Pre-lecture brain teaser

Consider the problem of a n-input AND function. The input (x) is a string n-digits long with $\Sigma = \{0, 1\}$ and has an output (y) which is the logical AND of all the elements of x.

Formulate a **language** that describes the above problem.

ECE-374-B: Lecture 1 - Regular Languages

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$L_{AND} = \{$ "0010111|0" → if a input has a 0 then string ends with 0

"1111|1" otherwise string ends with 1

Formats instances a string where 'Input | Output'

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~~0|1~~

$$L_{AND_N} = \left\{ \begin{array}{cccc} 0|0, & 1|1, & & \\ 0 \cdot 0|0, & 0 \cdot 1|0, & 1 \cdot 0|0, & 1 \cdot 1|1 \\ \vdots & \vdots & \vdots & \vdots \\ (0\cdot)^n|0, & (0\cdot)^{n-1}1|0, & \ldots & (1\cdot)^n|1\ldots \end{array} \right\} \tag{1}$$

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This is an example of a regular language which we'll be discussing today.

Terminology Review

- A character(a, b, c, x) is a unit of information represented by a symbol: (letters, digits, whitespace)
- A alphabet(Σ) is a set of characters
- A string(w) is a sequence of characters
- A language(A, B, C, L) is a set of strings

L = all strings

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Defining a language

How do we define a language? **Through grammars!**

What is a grammar?

Grammar (C~~F~~G) Definition

Definition
A C~~F~~G is a quadruple $G = (V, T, P, S)$

- V is a finite set of non-terminal (variable) symbols

$$G = \Big(\quad \text{Variables}, \quad \text{Terminals}, \quad \text{Productions}, \quad \text{Start var} \quad \Big)$$

Grammar (CFG) Definition

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A CFG is a quadruple $G = (V, T, P, S)$

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- T is a finite set of terminal symbols (alphabet)
- P is a finite set of productions, each of the form
  $A \to \alpha$
  where $A \in V$ and α is a string in $(V \cup T)^*$.
  Formally, $P \subset V \times (V \cup T)^*$.

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  where $A \in V$ and α is a string in $(V \cup T)^*$.

  Formally, $P \subset V \times (V \cup T)^*$.
- $S \in V$ is a start symbol

$$G = \Bigg(\quad \text{Variables}, \quad \text{Terminals}, \quad \text{Productions}, \quad \text{Start var} \quad \Bigg)$$

Example

L = all strings with 000 as a substring

- $V = \{S, A, B\}$
- $T = \{0, 1\}$
- $P = \left\{ \begin{array}{l} S \rightarrow 0S|1S|A \\ A \rightarrow 000B \\ B \rightarrow 0B|1B|\epsilon \end{array} \right\}$

  ($A \rightarrow B|C$ is abbreviation for $A \rightarrow B, A \rightarrow C$)

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What strings can S generate like this?

Example

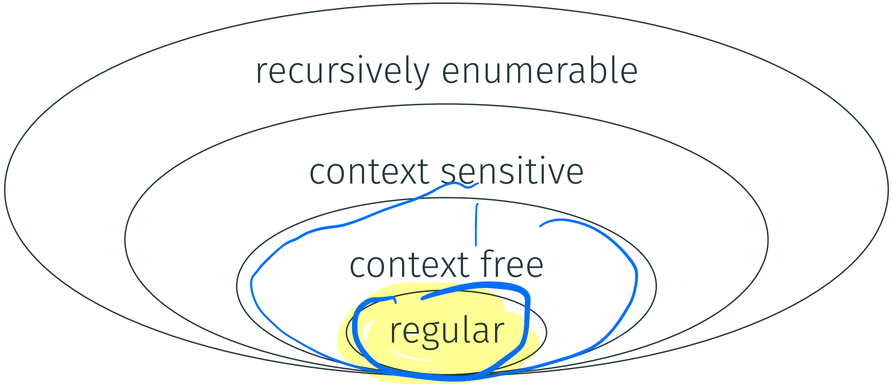
- $V = \{S, A, B\}$
- $T = \{0, 1\}$
- $P = \left\{ \begin{array}{l} S \to 0S|1S|A \\ A \to 000B \\ B \to 0B|1B|\epsilon \end{array} \right\}$

  $S \to 0S$
  $S \to 1S$
  $S \to A$

  ($A \to B|C$ is abbreviation for $A \to B, A \to C$)

$$S \rightsquigarrow 1S \rightsquigarrow 10S \rightsquigarrow 10A \rightsquigarrow 10000B \rightsquigarrow 10000\varepsilon \rightsquigarrow 10000$$

Chomsky Hierarchy



Grammar	Languages	Production Rules	Automation	Examples
Type-0	Recursively enumerable	$\gamma \to \alpha$ (no constraints)	Turing machine	$L = \{\langle M, w\rangle \mid M$ is a TM which halts on $w\}$
Type-1	Context-sensitive	$\alpha A \beta \to \alpha\gamma\beta$	Linear bounded Non-deterministic Turing machine	$L = \{a^n b^n c^n \mid n > 0\}$
Type-2	Context-free	$A \to \alpha$	Non-deterministic Push-down automata	$L = \{a^n b^n \mid n > 0\}$
Type-3	Regular	$A \to aB$	Finite State Machine	$L = \{a^n \mid n > 0\}$

Meaning of symbols: • a = terminal • A, B = variables • α, β, γ = string of $\{a \cup A\}^*$ • α, β = maybe empty —– γ = never empty

• Table borrowed from wikipedia: `https://en.wikipedia.org/wiki/Chomsky_hierarchy`

Regular Languages

Theorem (Kleene's Theorem)

A language is regular if and only if it can be obtained from finite languages by applying the three operations:

- *Union*
- *Concatenation*
- *Repetition*

a finite number of times.

Regular Languages

A class of simple but useful languages.

The set of regular languages over some alphabet Σ is defined inductively.

Base Case

- $\emptyset$ is a regular language.
- $\{\epsilon\}$ is a regular language.
- $\{a\}$ is a regular language for each $a \in \Sigma$. Interpreting a as string of length 1.

Regular Languages

$L_1^0 = \{\epsilon\}$

Inductive step:

$L_1 = \{a\}$

$L_2 = \{b\}$

We can build up languages using a few basic operations:

- If L_1, L_2 are regular then $L_1 \cup L_2$ is regular.
- If L_1, L_2 are regular then L_1L_2 is regular.
- If L is regular, then $L^* = \cup_{n\geq 0} L^n$ is regular.
  The $\cdot^*$ operator name is Kleene star.
- If L is regular, then so is $\overline{L} = \Sigma^* \setminus L$.

$L_1 \cup L_2 = \{a, b\}$

$L_1 \cdot L_2 = \{ab\} = L_3$

$L_5 = L_3 \cup L_1 \cup L_2 = \{a, b, ab\}$

Regular languages are closed under operations of union, concatenation and Kleene star.

$L_4 = L_1 \cdot L_1 = \{aa\}$

$L_4 \cup L_5 = \{a, b, aa, ab\}$

$L_1^* = \{\epsilon, a, aa, aaa, aaaa, aaaaa, \ldots\}$

Some simple regular languages

Lemma
If w is a string then $L = \{w\}$ is regular.

Example: $\{aba\}$ or $\{abbabbab\}$. Why?

$\Sigma = \{a, b\}$

$L_a = \{a\}$

$L_b = \{b\}$

$L_a \cdot L_b \cdot L_a$

$L_a \cdot L_b \cdot L_b \cdot L_a \cdots$

$\{\underbrace{a^{z^{z^{z \cdots}}}}\}$

Some simple regular languages

Lemma
If w is a string then $L = \{w\}$ is regular.

$L_a = \{a\}$

$L_b = \{b\}$

Example: $\{aba\}$ or $\{abbabbab\}$. Why?

Lemma
Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

$L_{abaab} = L_a \cdot L_b \cdot L_a \cdot L_a \cdot L_b$

$L_{aba} = L_a \cdot L_b \cdot L_a$

$L_a \cup L_{abaab} \cup L_{aba}$

Have basic operations to build regular languages.

Important: Any language generated by a finite sequence of such operations is regular.

Lemma
Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

Have basic operations to build regular languages.

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Lemma
Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

Note:Kleene star (repetition) is a **single** operation!

Regular Languages - Example

Example: The language $L_{01} = 0^i1^j|$ for all $i, j \geq 0$ is regular: $\Sigma = \{0, 1\}$

$L_0 = \{0\}$ $L_{0S} = L_0^*$

$L_1 = \{1\}$ $L_{1S} = L_1^*$

$L_{01} = L_{0S} \cdot L_{1S}$

$L_{0S} = \{ \varepsilon, 0, 00, 000, 0000 \}$

$L_{1S} = \{ \varepsilon, 1, 11, 111$

$L_{01} = \{ \varepsilon\varepsilon, \varepsilon 1, \varepsilon 11, 0\varepsilon, 01, 011, 00\varepsilon, 001, 0011, 000\varepsilon$

Rapid-fire questions - regular languages

1. $L_1 = \left\{ 0^i \,\middle|\, i = 0, 1, \ldots, \infty \right\}$. The language L_1 is regular. T/F?

L_0^*

1. $L_1 = \left\{ 0^i \,\middle|\, i = 0, 1, \ldots, \infty \right\}$. The language L_1 is regular. T/F?
2. $L_2 = \left\{ 0^{17i} \,\middle|\, i = 0, 1, \ldots, \infty \right\}$. The language L_2 is regular. T/F?

$L_{170's} = \{ \text{"}00000000000000000\text{"} \}$

$L_{170's} = L_0 \cdot L_0 \cdot L_0 \cdot \;\cdot\;\cdot$

16 ops

$L_2 = L_{170's}^*$

Rapid-fire questions - regular languages

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2. $L_2 = \left\{ 0^{17i} \;\middle|\; i = 0, 1, \ldots, \infty \right\}$. The language L_2 is regular. T/F?
3. $L_3 = \left\{ 0^i \;\middle|\; i \text{ is divisible by } 2, 3, \text{or } 5 \right\}$. L_3 is regular. T/F?

$$L_3 = \bigcup \begin{cases} L_{\%2} = (L_0 \cdot L_0)^* \\ L_{\%3} = (L_0 \cdot L_0 \cdot L_0)^* \\ L_{\%5} = (L_0 \cdot L_0 \cdot L_0 \cdot L_0 \cdot L_0)^* \end{cases}$$

Rapid-fire questions - regular languages

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3. $L_3 = \left\{ 0^i \;\middle|\; i \text{ is divisible by } 2, 3, \text{or } 5 \right\}$. L_3 is regular. T/F?
4. $L_4 = \{ w \in \{0, 1\}^* \mid w \text{ has at most 2 1s} \}$. L_4 is regular. T/F? $\Sigma = \{0, 1\}$

$L_4 = L_{\text{HAS 0 1's}} \cup L_{\text{HAS 1 1's}} \cup L_{\text{HAS 2 1's}}$

$L_{\text{HAS 0 1's}} \to L_0^*$

$L_{\text{HAS 1 1's}} \to L_0^* \cdot L_1 \cdot L_0^*$

$L_{\text{HAS 2 1's}} \to L_0^* L_1 L_0^* L_1 L_0^*$

$L_\epsilon = \{\epsilon\}$

$L_0^* \cdot L_1 \cdot L_0^* \to L_0^* \cdot L_1^* \cdot L_1^* \cdot \epsilon \cdot L_0^*$

$L_{\text{maybe } 1?} = \{1, \epsilon\} = L_\epsilon \cup L_1$

$L_4 = L_0^* L_{1?} L_0^* L_{1?} L_0^* \cdot L_\epsilon \cdot L_\epsilon$

$L_0^* \cdot \epsilon \cdot L_0^* \cdot L_1 \cdot L_0^*$

$L_0^* \cdot \epsilon \cdot L_0^* \cdot \epsilon \cdot L_0^*$

Regular Expressions

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star names after him [1].

Inductive Definition

A regular expression $\mathbf{r}$ over an alphabet Σ is one of the following:

Base cases:

- $\emptyset$ denotes the language $\emptyset$
- ϵ denotes the language $\{\epsilon\}$.
- a denote the language $\{a\}$.

Inductive cases: If $\mathbf{r}_1$ and $\mathbf{r}_2$ are regular expressions denoting languages R_1 and R_2 respectively then,

- $(\mathbf{r}_1 + \mathbf{r}_2)$ denotes the language $R_1 \cup R_2$
- $(\mathbf{r}_1 \bullet \mathbf{r}_2) = r_1 \bullet r_2 = (\mathbf{r}_1\mathbf{r}_2)$ denotes the language R_1R_2
- $(\mathbf{r}_1)^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages	Regular Expressions
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	$\mathbf{a}$ denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r_1} + \mathbf{r_2}$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r_1} \bullet \mathbf{r_2}$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

Notation and Parenthesis

- For a regular expression **r**, $L(\mathbf{r})$ is the language denoted by **r**. Multiple regular expressions can denote the same language!
  Example: $(0+1)$ and $(1+0)$ denotes same language $\{0,1\}$

$L = \{\text{"0"}, \text{"1"}\}$

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  Example: $rst = (rs)t = r(st)$, $r + s + t = r + (s + t) = (r + s) + t$.

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- Omit parenthesis by associativity of each operation.
  Example: $rst = (rs)t = r(st)$, $r + s + t = r + (s + t) = (r + s) + t$.
- Superscript $+$. For convenience, define $\mathbf{r}^+ = \mathbf{r}\mathbf{r}^*$. Hence if $L(\mathbf{r}) = R$ then $L(\mathbf{r}^+) = R^+$.
- Other notation: $r + s$, $r \cup s$, $r|s$ all denote union. rs is sometimes written as $r \bullet s$.

Some examples of regular expressions

Creating regular expressions

1. All strings that end in 1011?

$$\Sigma^* \cdot 1011$$

$$(0+1)^* 1011$$

Creating regular expressions

1. All strings that end in 1011?
2. All strings except 11?

111

$r = \varepsilon + 0 + 1 + 00 + 01 + 10 + \Sigma\Sigma\Sigma\Sigma^*$

$1\,1\,1\,\varepsilon$

$\Sigma^* \setminus 11$

$\Sigma\Sigma\Sigma$

Creating regular expressions

1. All strings that end in 1011?
2. All strings except 11?
3. All strings that do not contain 000 as a subsequence?

Creating regular expressions

1. All strings that end in 1011?
2. All strings except 11?
3. All strings that do not contain 000 as a subsequence?
4. All strings that do not contain the substring 10?

Interpreting regular expressions

1. $(0+1)^*$:

Interpreting regular expressions

1. $(0+1)^*$:
2. $(0+1)^*001(0+1)^*$:

Interpreting regular expressions

1. $(0+1)^*$:
2. $(0+1)^*001(0+1)^*$:
3. $0^* + (0^*10^*10^*10^*)^*$:

Interpreting regular expressions

1. $(0+1)^*$:
2. $(0+1)^*001(0+1)^*$:
3. $0^* + (0^*10^*10^*10^*)^*$:
4. $(\epsilon+1)(01)^*(\epsilon+0)$:

Tying everything together

Consider the problem of a n-input AND function. The input (x) is a string n-digits long with an input alphabet $\Sigma_i = \{0, 1\}$ and has an output (y) which is the logical AND of all the elements of x. We knwo the language used to describe it is:

$$L_{AND_N} = \left\{ \begin{matrix} 0 \cdot |0, & 1 \cdot |1, & & \\ 0 \cdot 0 \cdot |0, & 0 \cdot 1 \cdot |0, & 1 \cdot 0 \cdot |0, & 1 \cdot 1 \cdot |1 \\ \vdots & \vdots & \vdots & \vdots \\ (0 \cdot)^n|0, & (0 \cdot)^{n-1}1|0, & \ldots & (1 \cdot)^n|1 \ldots \end{matrix} \right\}$$

Formulate the regular expression which describes the above language:

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Formulate the regular expression which describes the above language:

$$\Sigma = \{0, 1, `\cdot', `|'\}\ r_{AND_N} = \underbrace{(\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* \text{“}0\cdot\text{”}(\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* \text{“}|0\text{”}}_{\text{all output 0 instances}} + \overbrace{(\text{“}1\cdot\text{”})^* \text{“}|1\text{”}}^{\text{all output 1 instances}}$$

Regular expressions in programming

One last expression....

Bit strings with odd number of 0s and 1s

Bit strings with odd number of 0s and 1s

The regular expression is

$$(00+11)^*(01+10)$$
$$\Big(00+11+(01+10)(00+11)^*(01+10)\Big)^*$$

Bit strings with odd number of 0s and 1s

The regular expression is

$$(00+11)^*(01+10)$$
$$\Big(00+11+(01+10)(00+11)^*(01+10)\Big)^*$$

(Solved using techniques to be presented in the following lectures...)