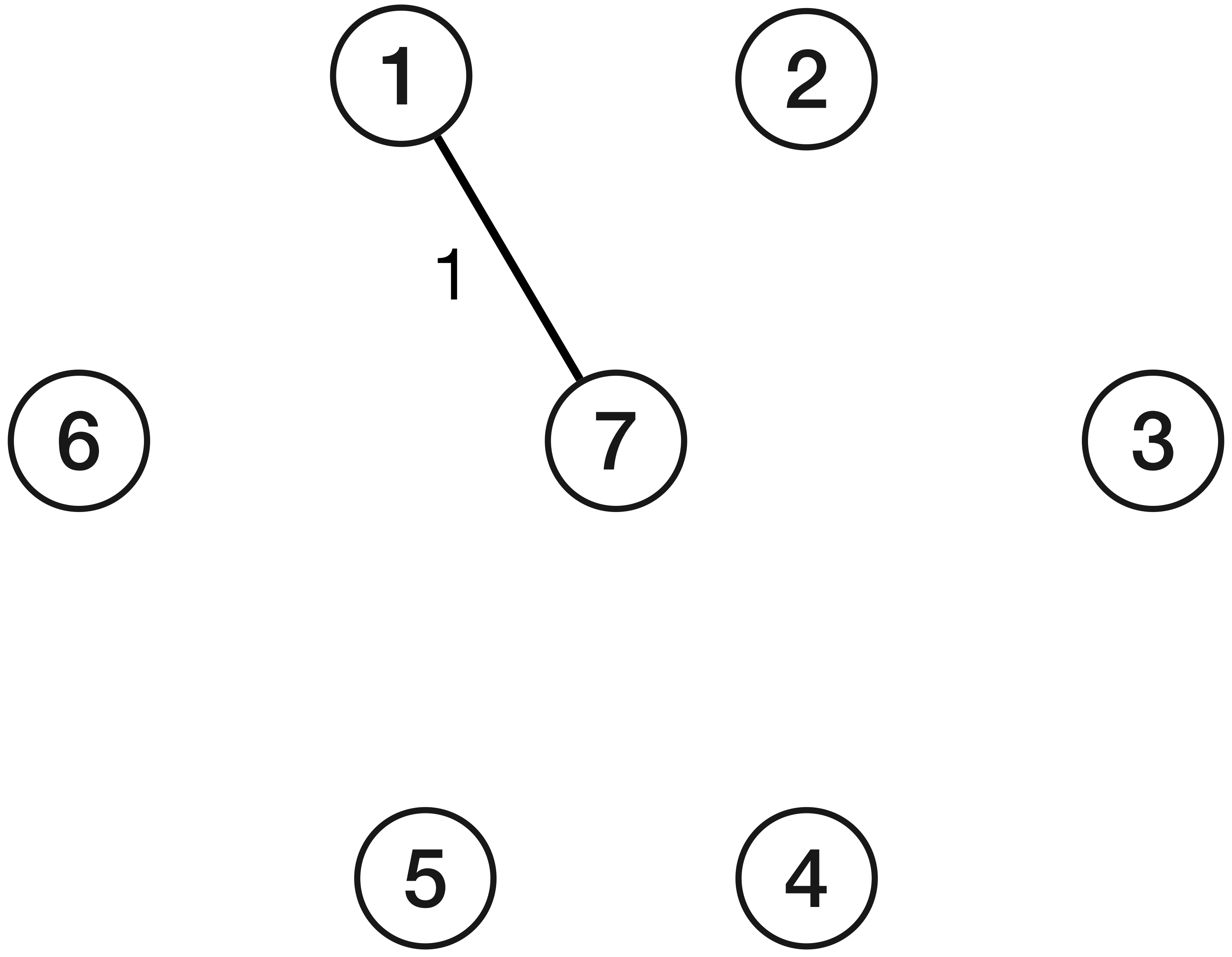

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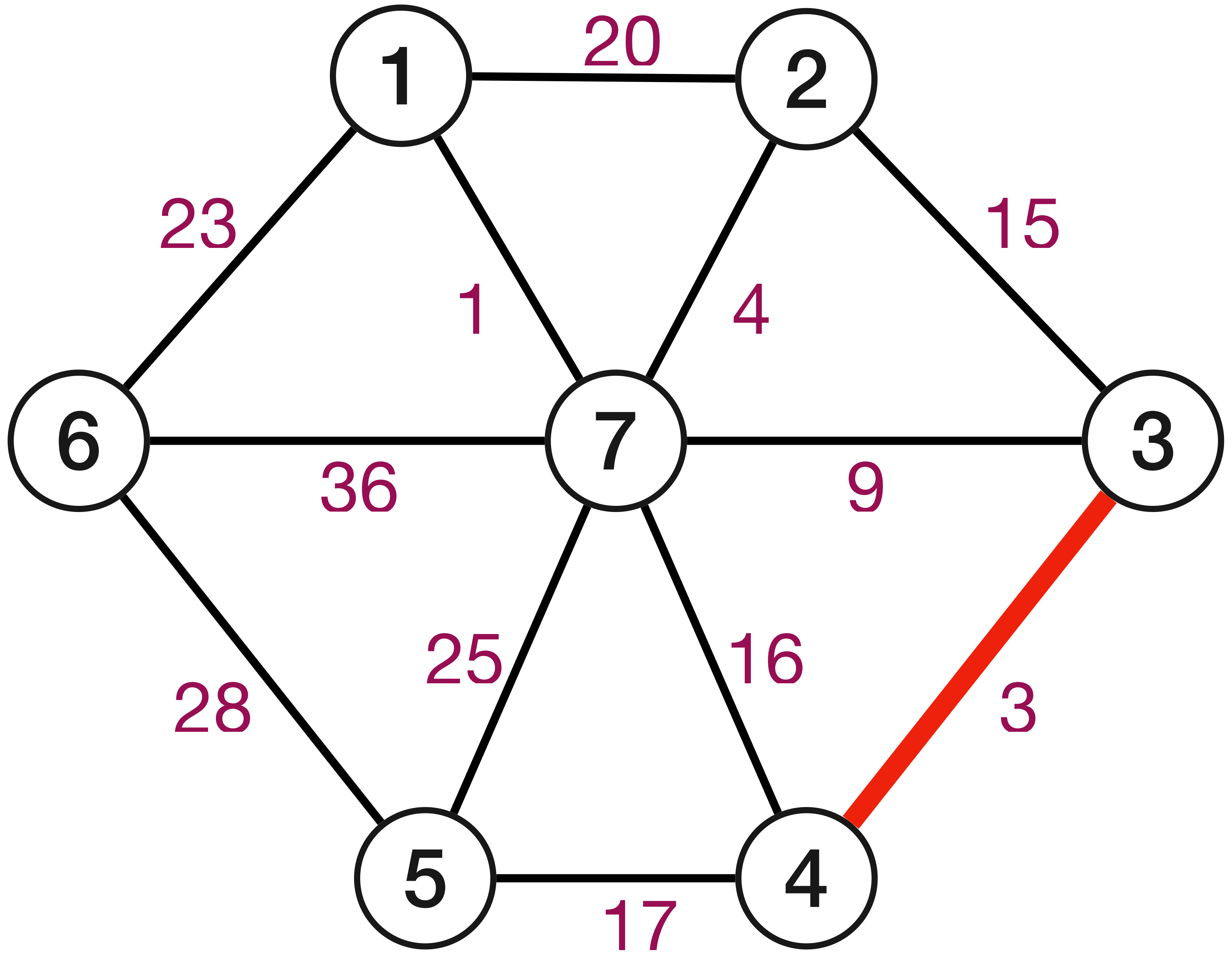

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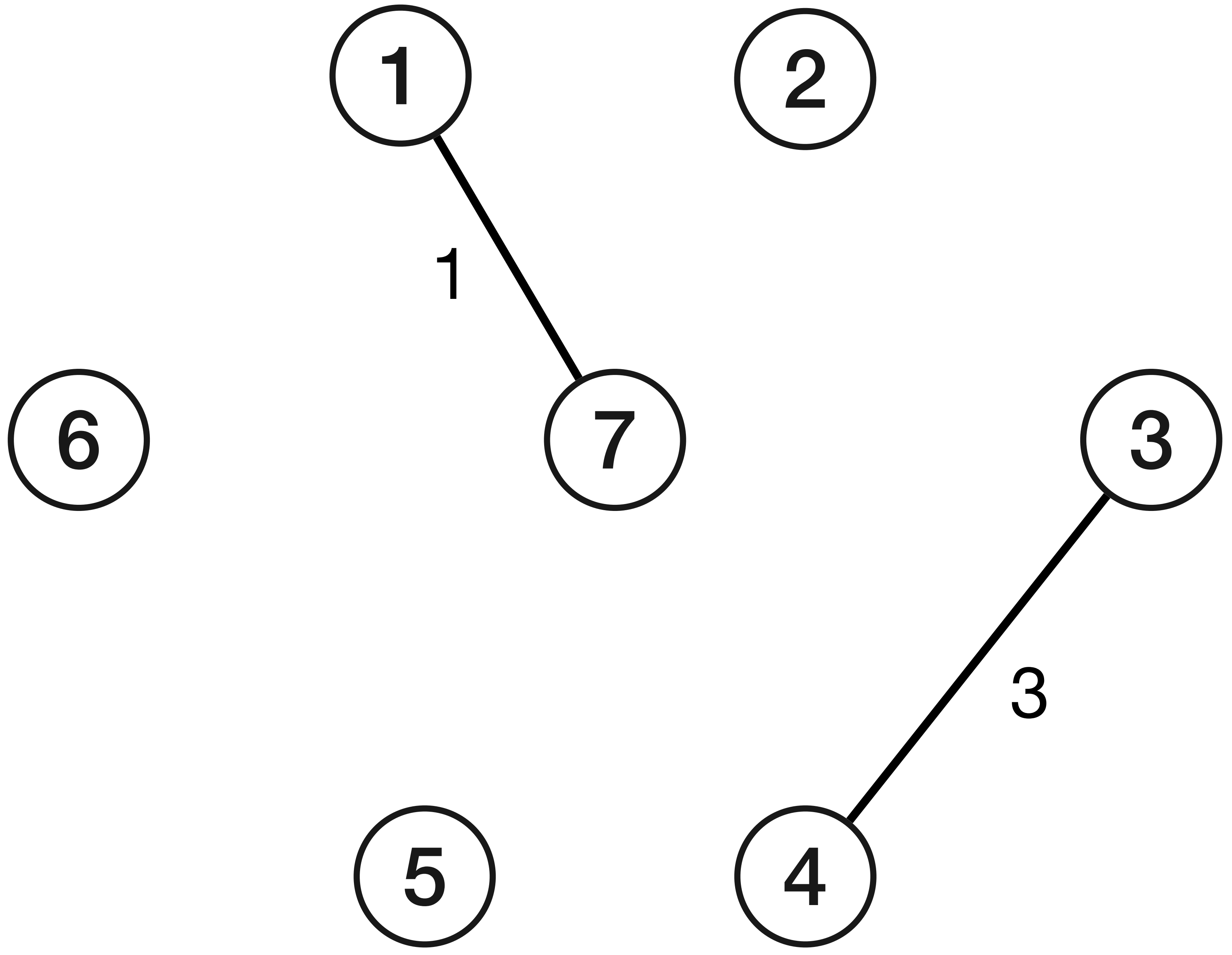

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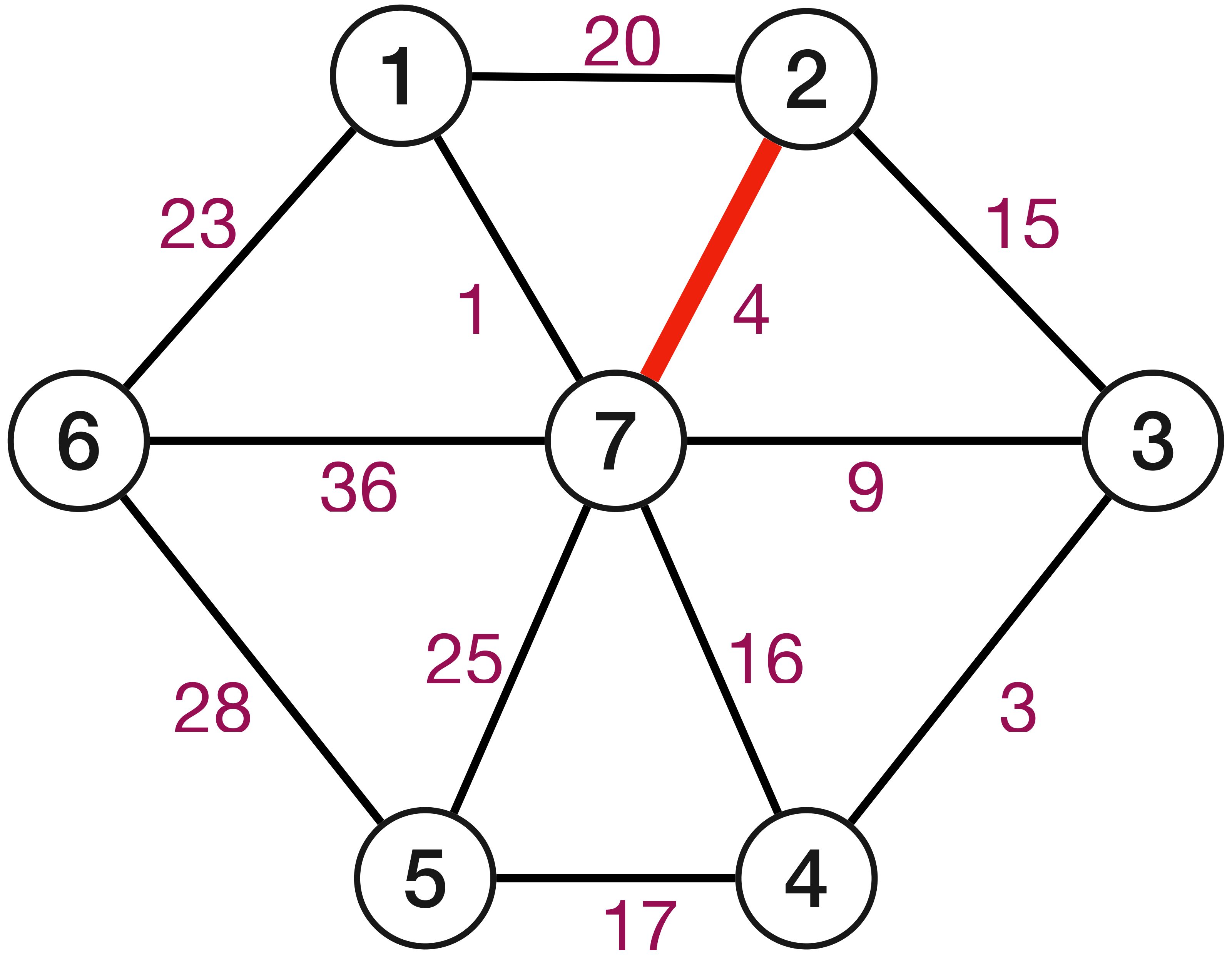

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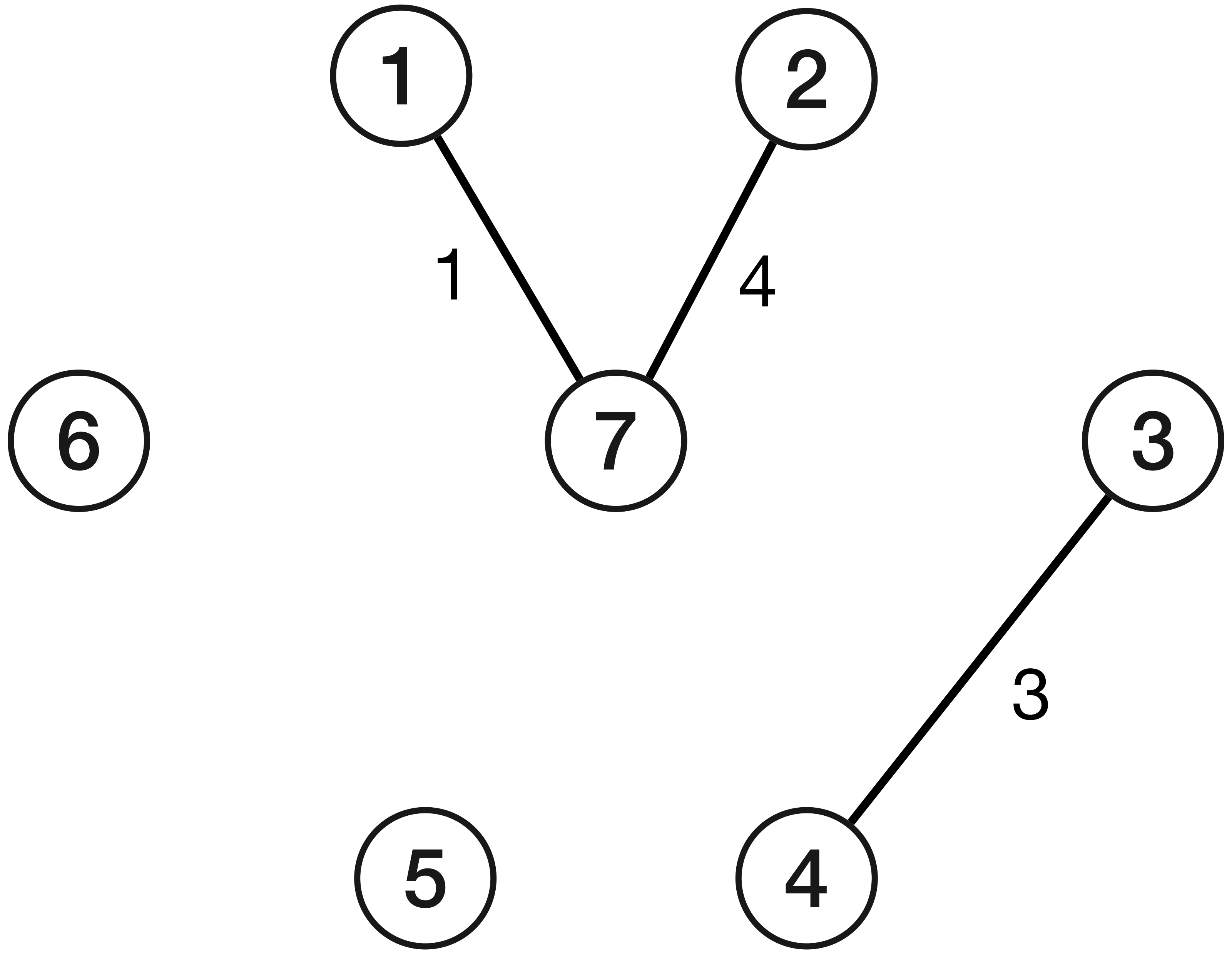

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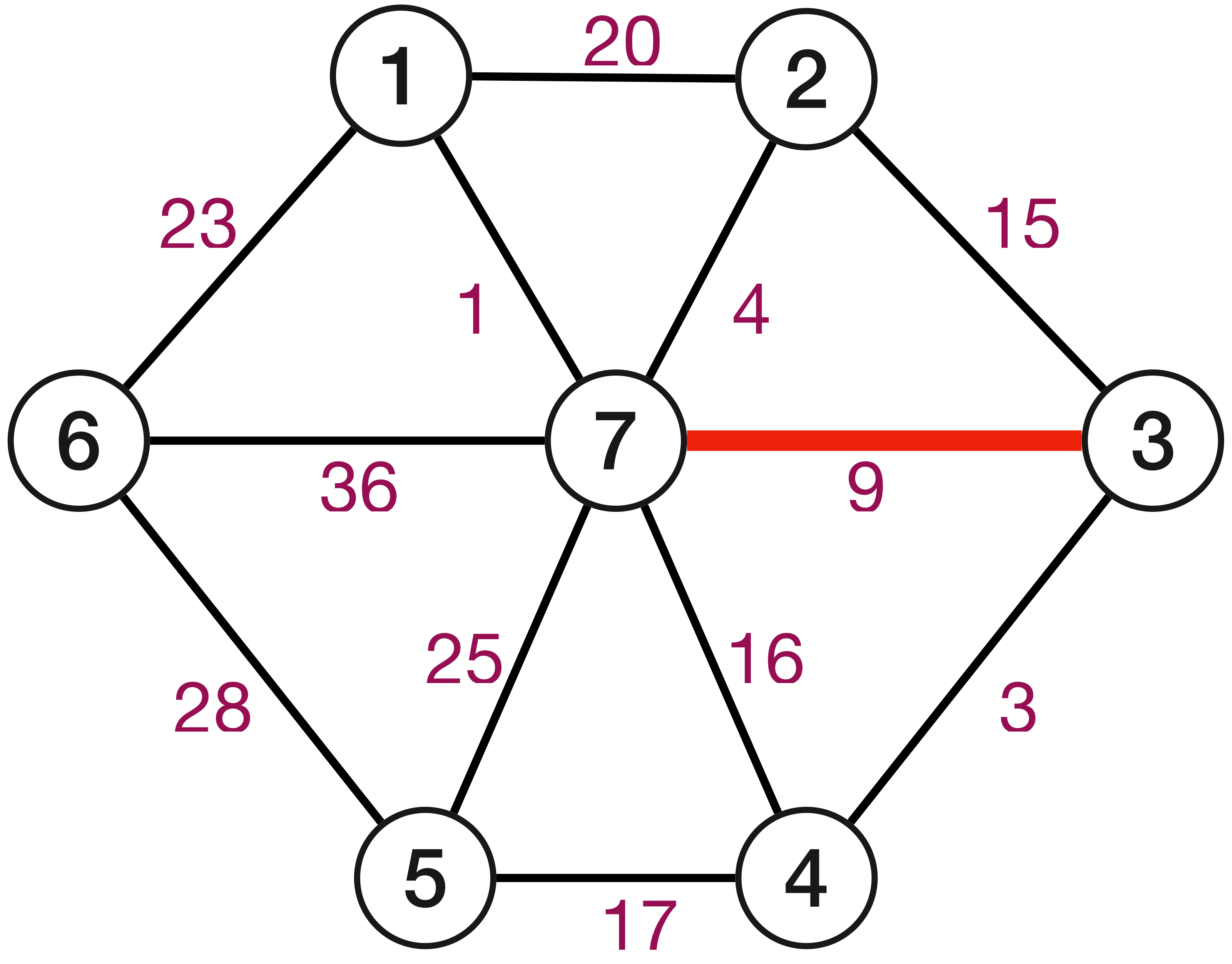

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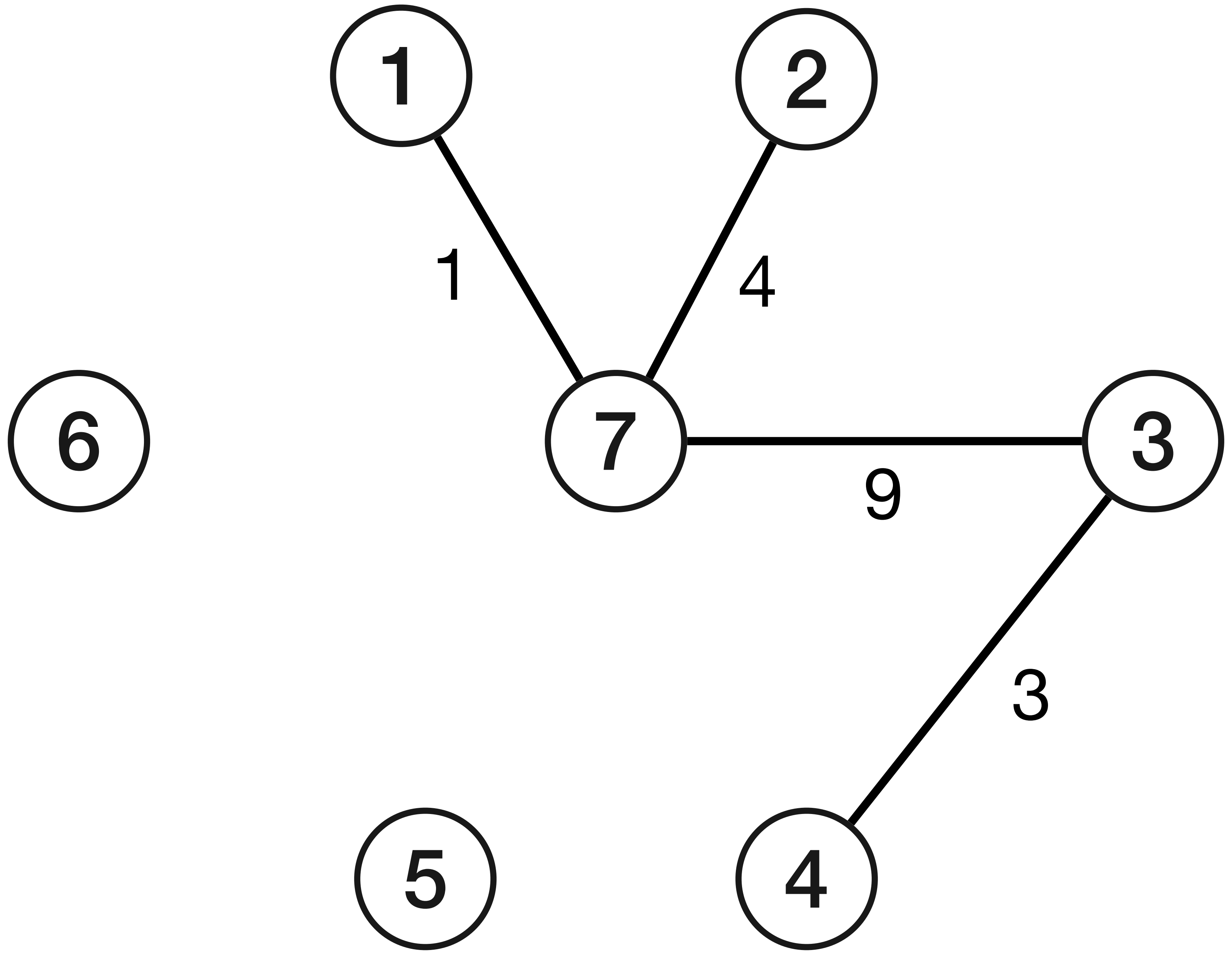

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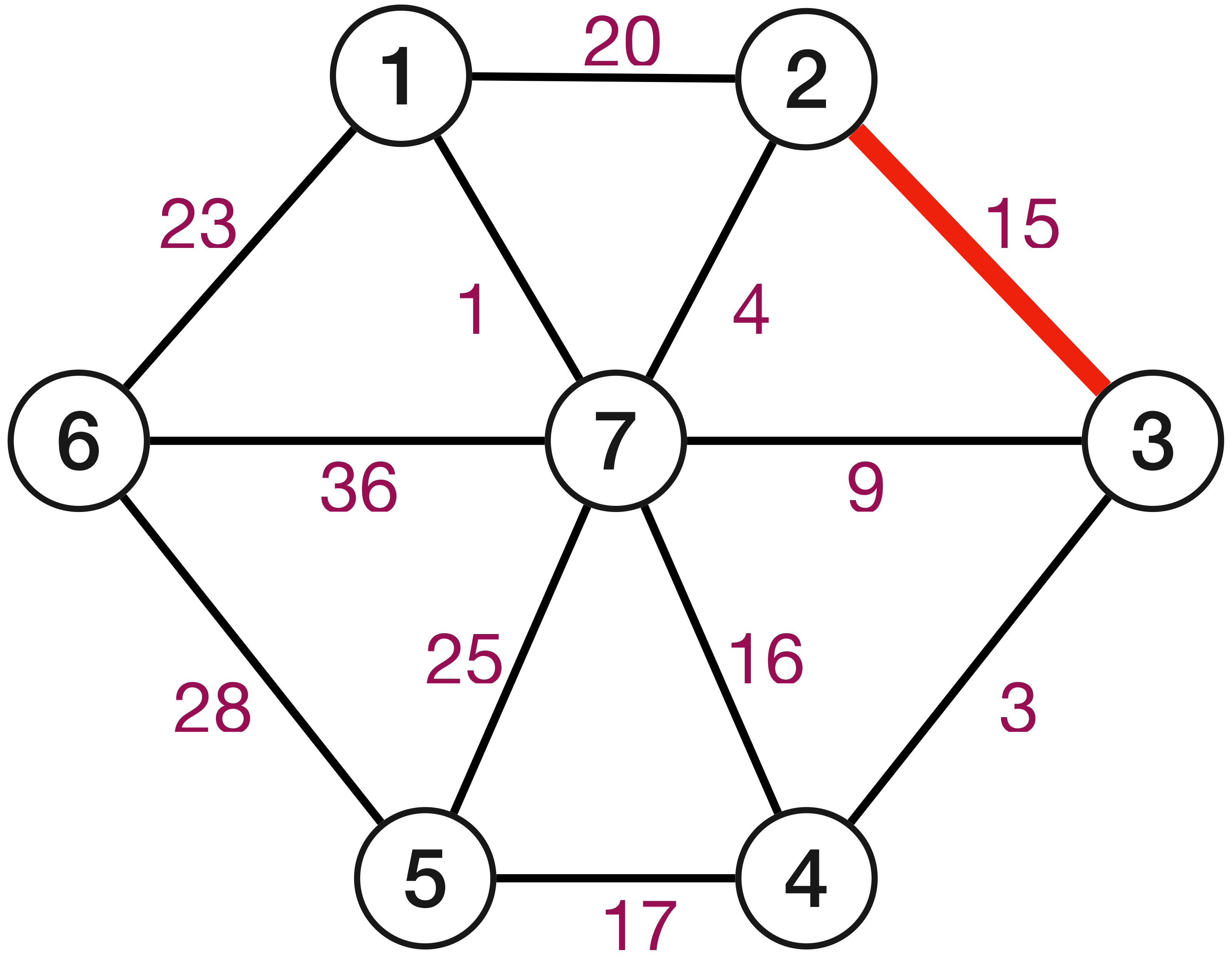

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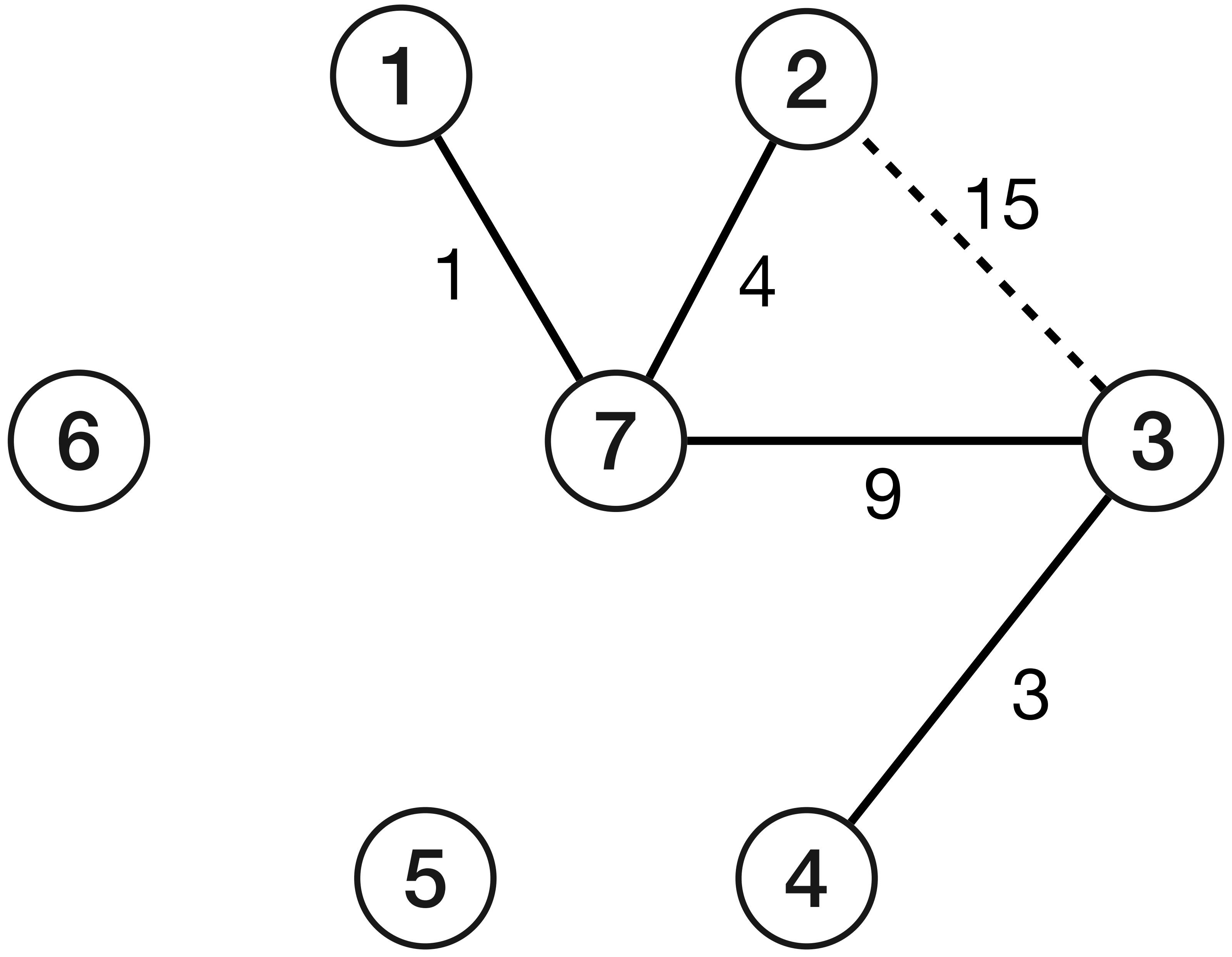

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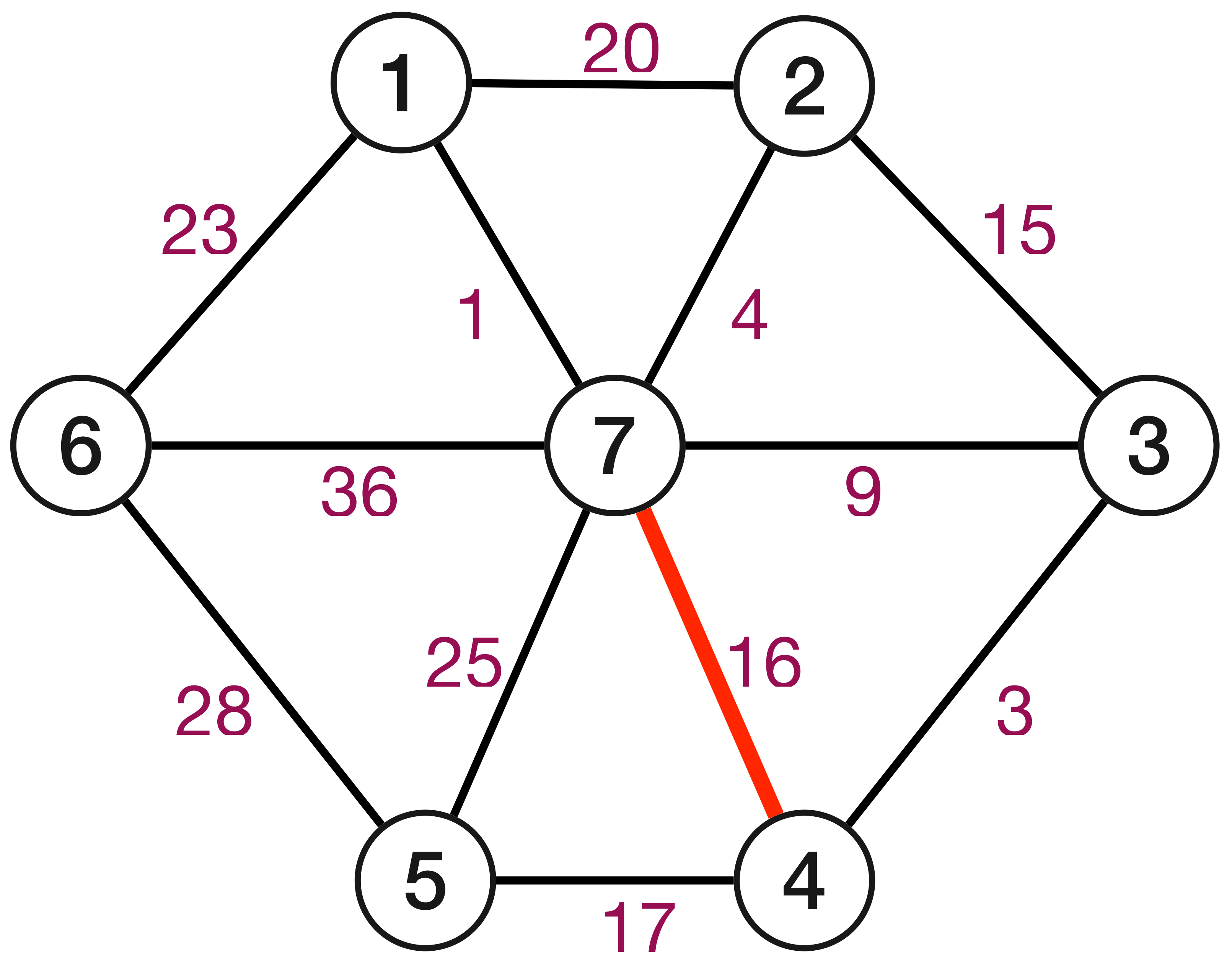

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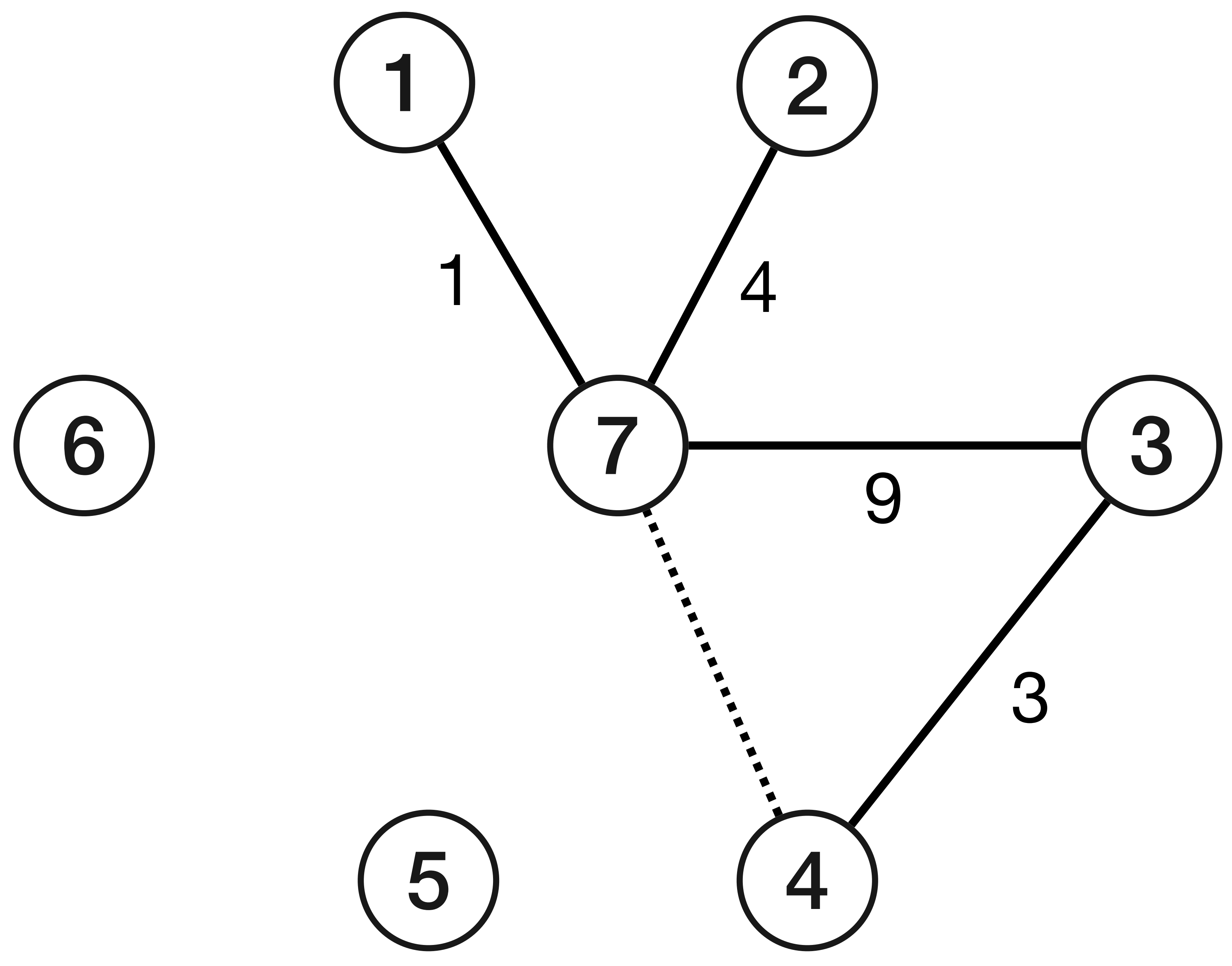

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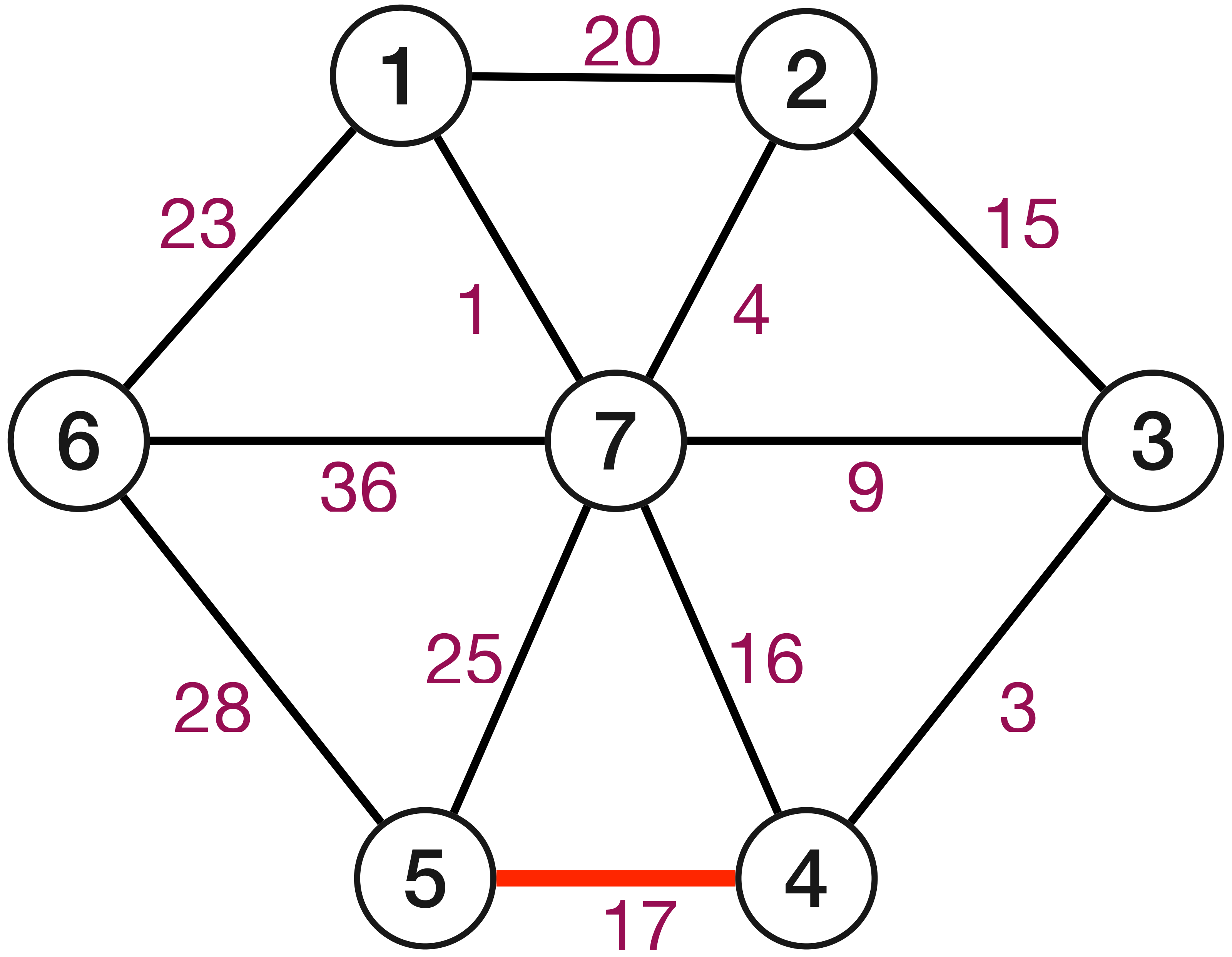

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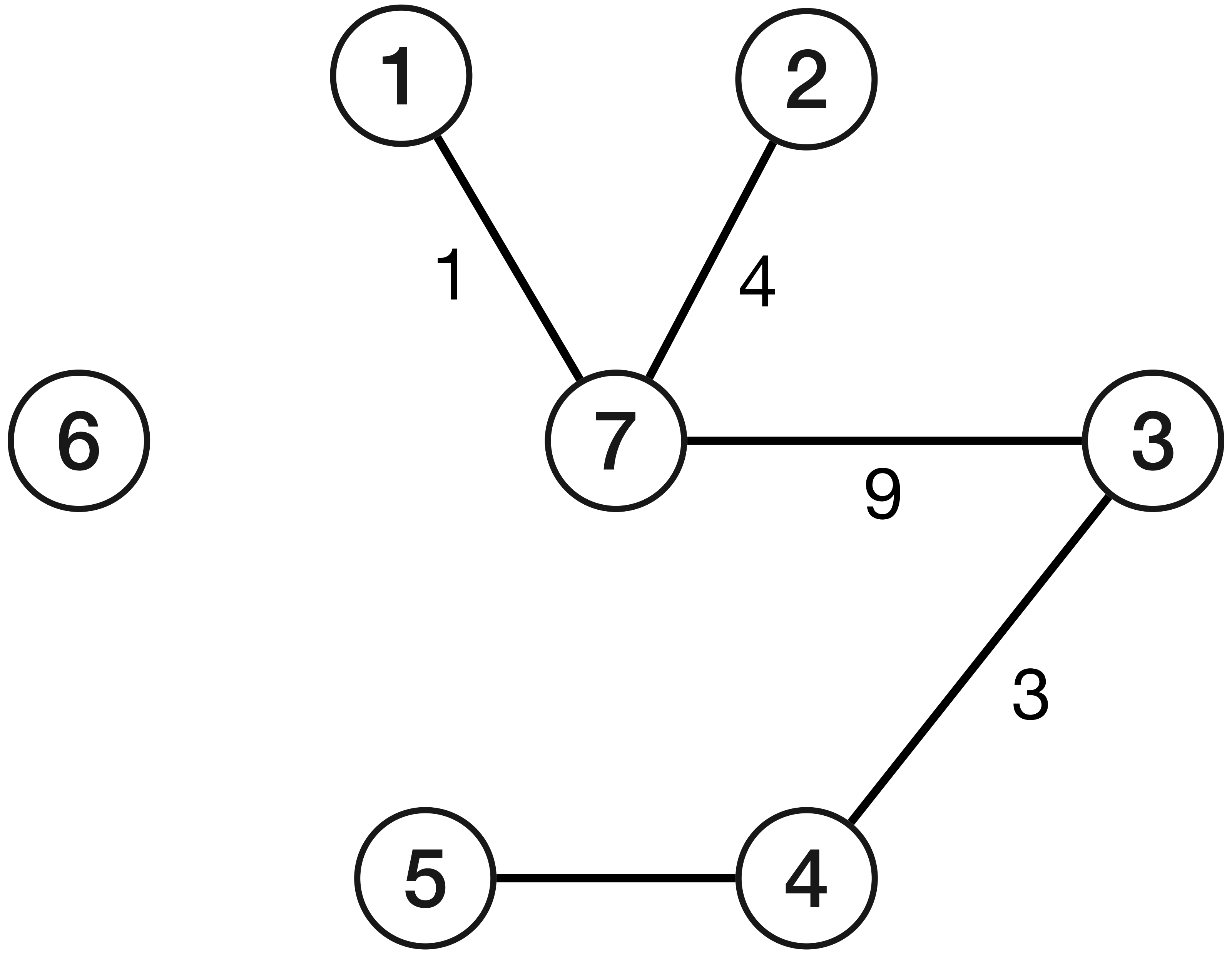

MST of G

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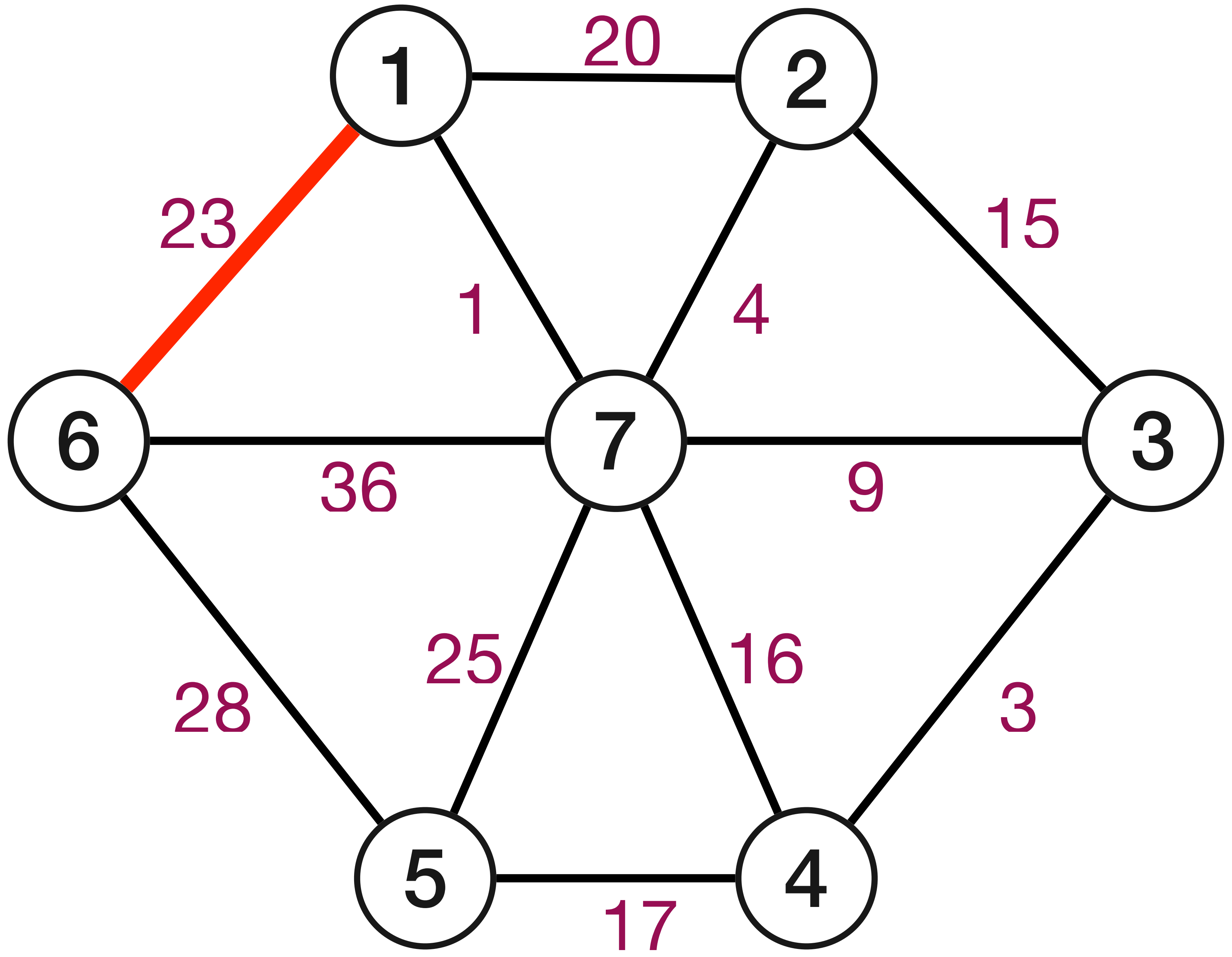

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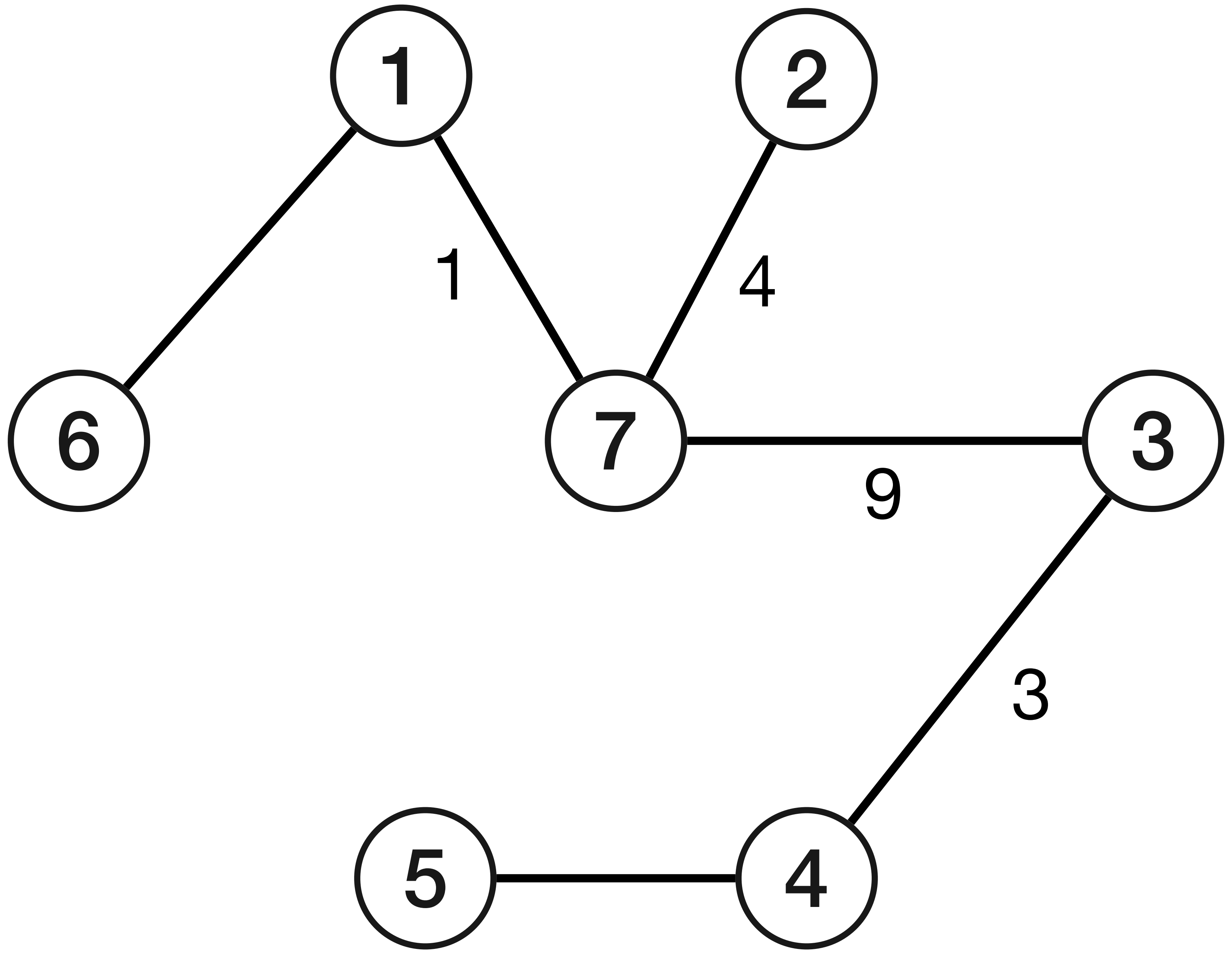


MST of G

Correctness of Kruskal's Algorithm

Kruskal's Algorithm: Picking the edge of lowest cost and adding if it does not form a cycle with existing edges generates a MST.

Proof: If $e = (u, v)$ is added to tree, then e is safe

- When algorithm adds e let S and S' be the connected components containing u and v respectively
- e is the lowest cost edge crossing S (and also S').
- If there is an edge e' crossing S and has lower cost than e, then e' would come before e in the sorted order and would be added by the algorithm to T
- Set of edges output is a spanning tree

Kruskal's Algorithm

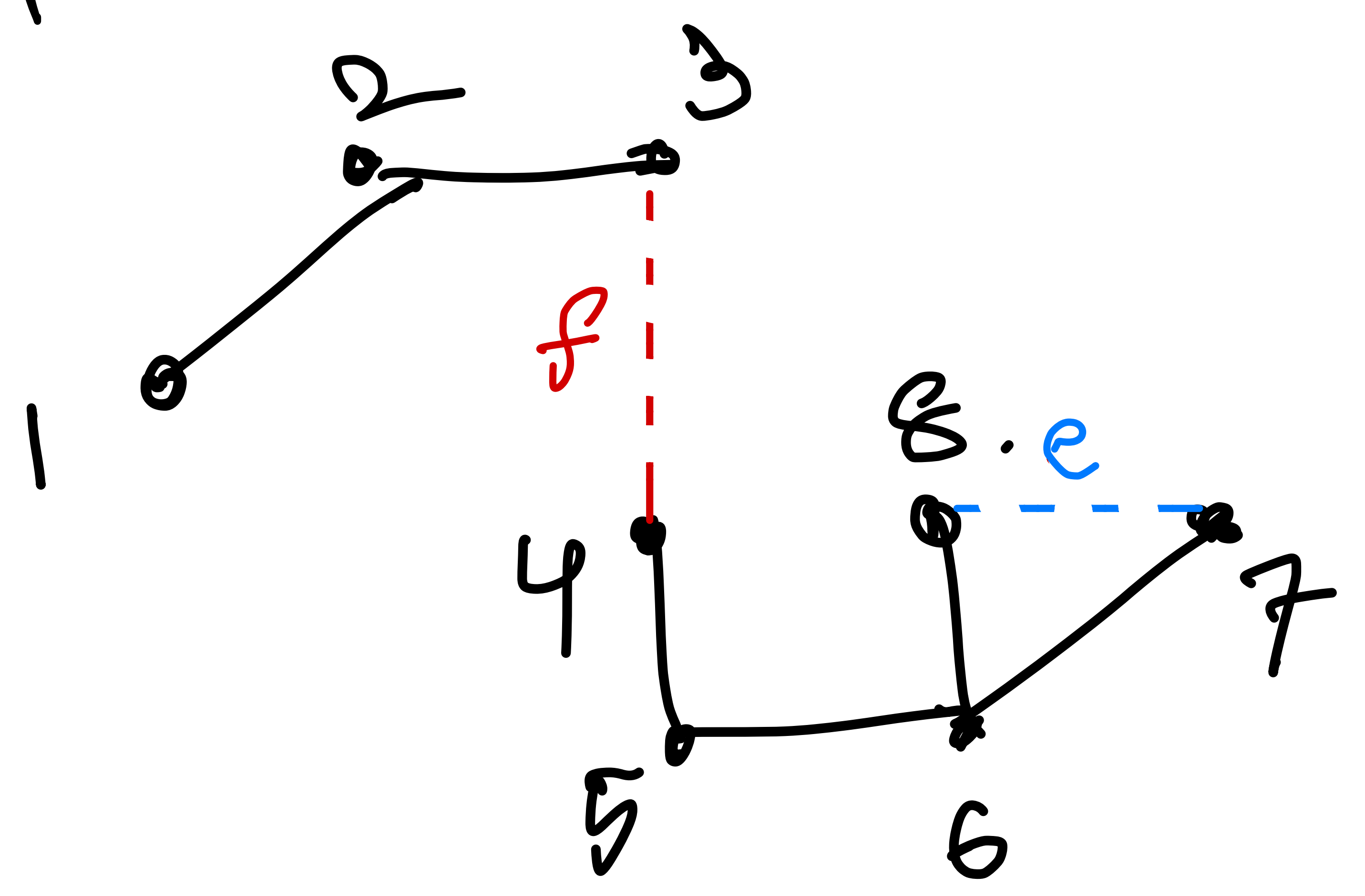
```
Kruskal_ComputeMST
    Initially E is the set of all edges in G
    T is empty (* T will store edges of a MST *)
    while E is not empty do
          choose e ∈ E of minimum cost
          remove e from E
          if (T ∪ {e} does not have cycles)
            add e to T
    return the set T
```

cycle detection

I answered a student question incorrectly in class. Faizi asked "why can't we store a list of vertices from edges added to T in a list L and check for a cycle from an incoming edge {u,v} by seeing if $u \in L, v \in L$?"

Because: L=[1, 2, 3, 4, 5, 7, 8]

see f vs e.



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- Presort edges based on cost. Choosing minimum can be done in $O(1)$ time
- Do BFS/DFS on $T \cup \{e\}$. Takes $O(n)$ time
- Total time $O(m \log m) + O(mn) = O(mn)$

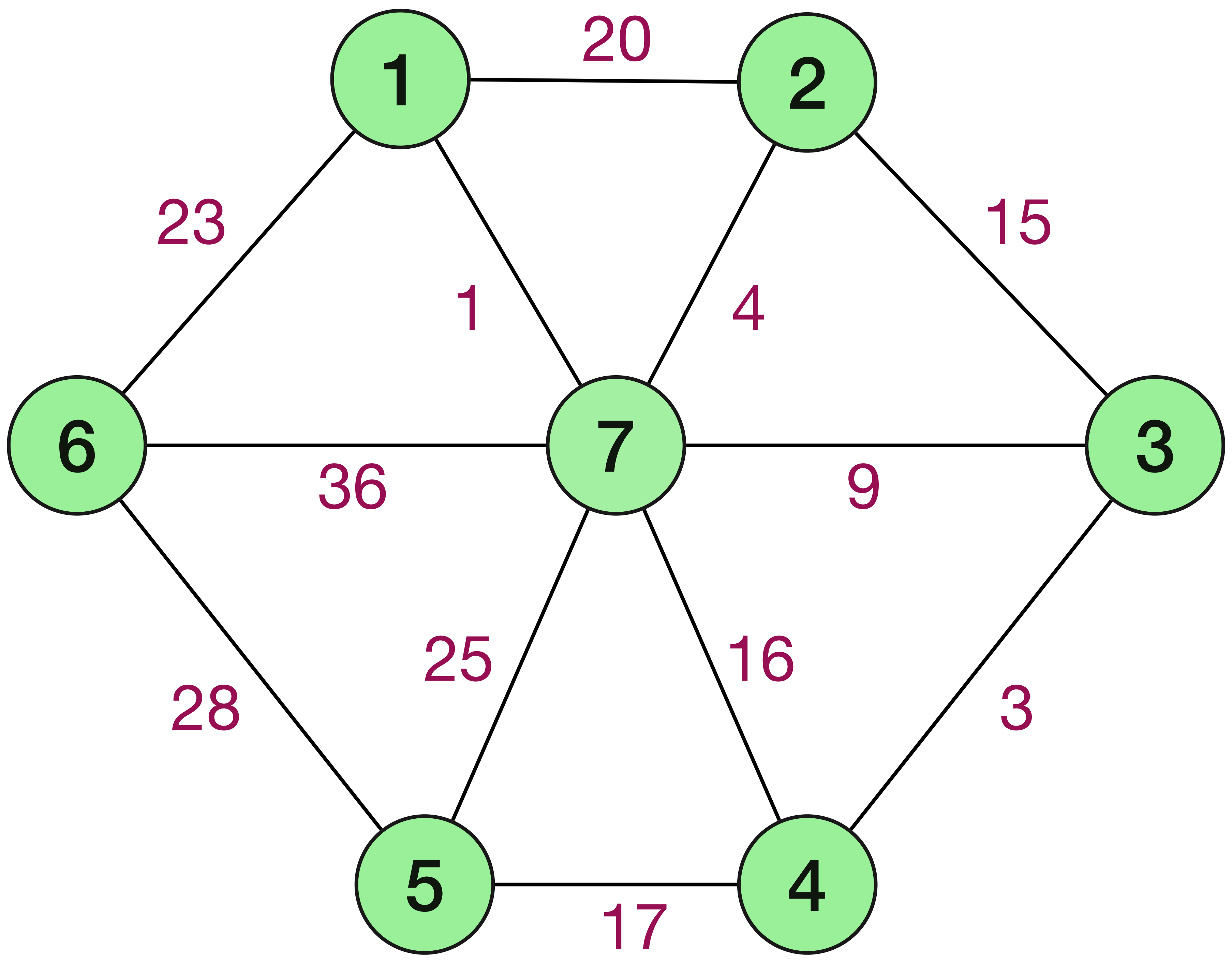
Kruskal's Algorithm (efficiently)

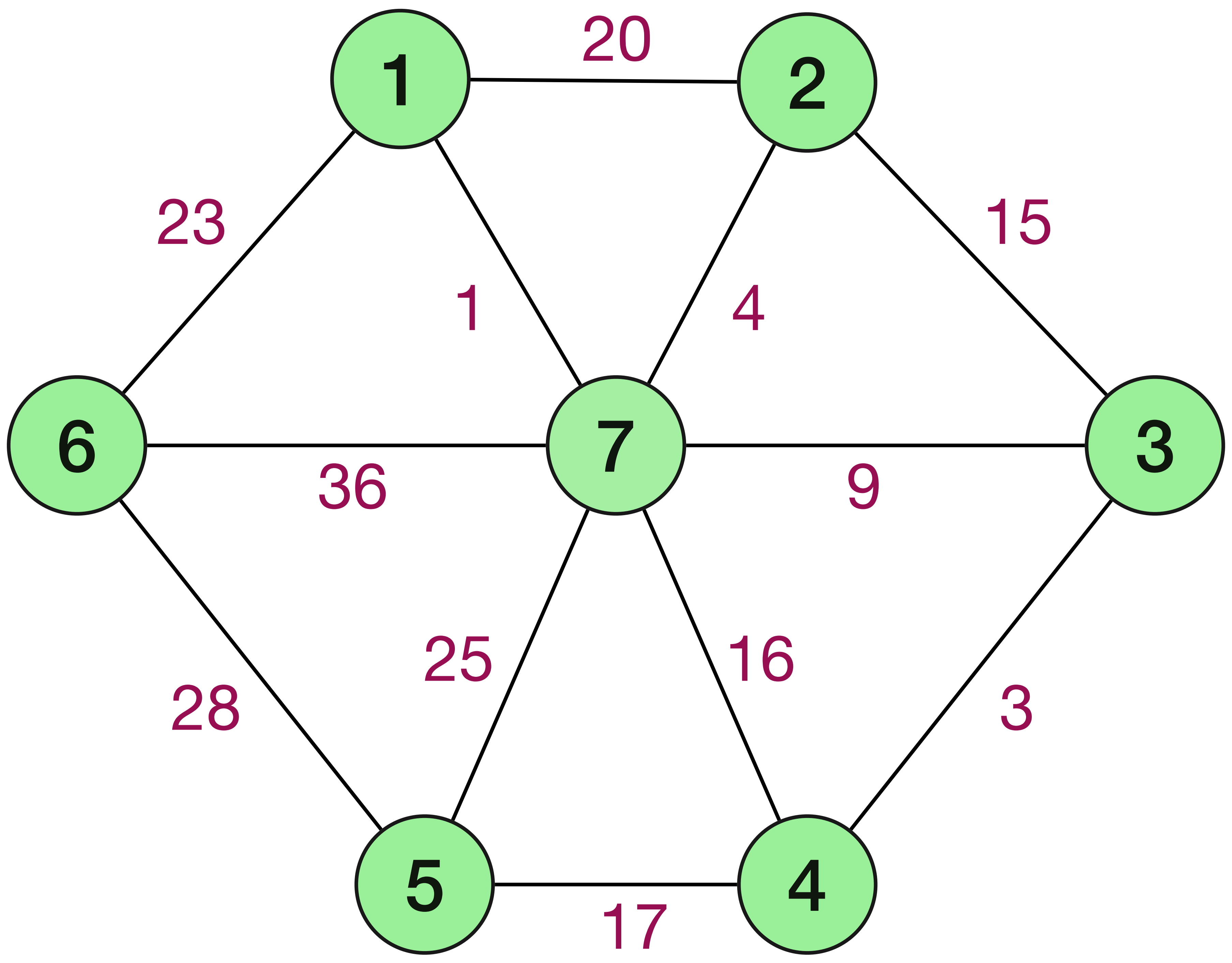
```
Kruskal_ComputeMST
    Sort edges in E based on cost
    T is empty (* T will store edges of a MST *)
    each vertex u is placed in a set by itself
    while E is not empty do
          pick e = (u,v)∈ E of minimum cost
          if u and v belong to different sets
            add e to T
            merge the sets containing u and v
    return the set T
```

- Need a data structure to check if two elements belong to same set and to merge two sets.
- Using Union-Find (disjoint-set) data structure can implement Kruskal's algorithm in $O((m+n)\log m)$ time.

Prim's algorithm

T maintained by algorithm will be a tree. Start with a node in T. In each iteration, pick edge with least attachment cost to T.





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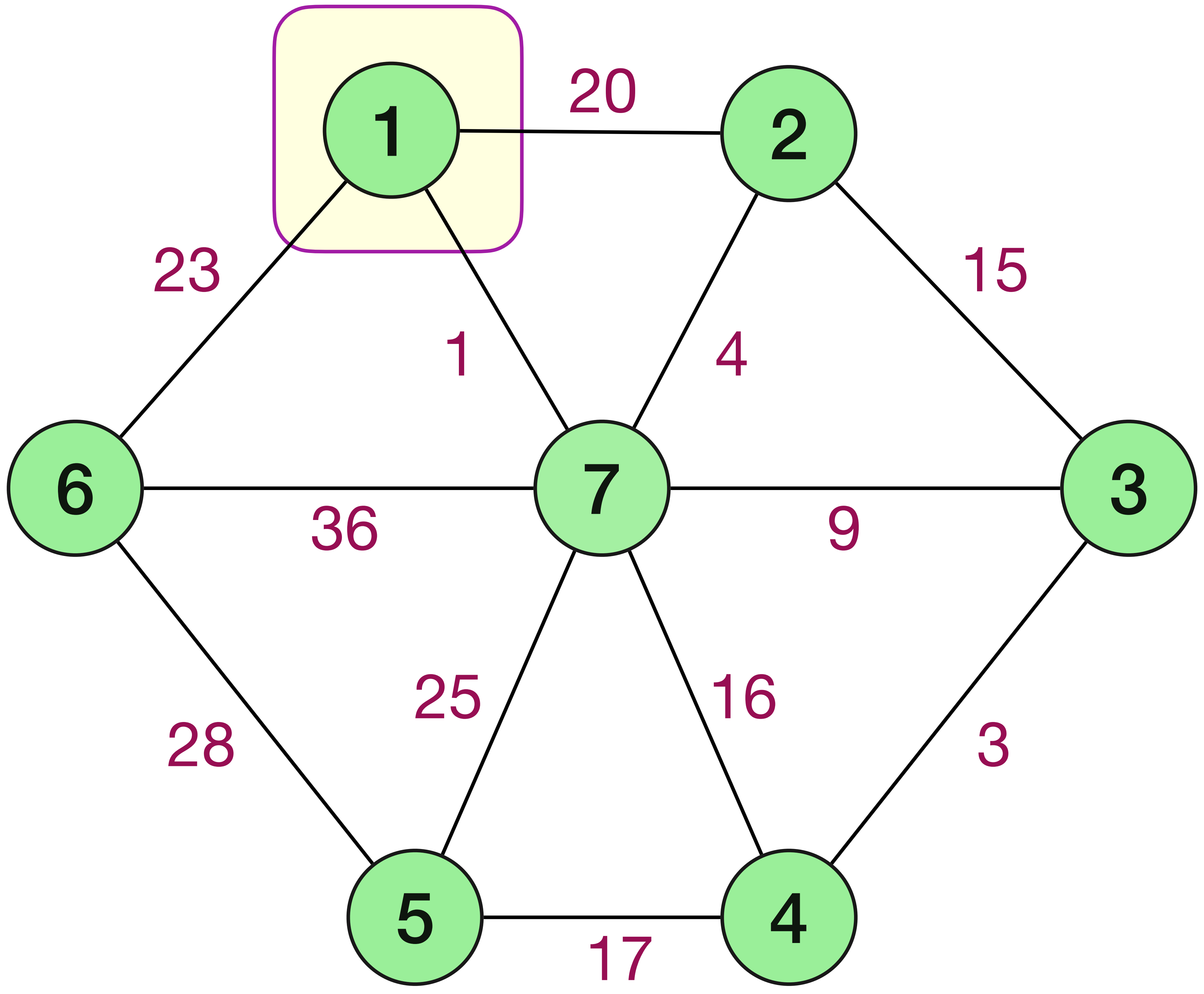

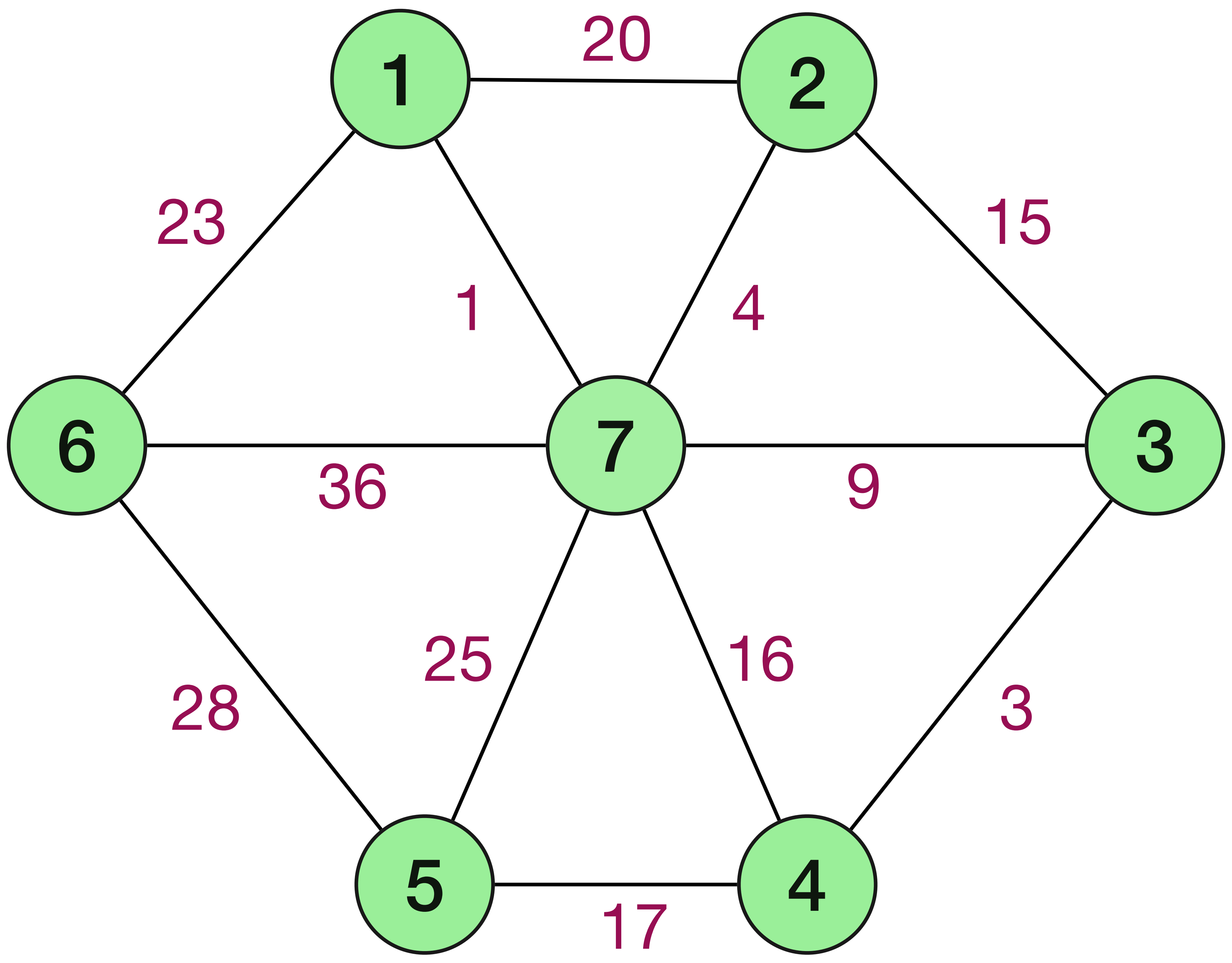

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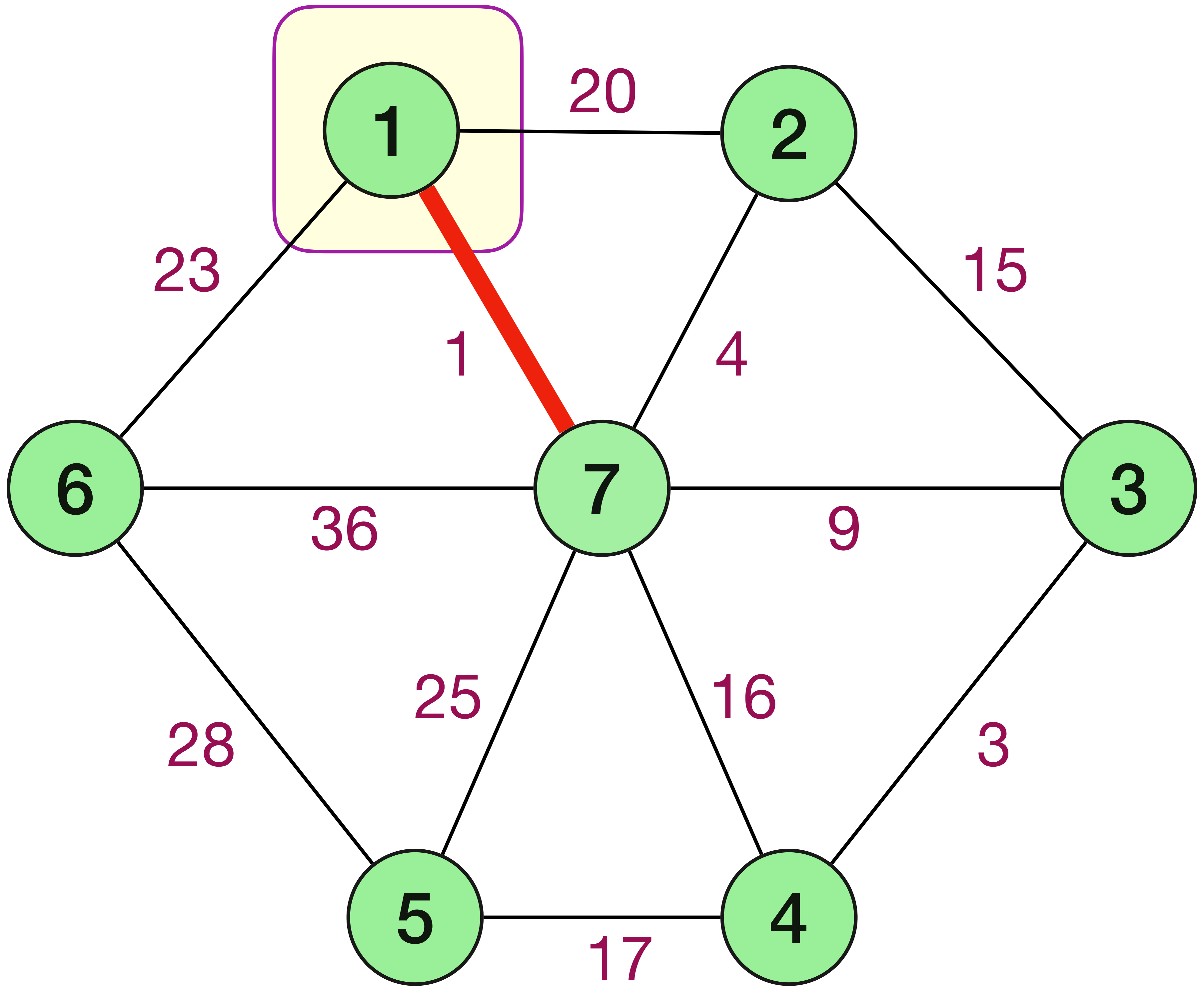

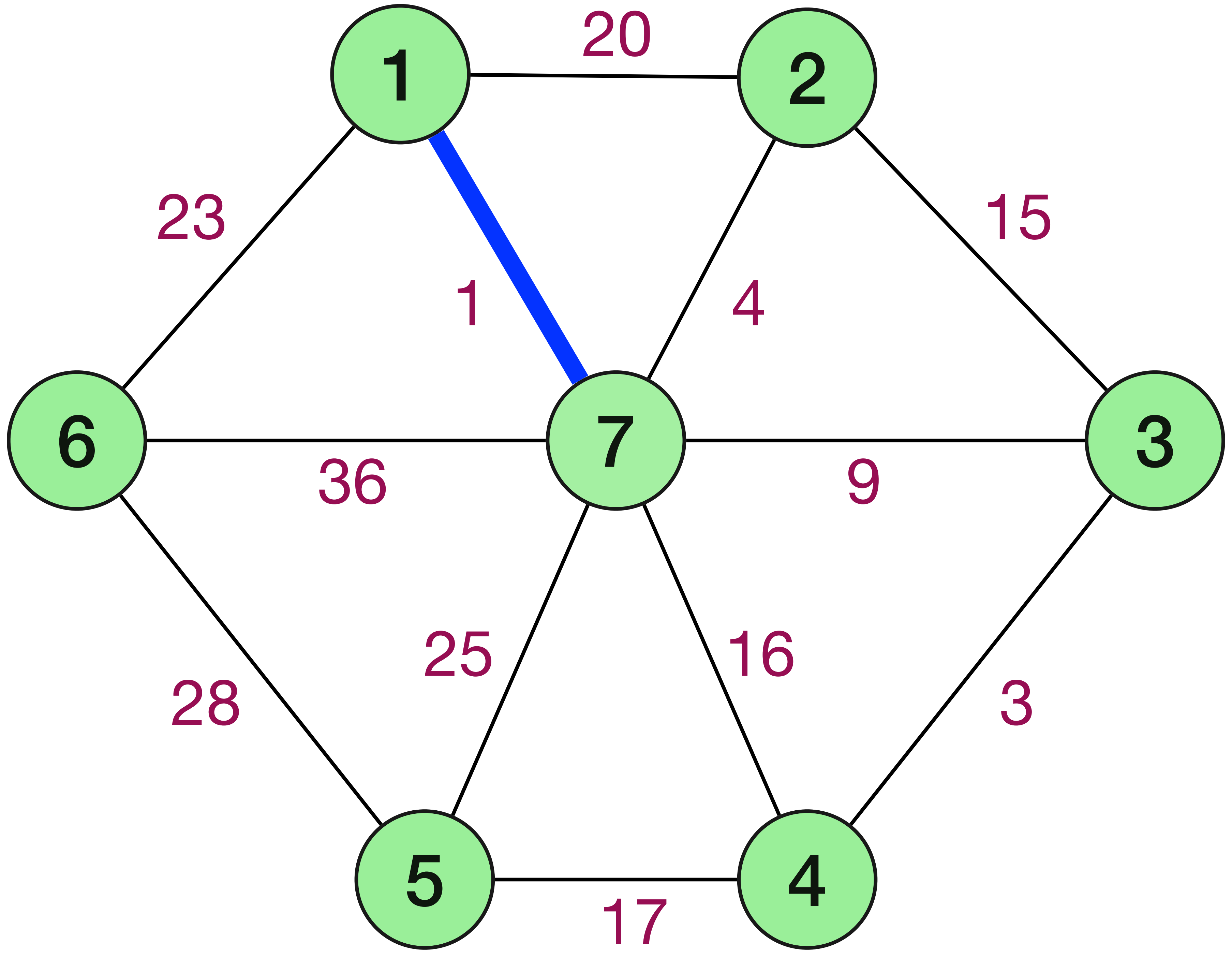

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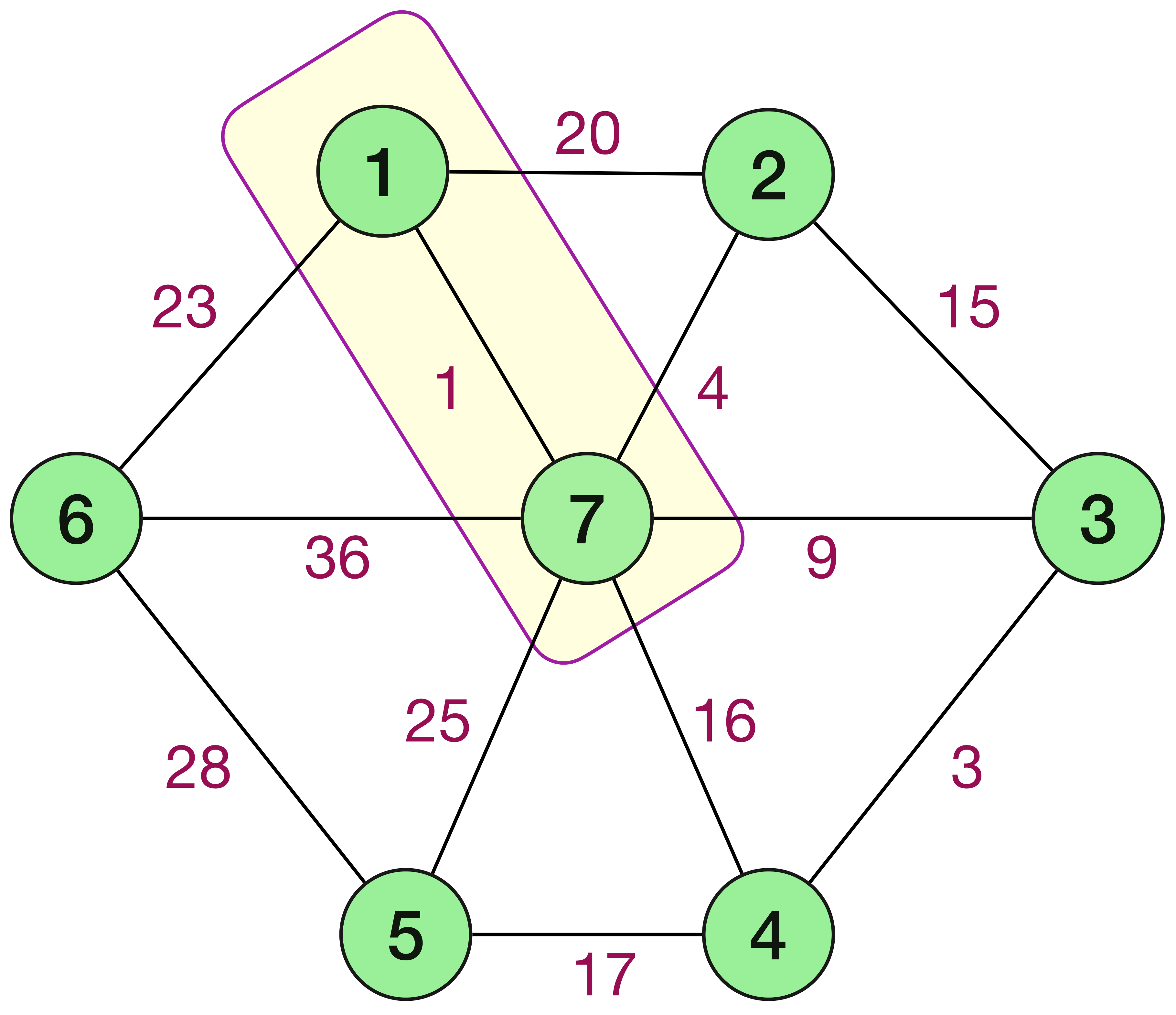

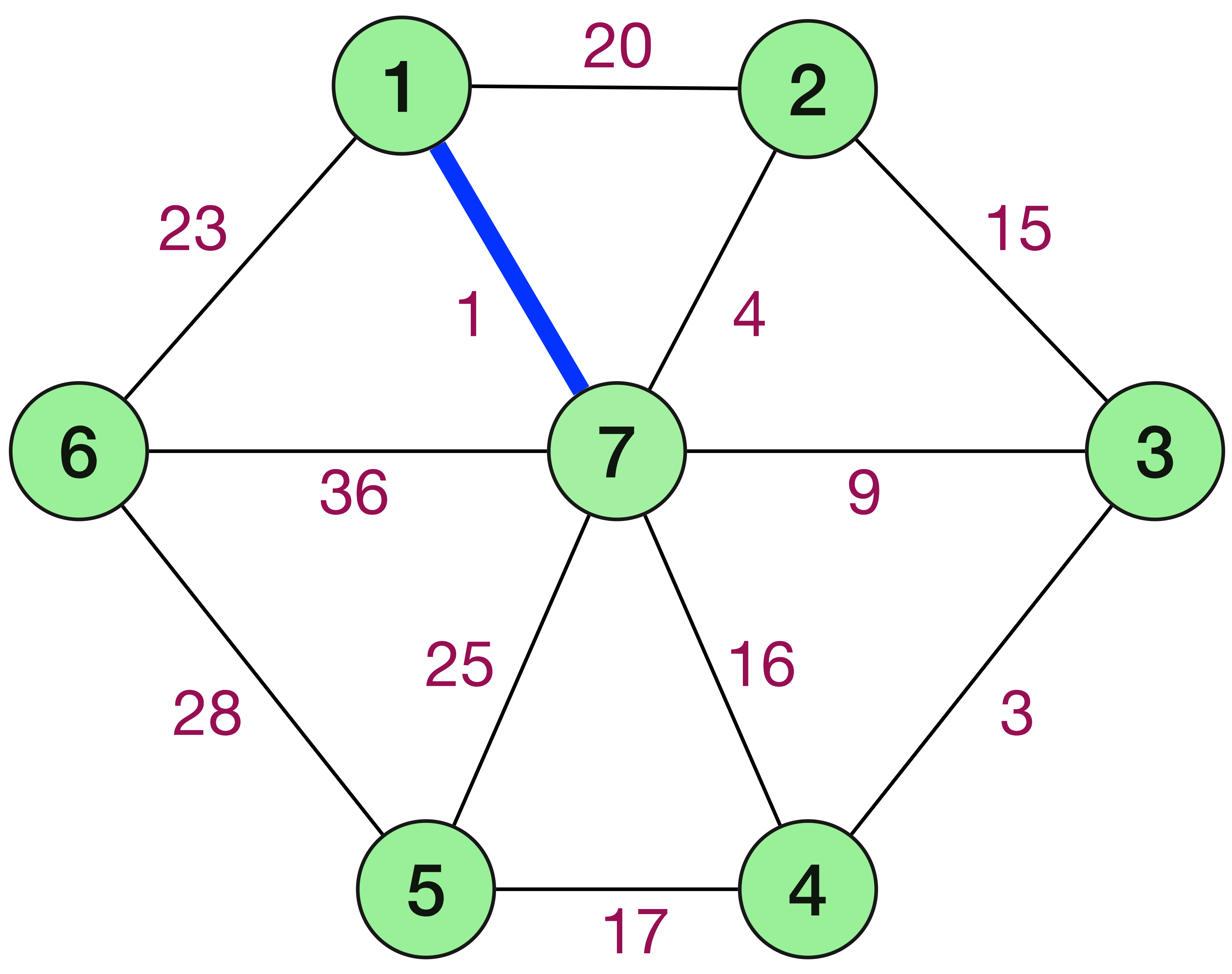

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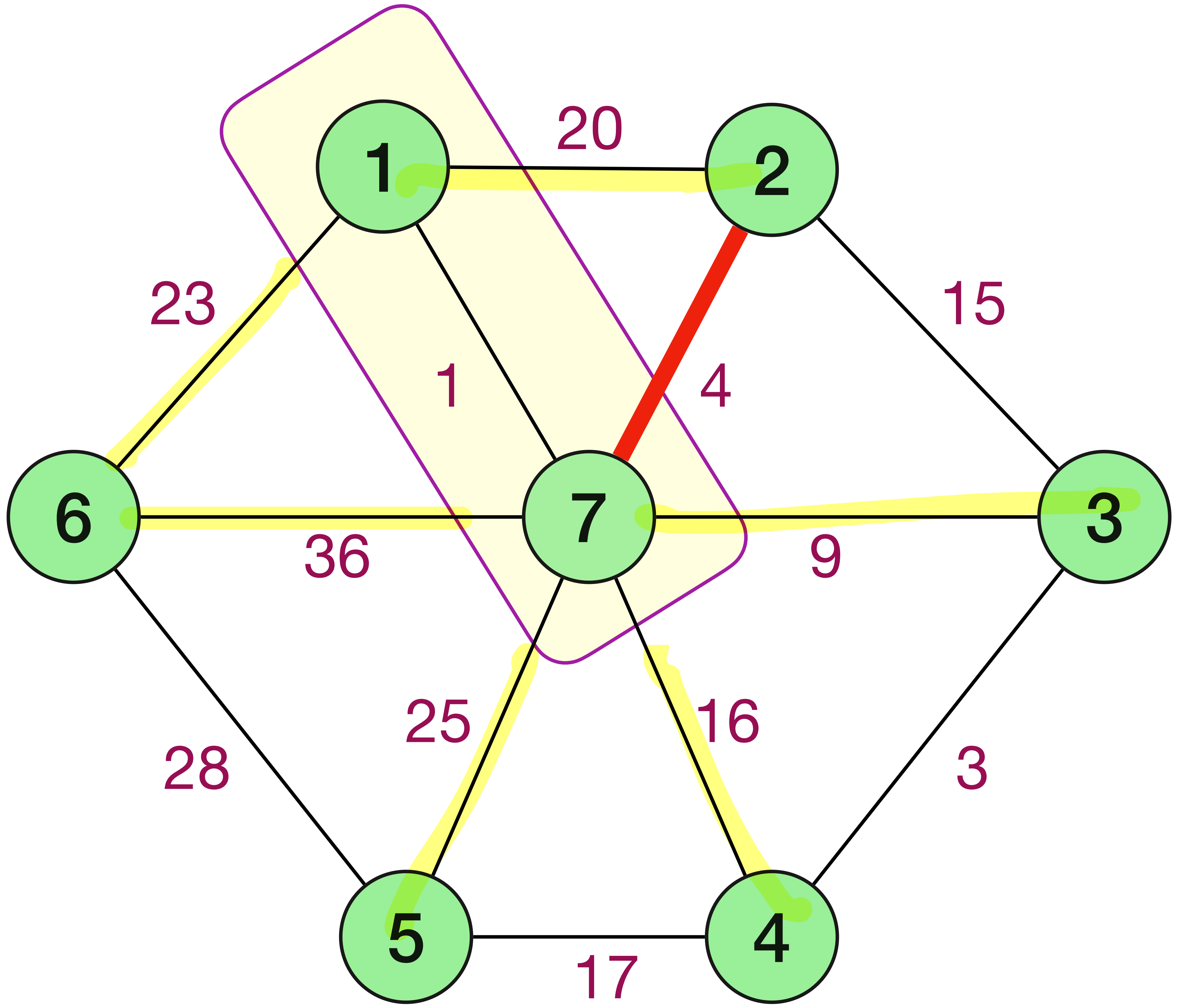

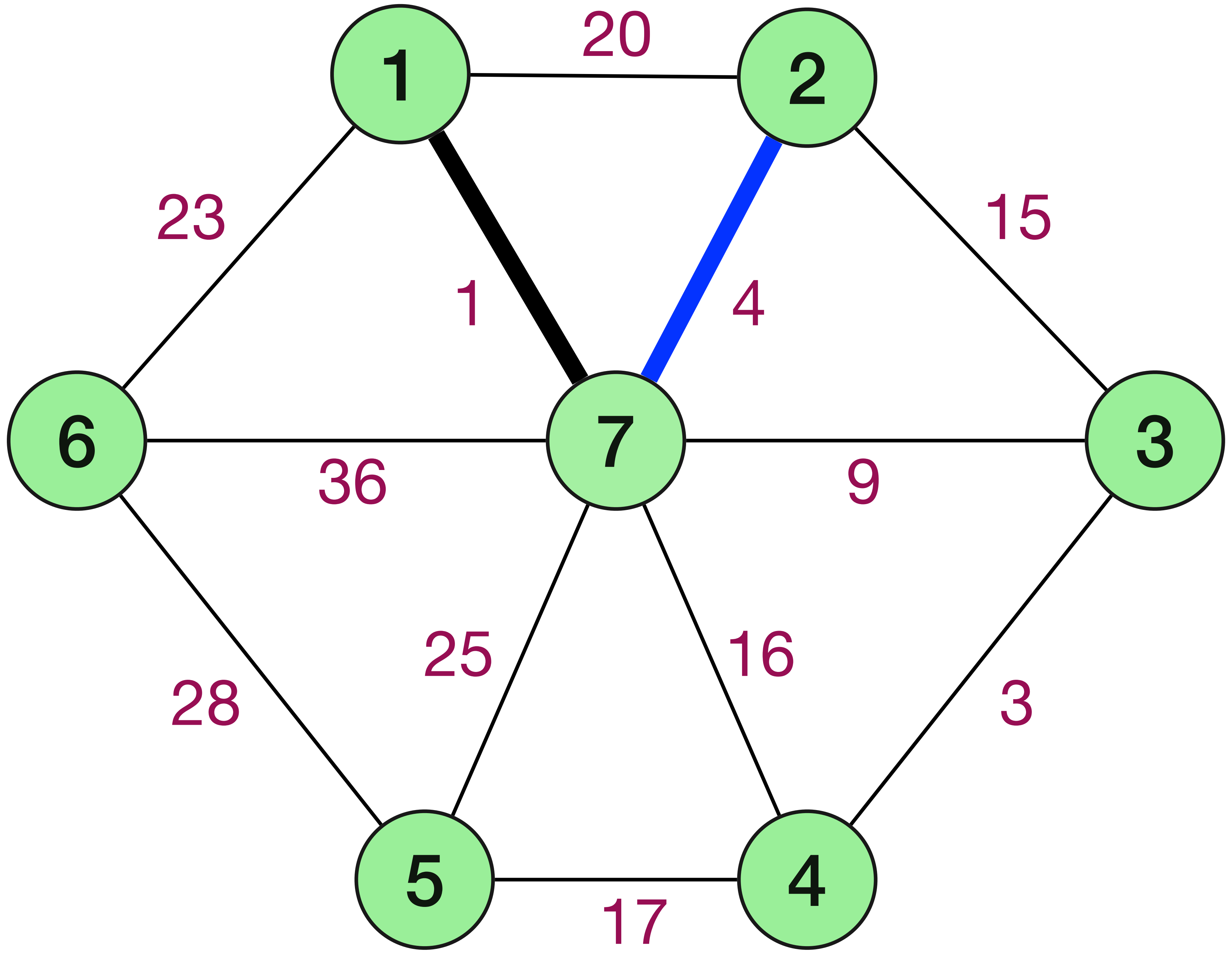

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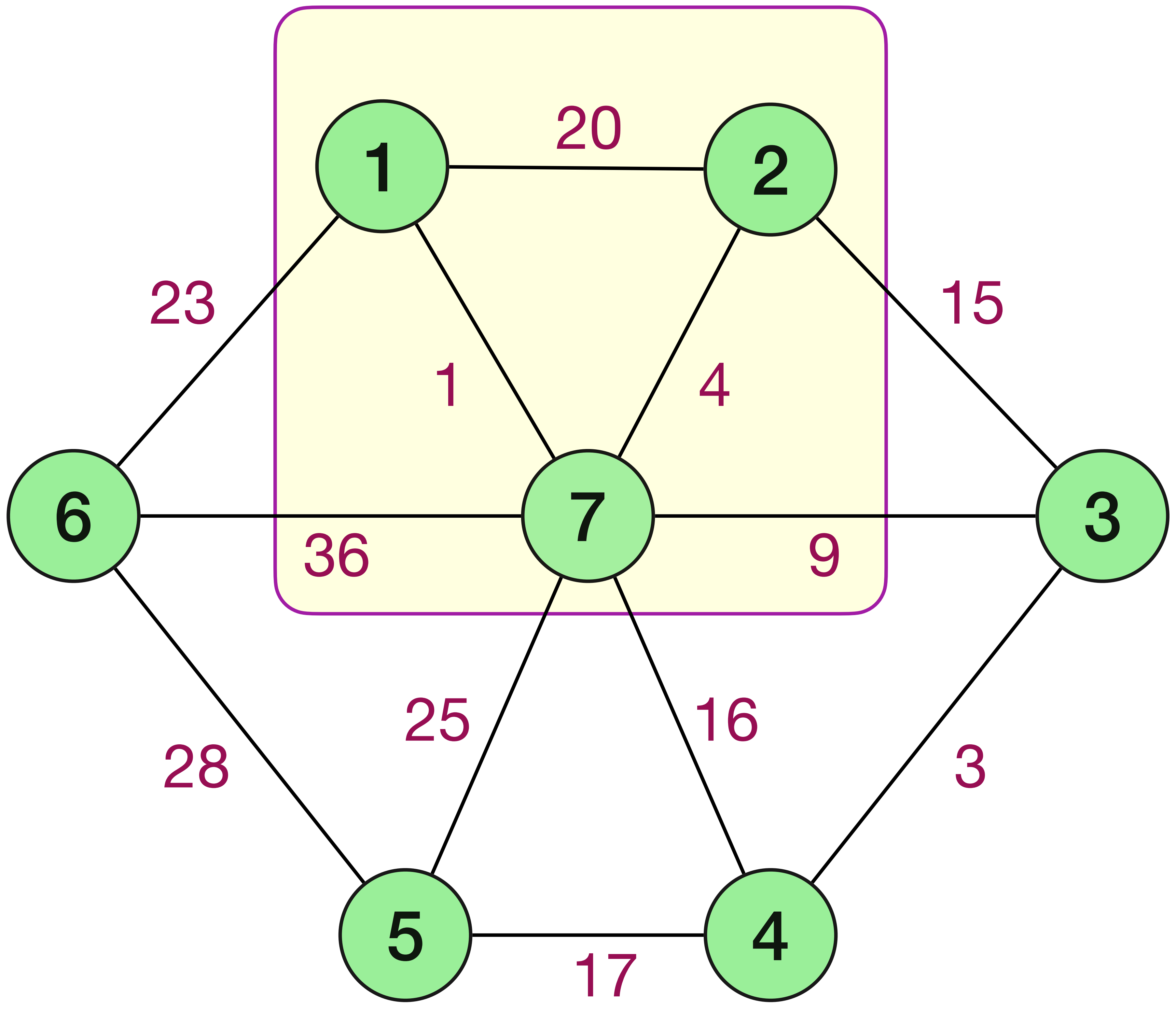

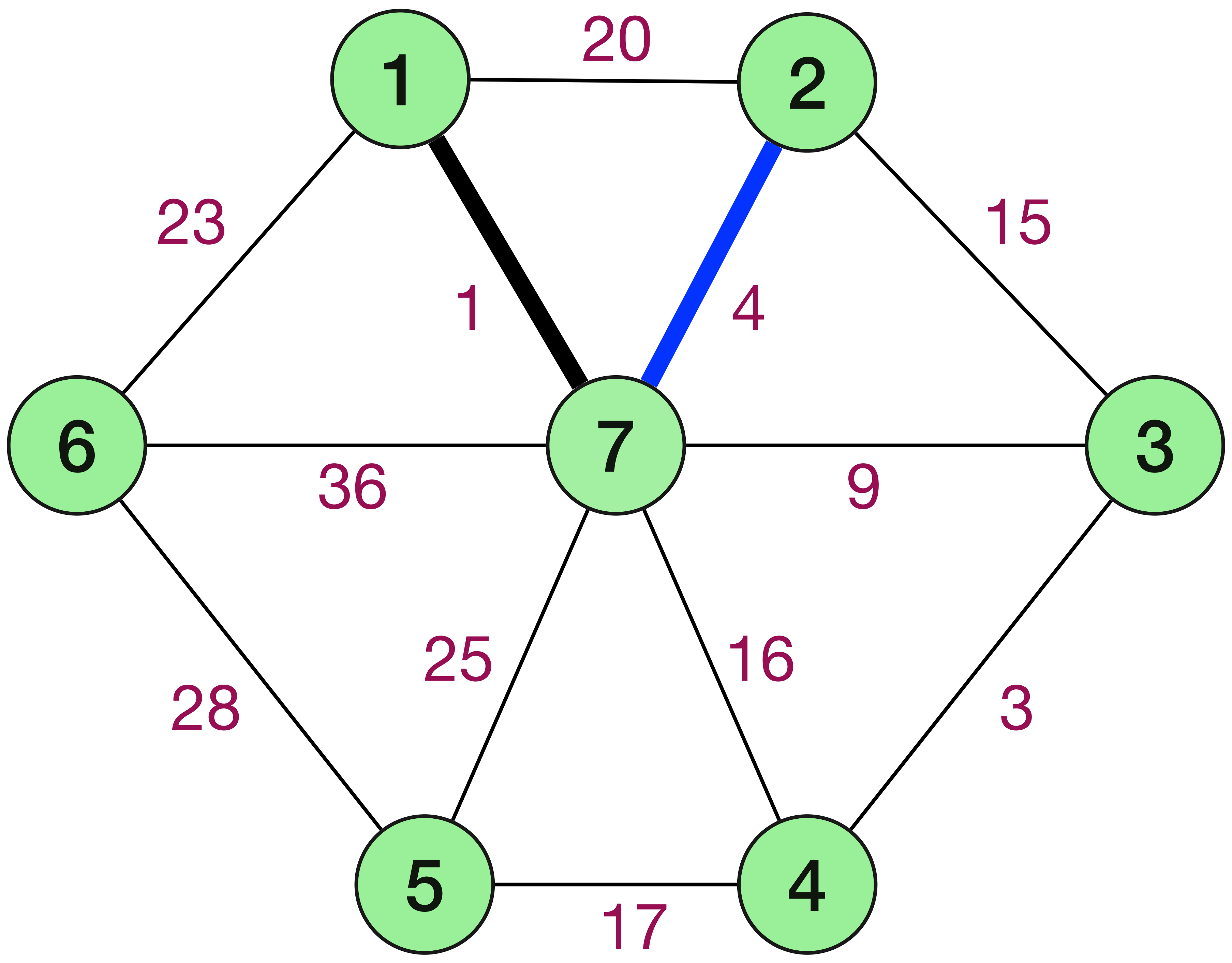

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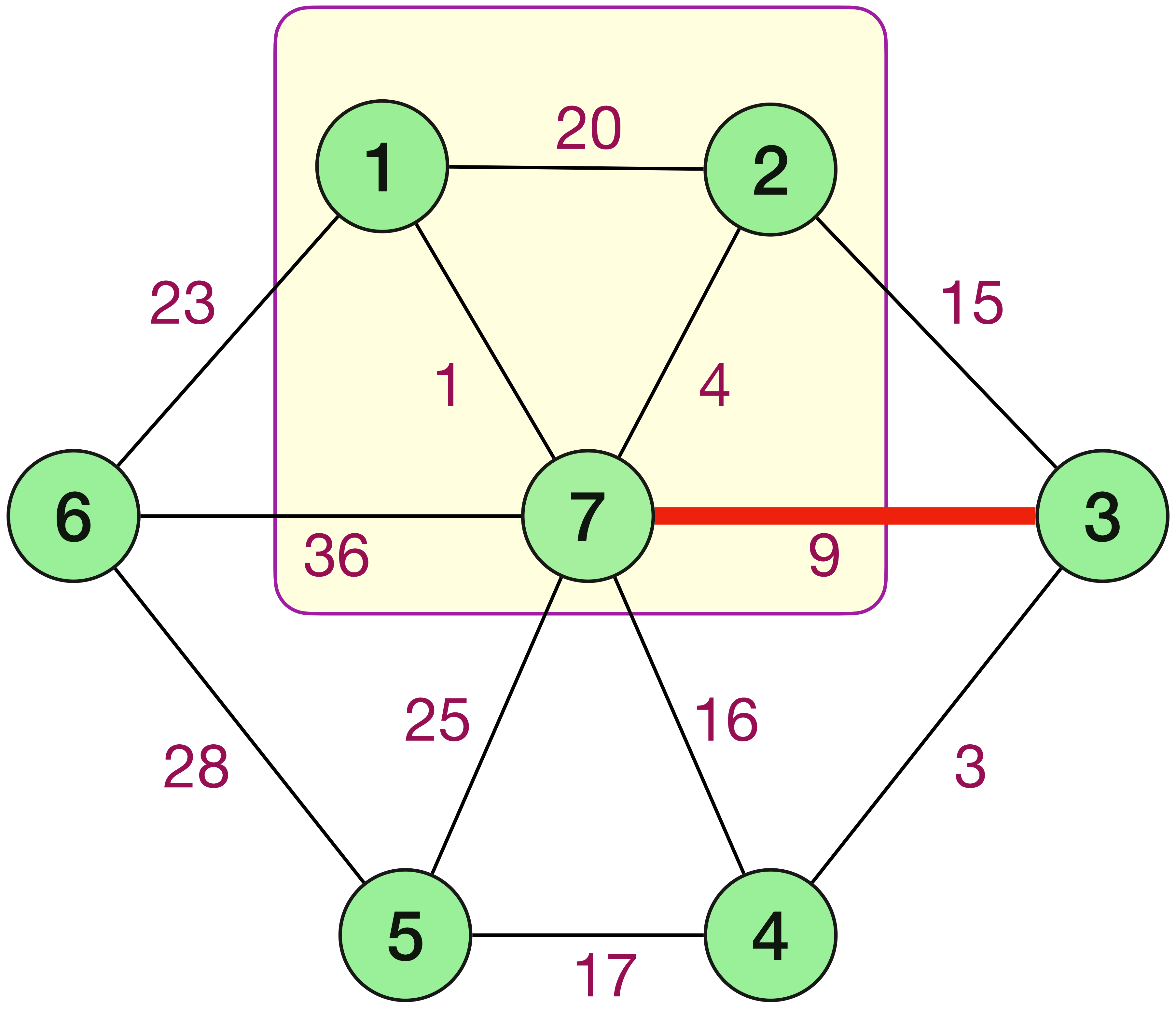

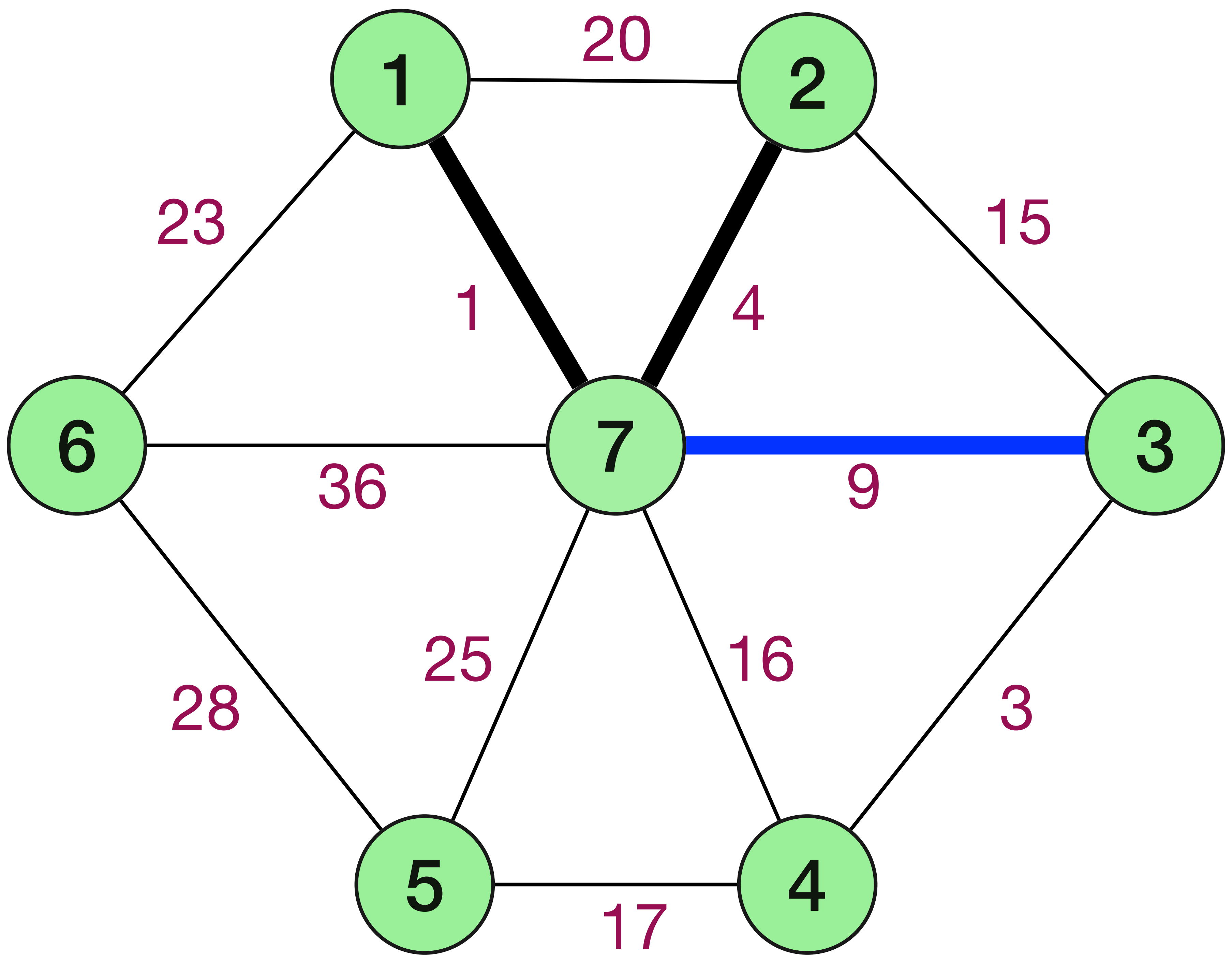

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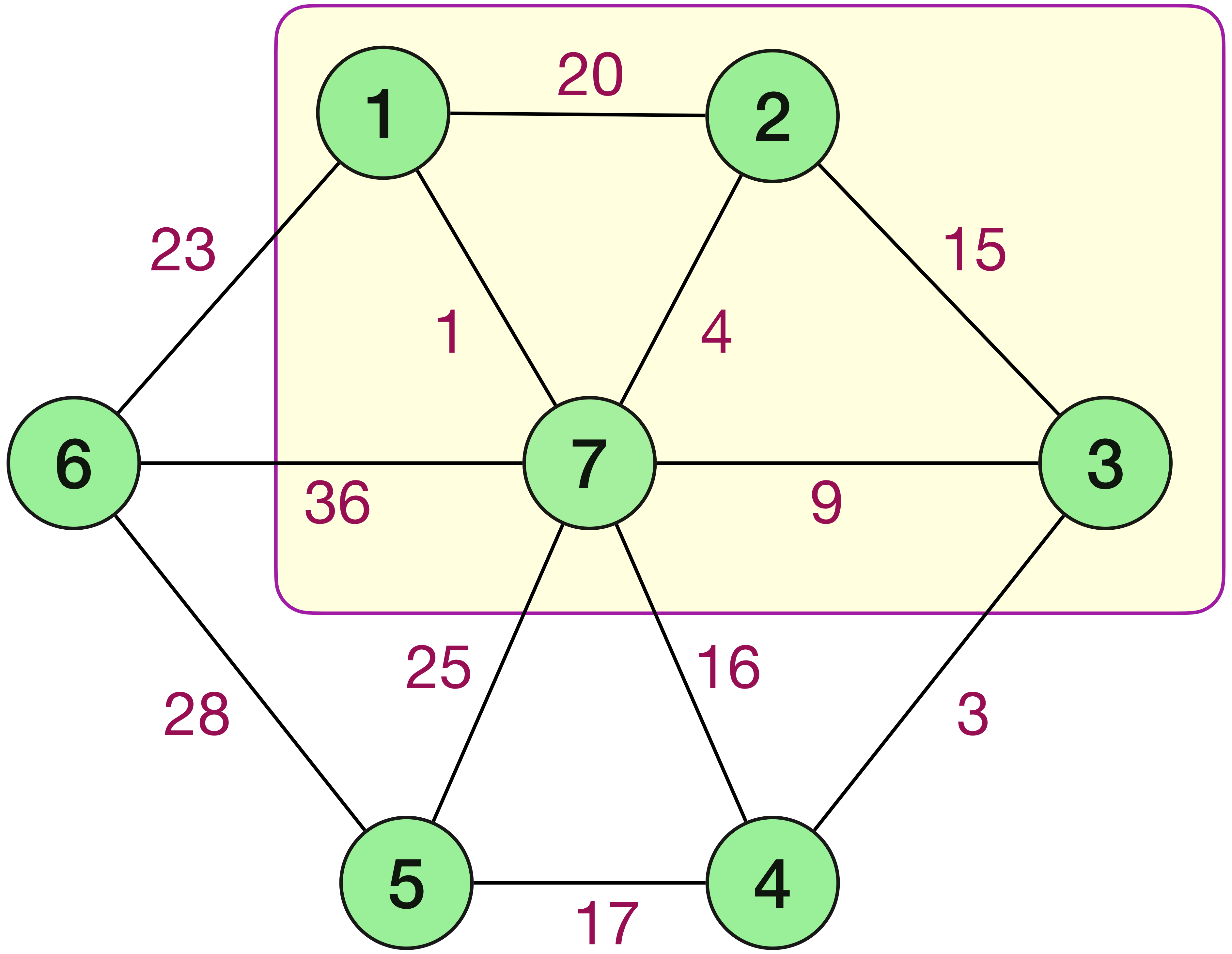

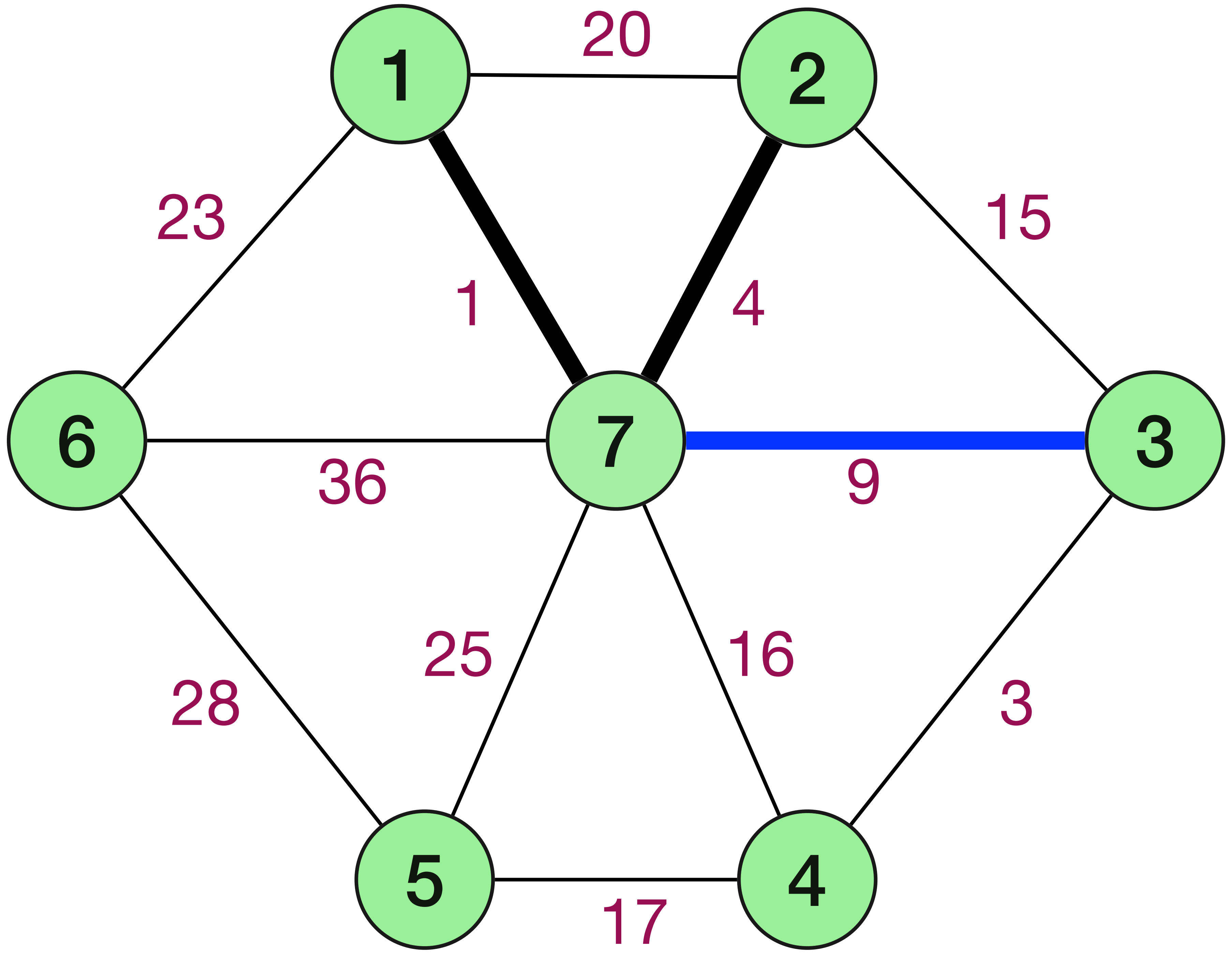

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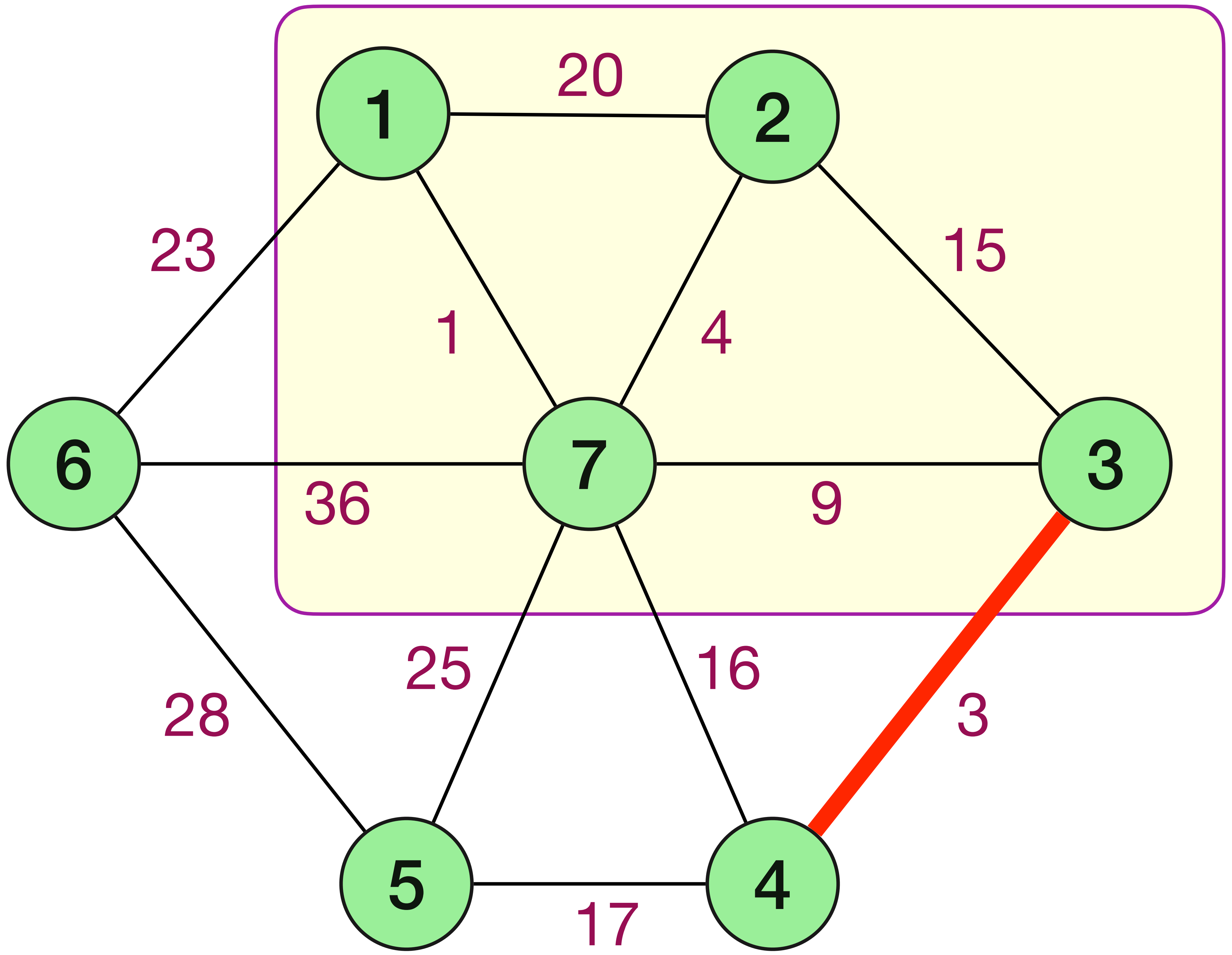

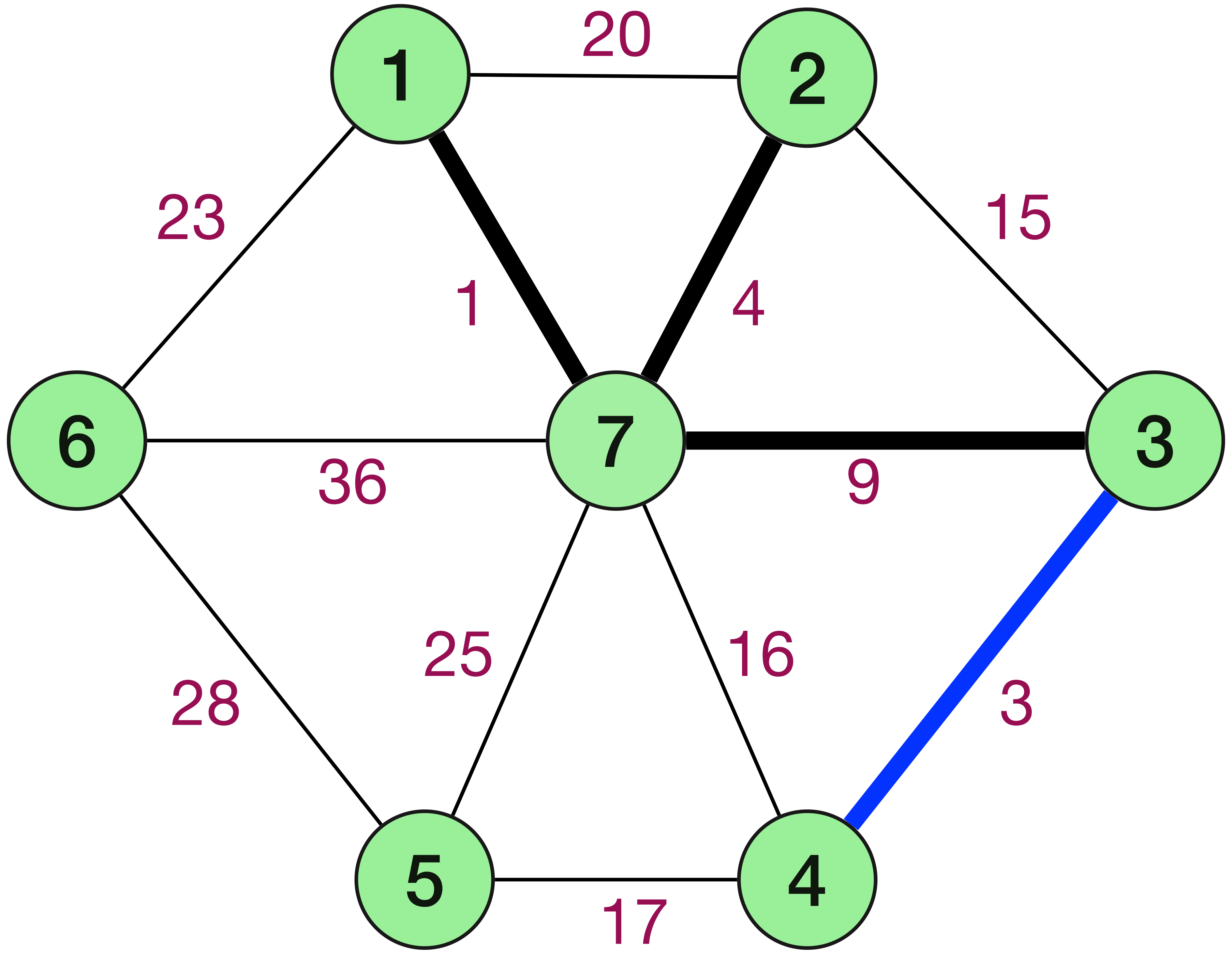

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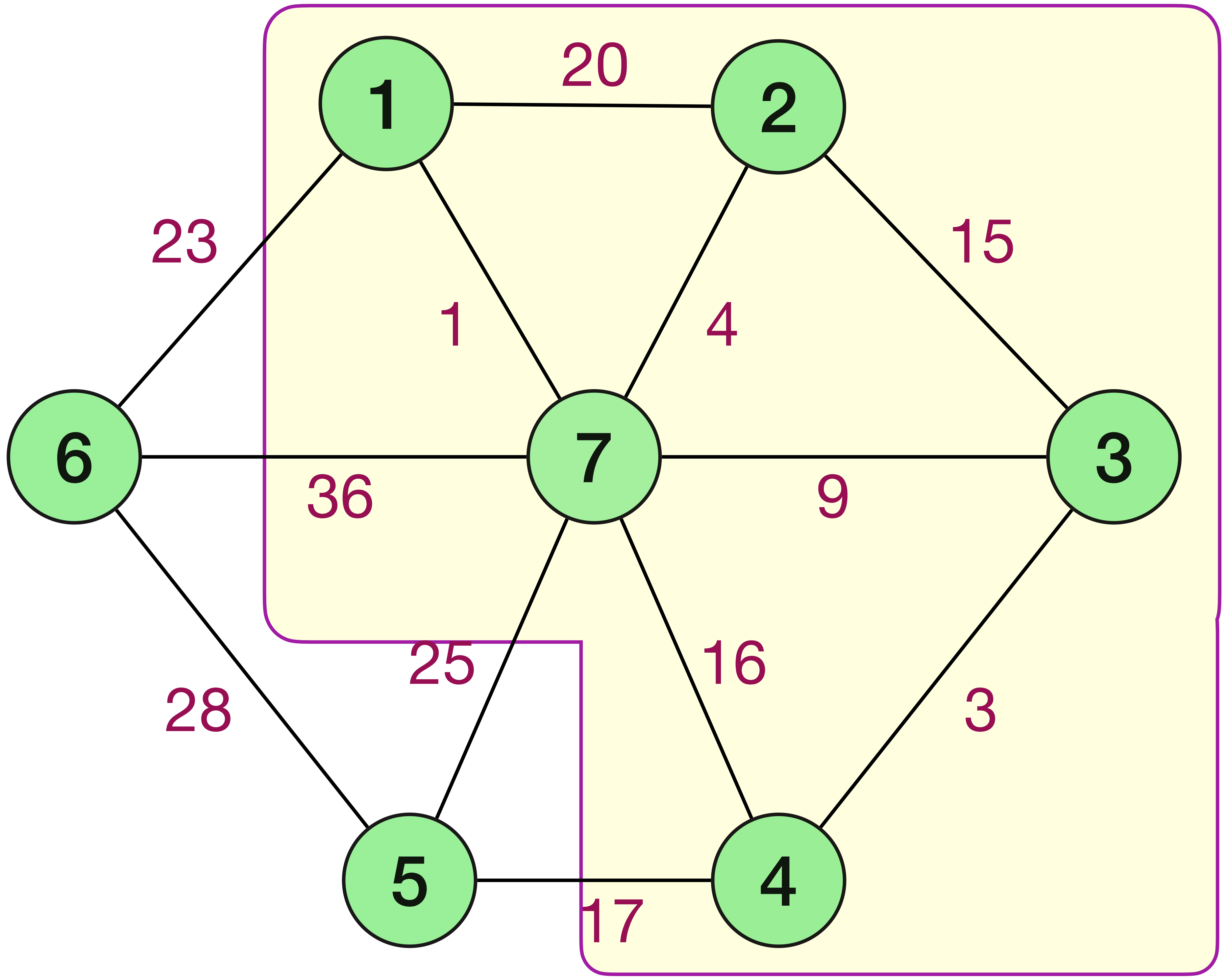

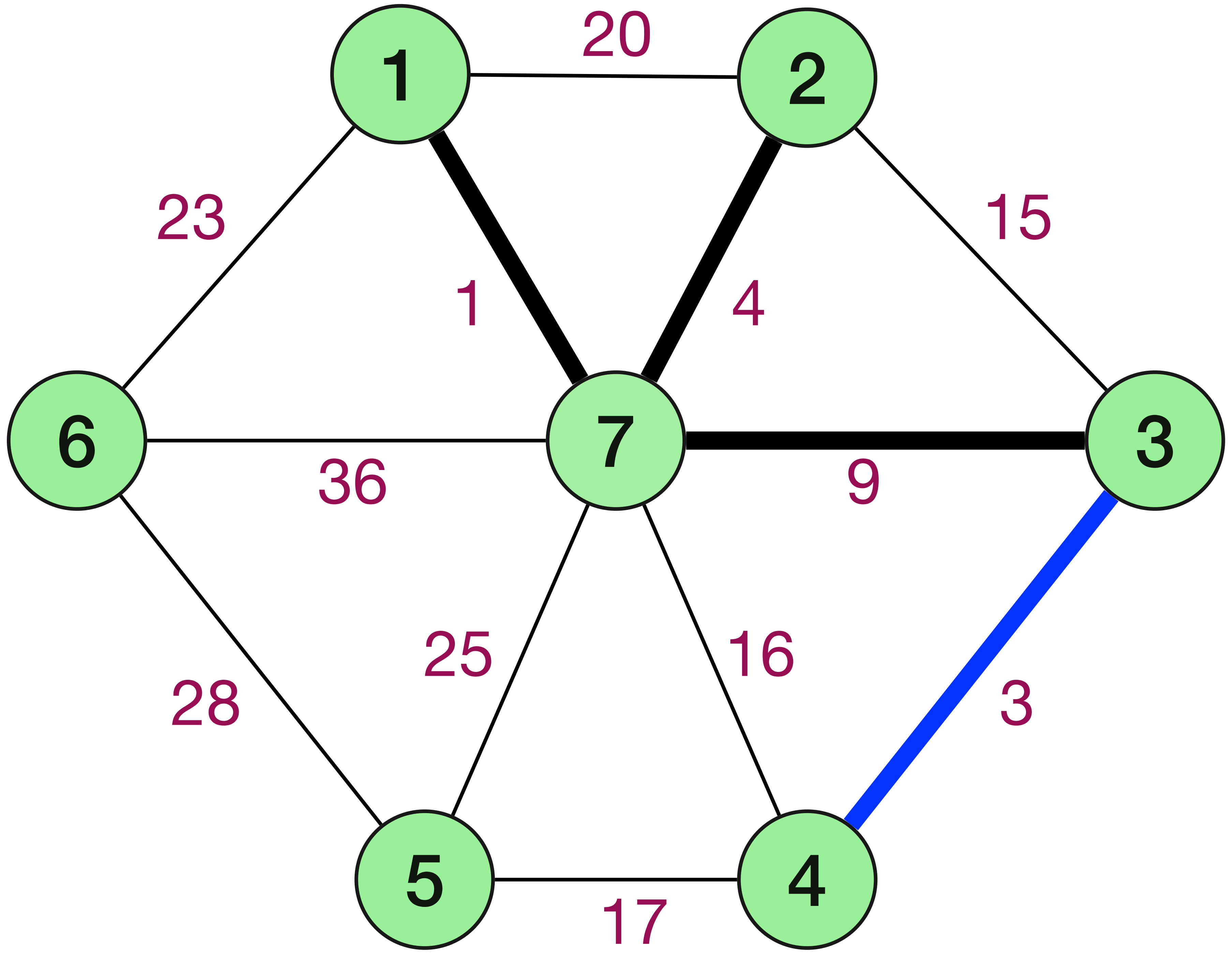

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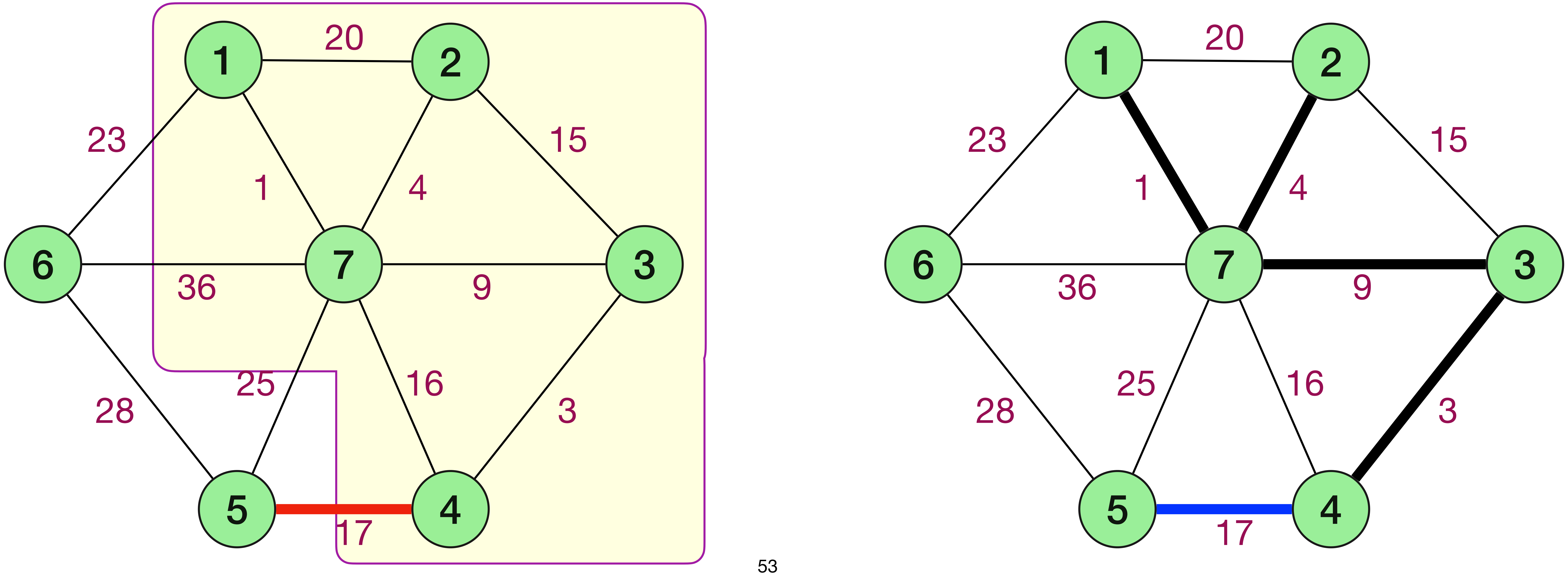

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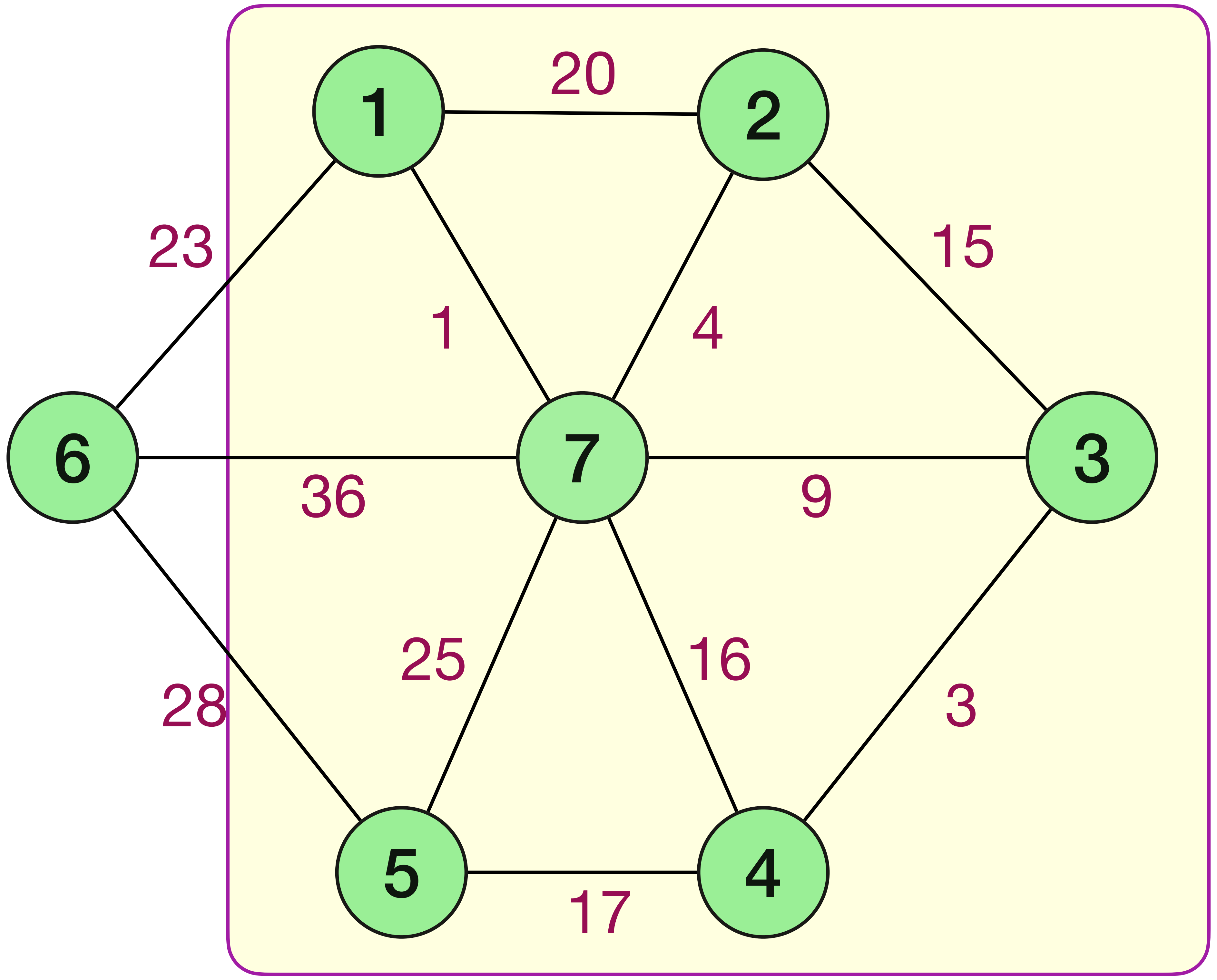

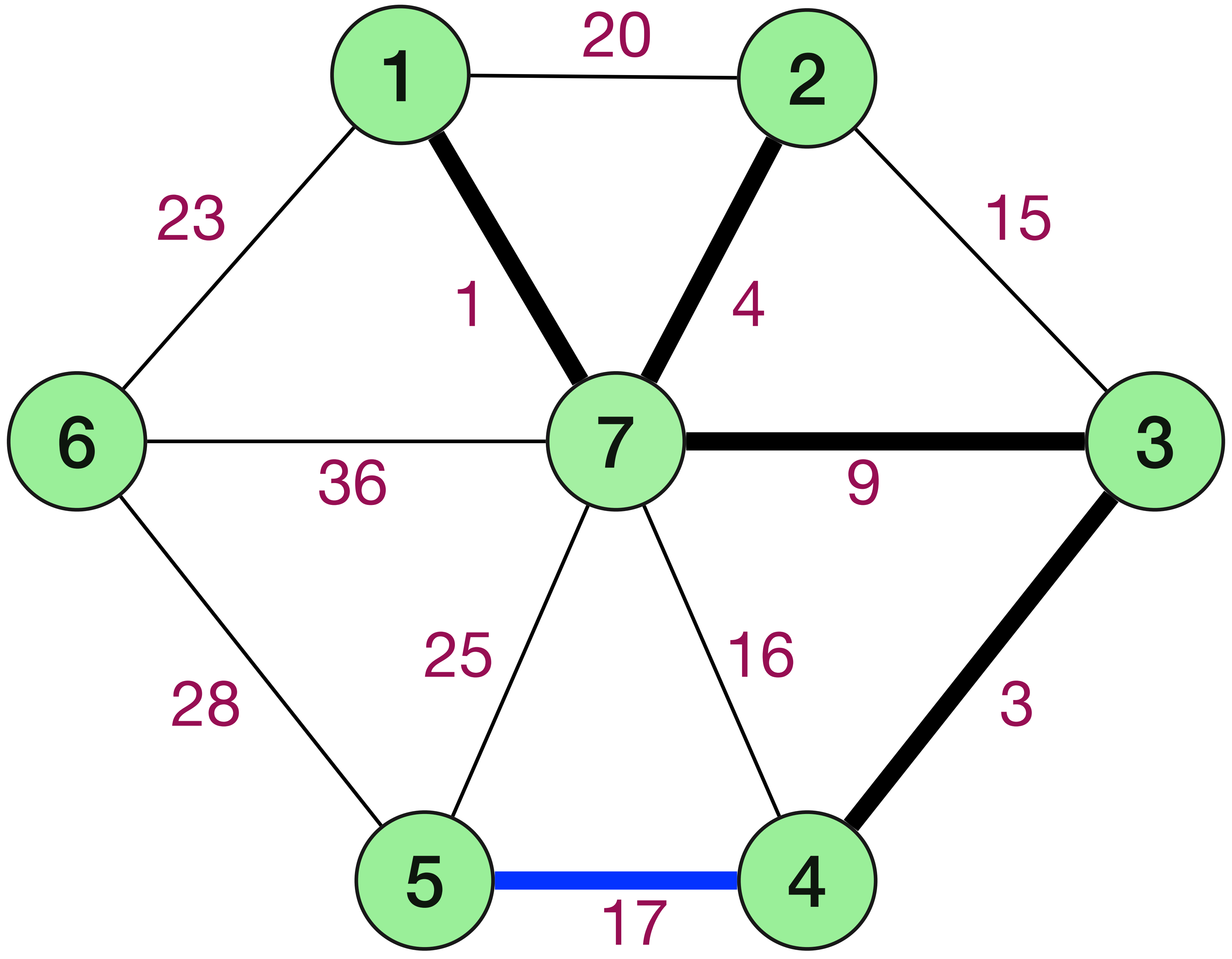

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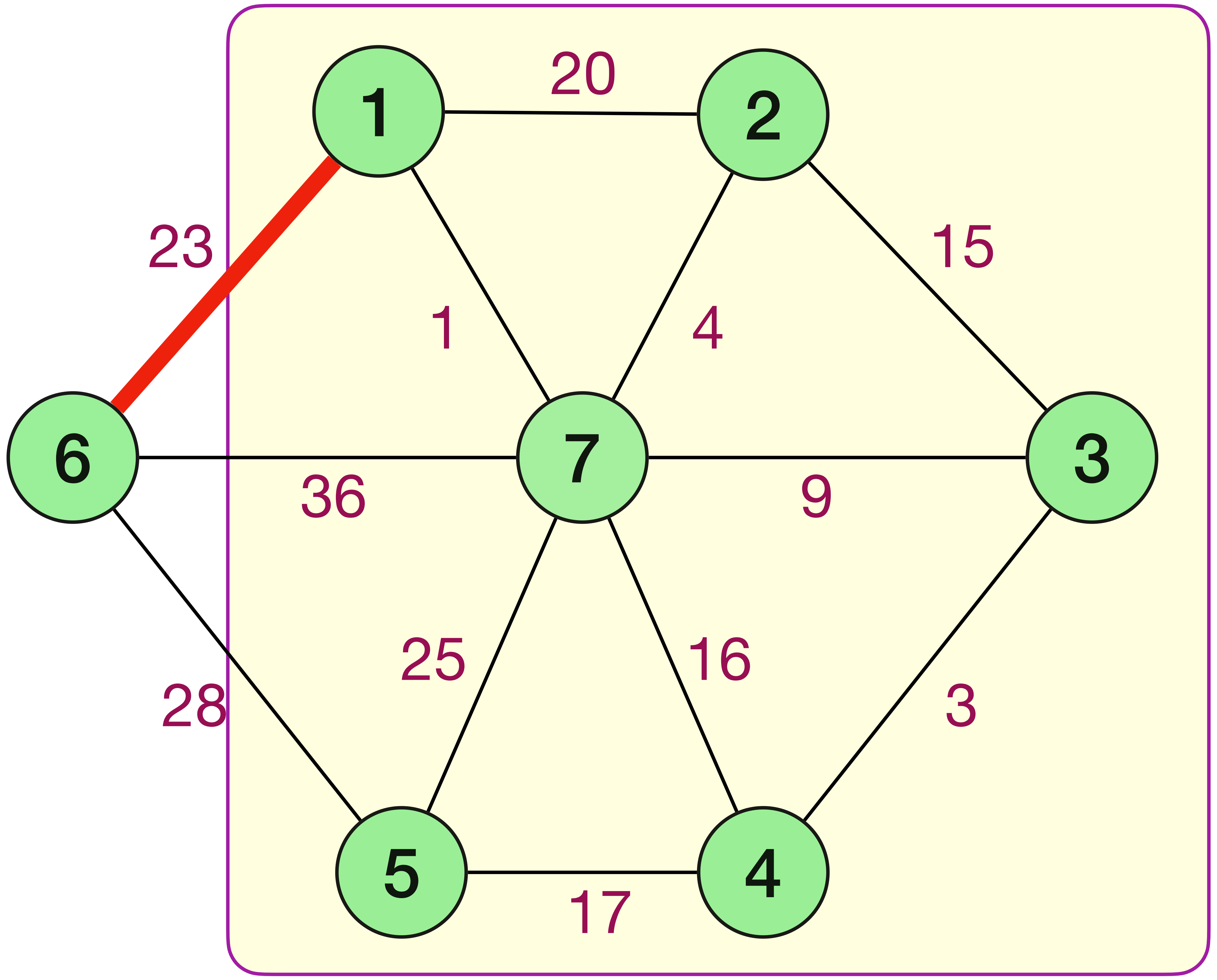

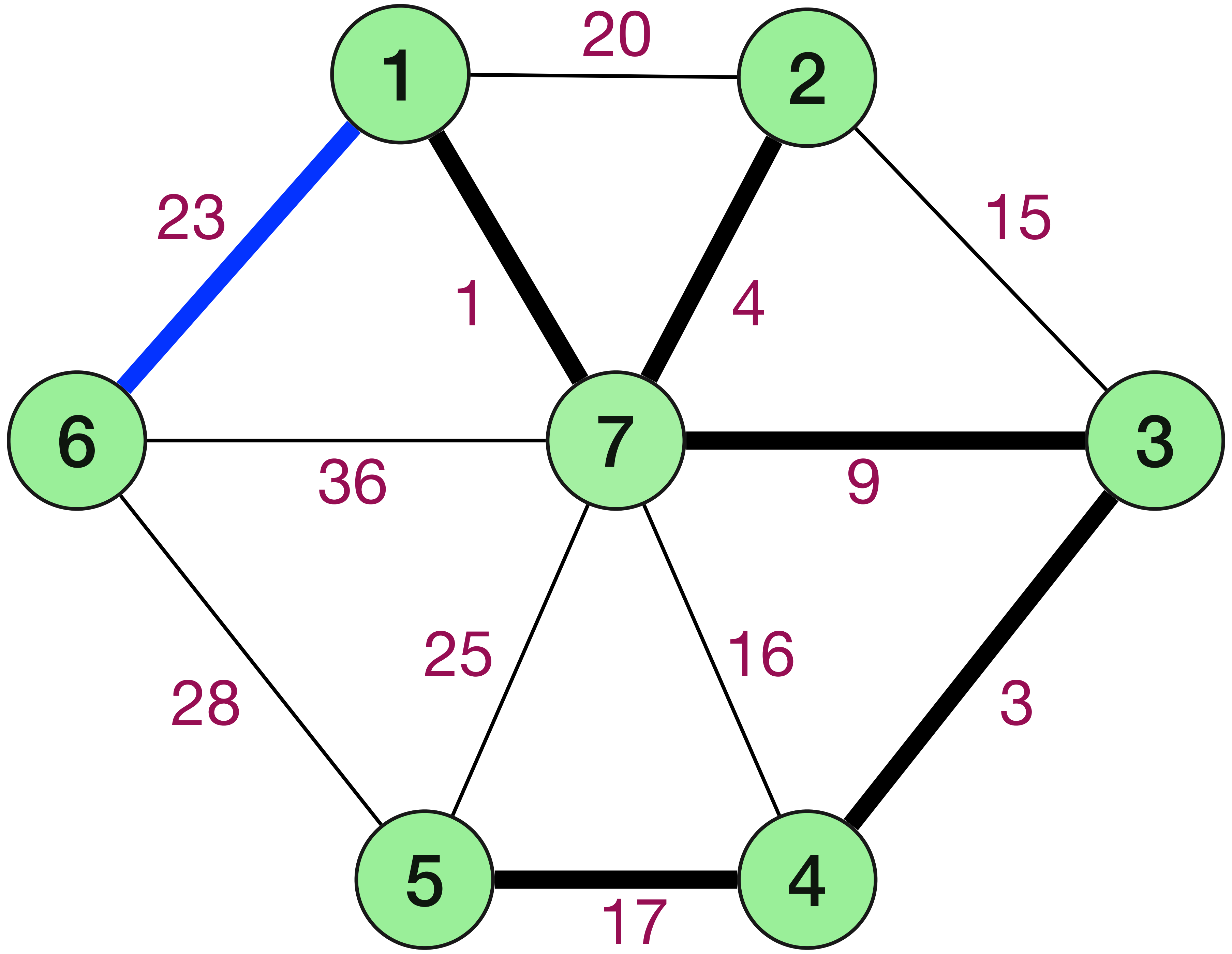

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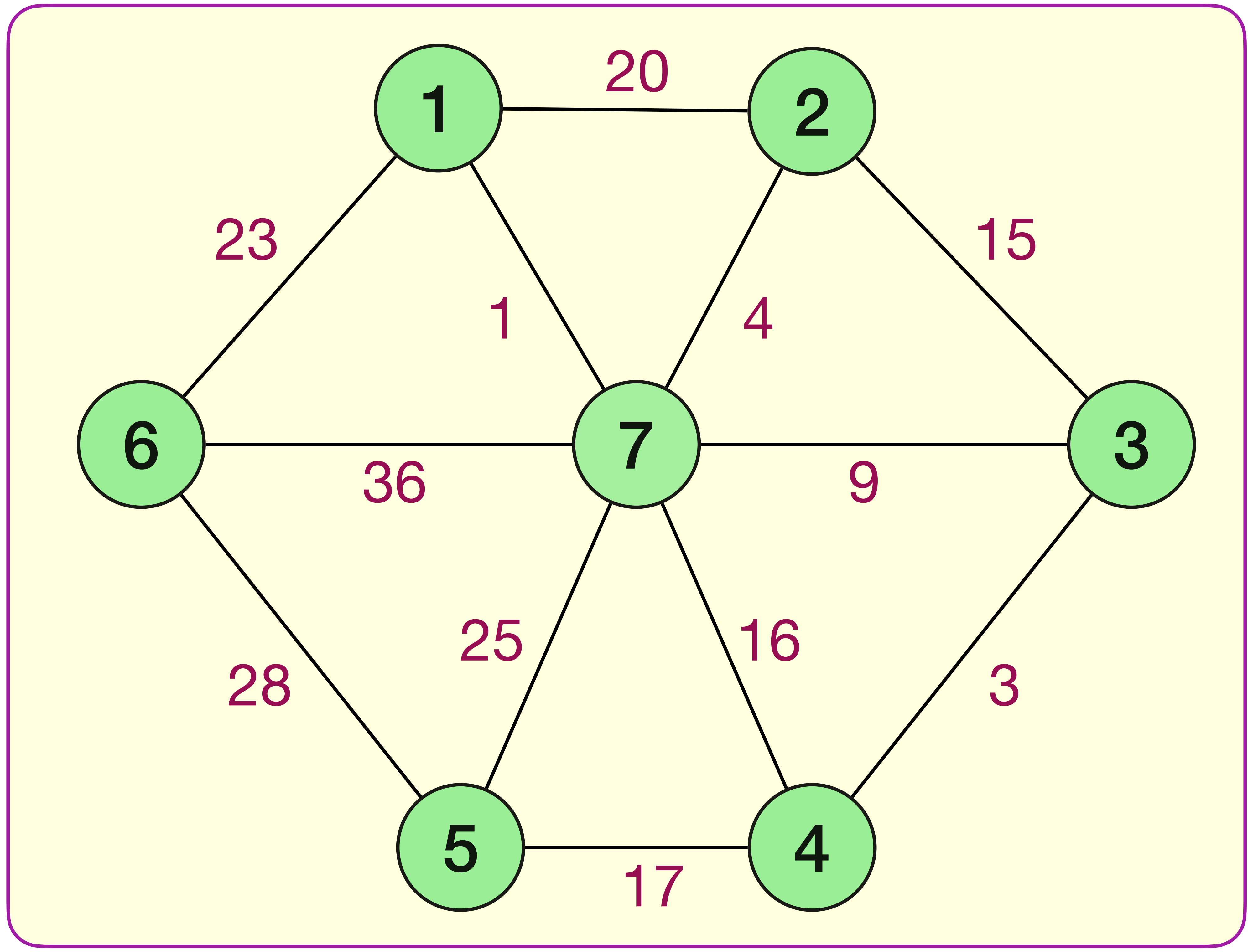

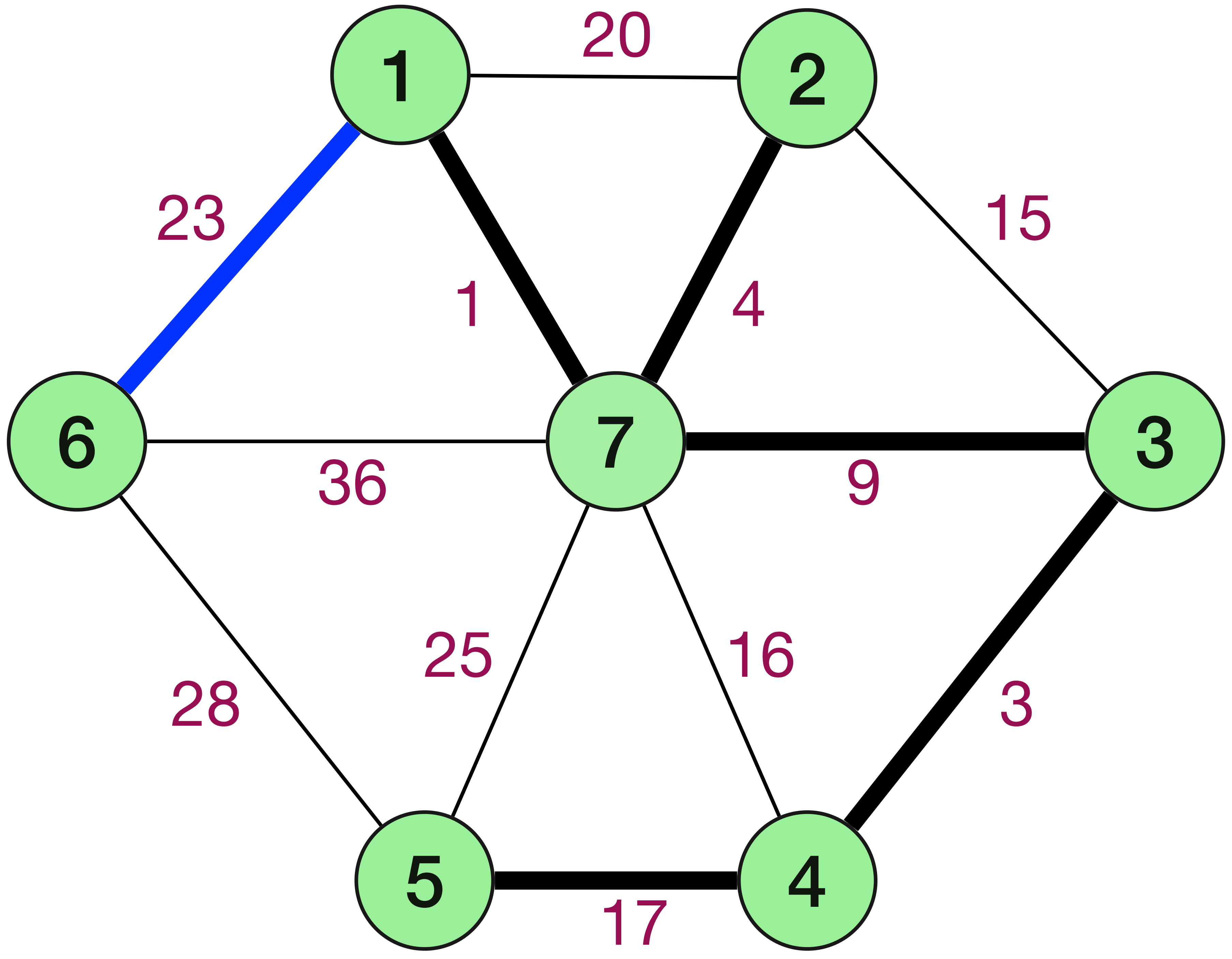

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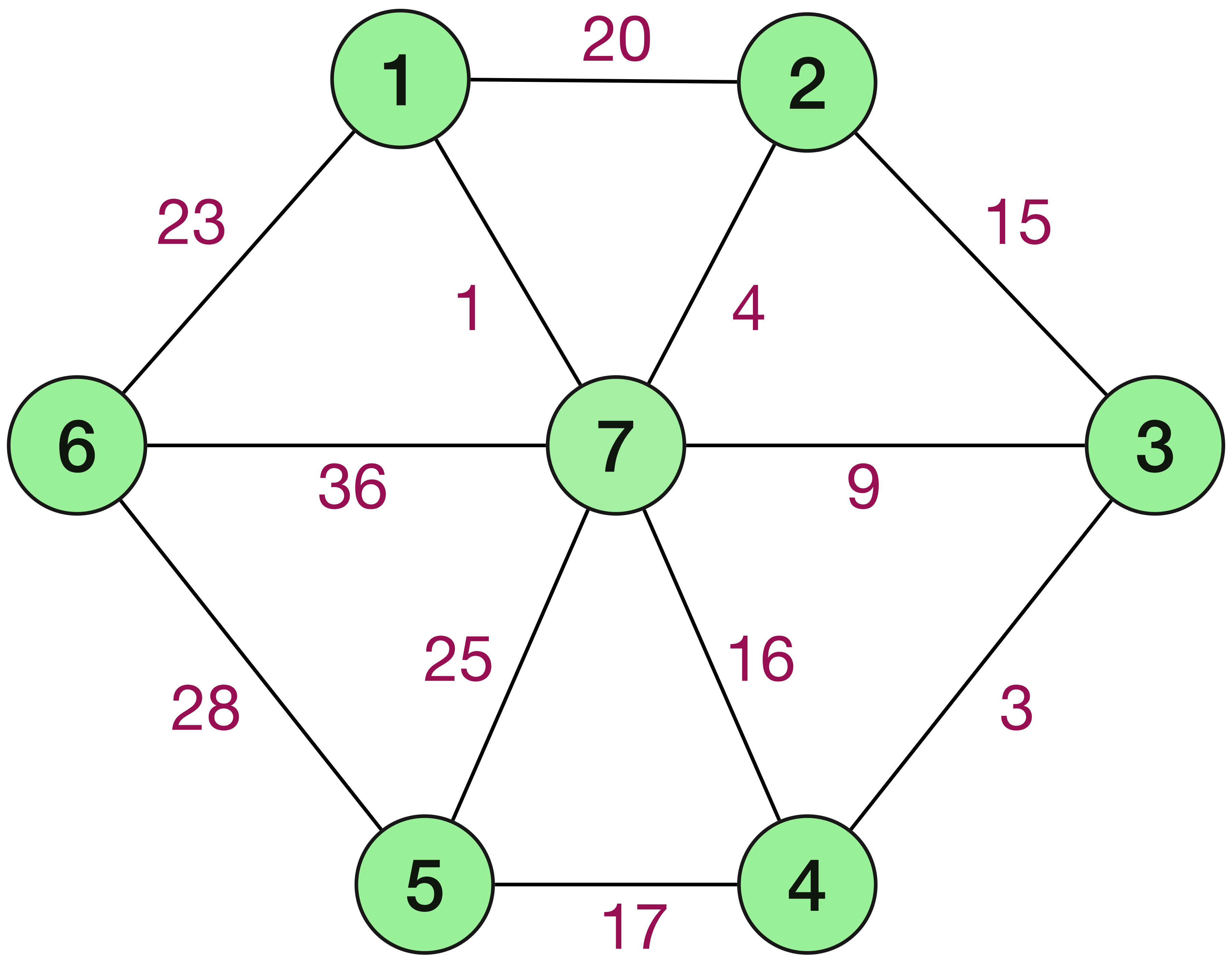

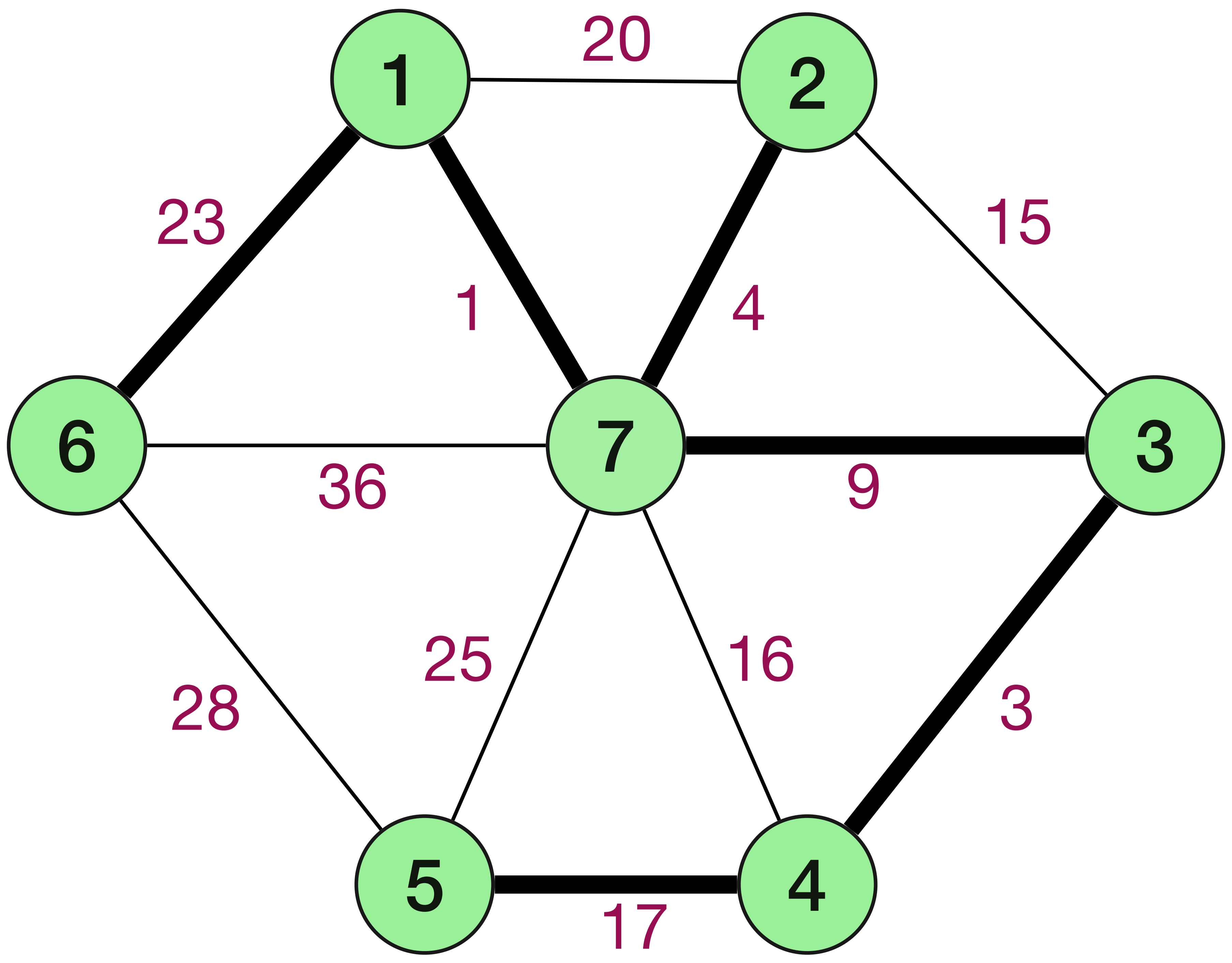

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  - Only safe edges added and they do not have a cycle

Implementing Prim's Algorithm

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Prim_ComputeMST
    E is the set of all edges in G
    S = {1}
    T is empty (* T will store edges of a MST *)
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- Total time $O(nm)$

MST algorithm for negative weights, and non-distinct costs

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Prim's and Kruskal's Algorithms are optimal with respect to lexicographic ordering.

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Question: Why does this not work for shortest paths?

MST: An epilogue

Best Known Asymptotic Running Times for MST

Prim's algorithm using Fibonacci heaps: $O(n \log n + m)$.

If m is $O(n)$ then running time is $\Omega(n \log n)$.

Question: Is there a linear time ($O(m+n)$ time) algorithm for MST?

- $O(m \log^* m)$ time **[Fredman and Tarjan 1987]**
- $O(m+n)$ time using bit operations in RAM model **[Fredman, Willard 1994]**
- $O(m+n)$ expected time (randomized algorithm) **[Karger, Klein, Tarjan 1995]**
- $O((n+m)\alpha(m,n))$ time **[Chazelle 2000]**
- Still open: Is there an $O(n+m)$ time deterministic algorithm in the comparison model?

we discussed Kosaraju's

same guy known for strongly connected component algorithms