NP-C problems & reductions redux

Sides based on material by Kani, Erickson, Chekuri, et. al.

All mistakes are my own! - Ivan Abraham (Fall 2024)

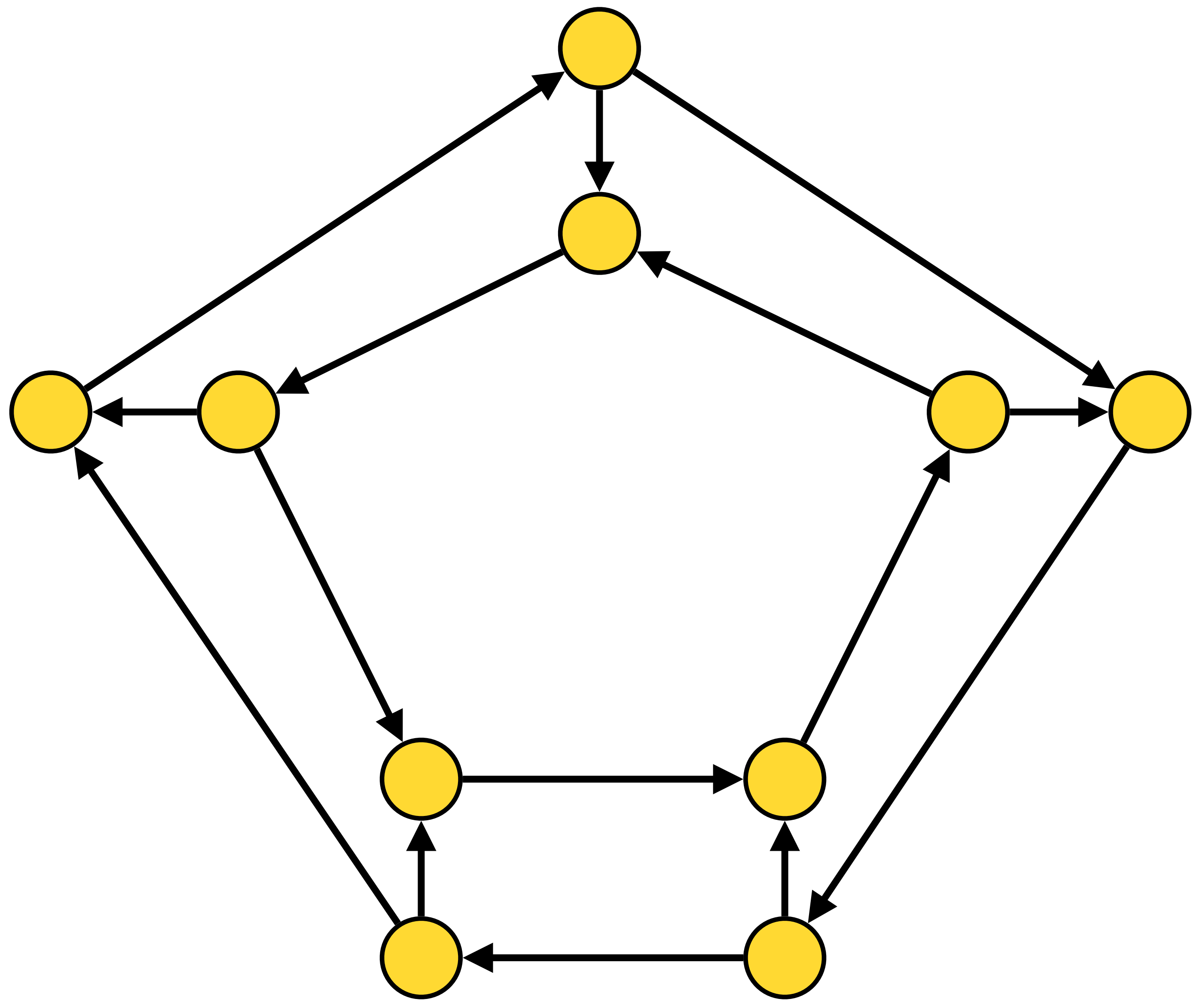
Image by ChatGPT (probably collaborated with DALL-E)

Reduction from 3SAT to Hamiltonian cycle

Directed Hamiltonian cycle

Input: Given a directed graph $G = (V, E)$ with n vertices.

Goal: Does G have a Hamiltonian cycle?



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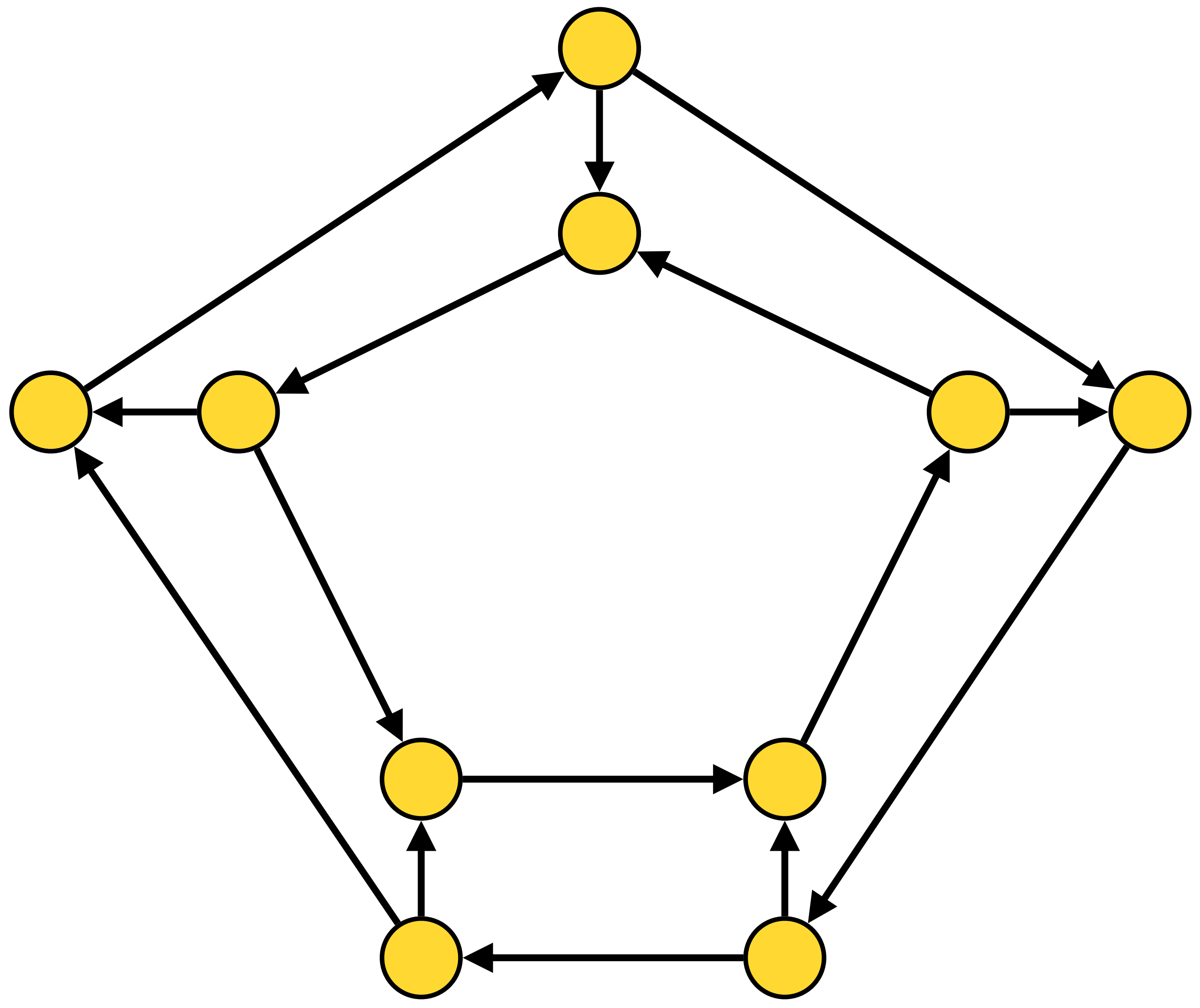
A ***Hamiltonian cycle*** is a cycle in the graph that visits every vertex in G exactly once



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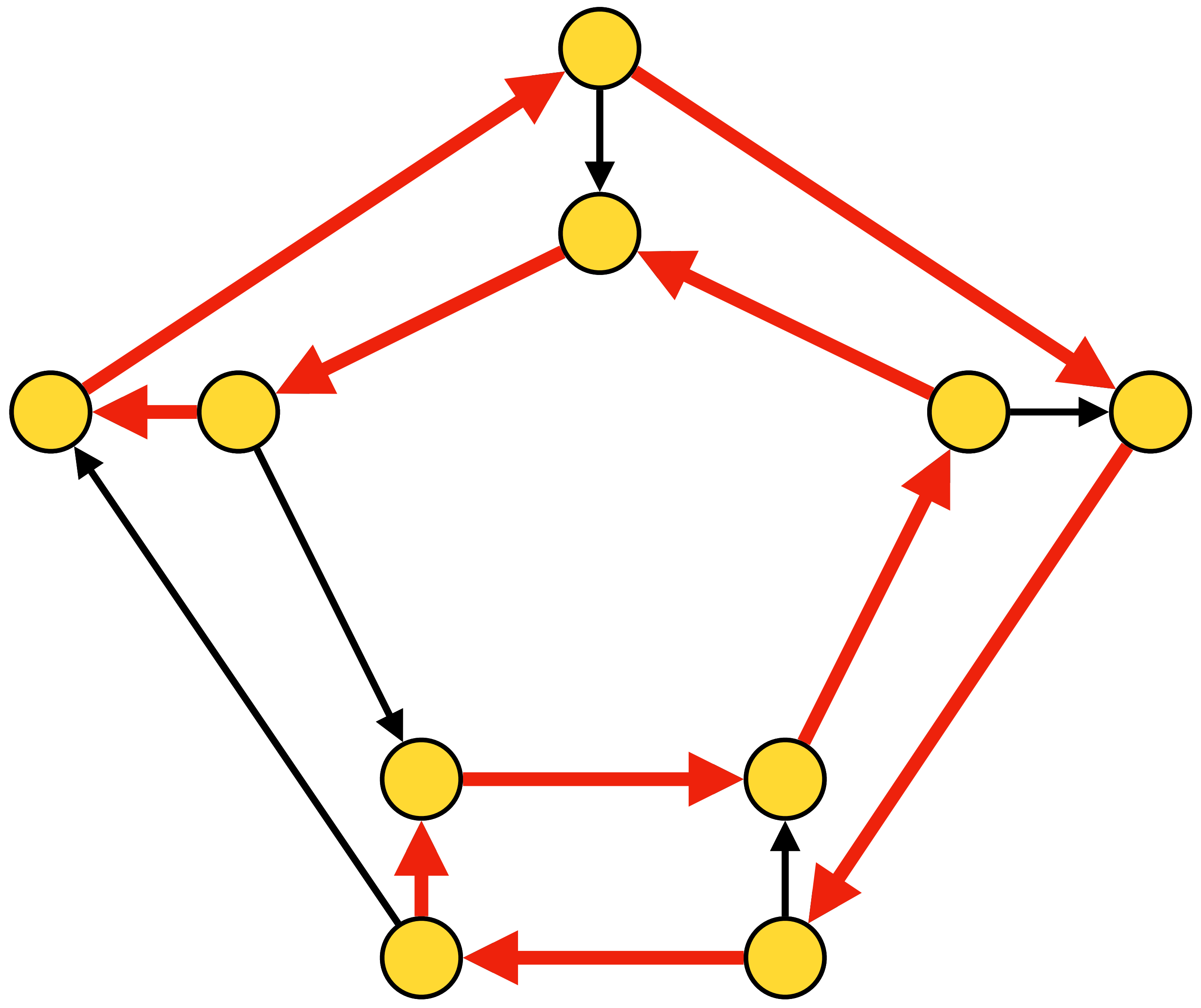
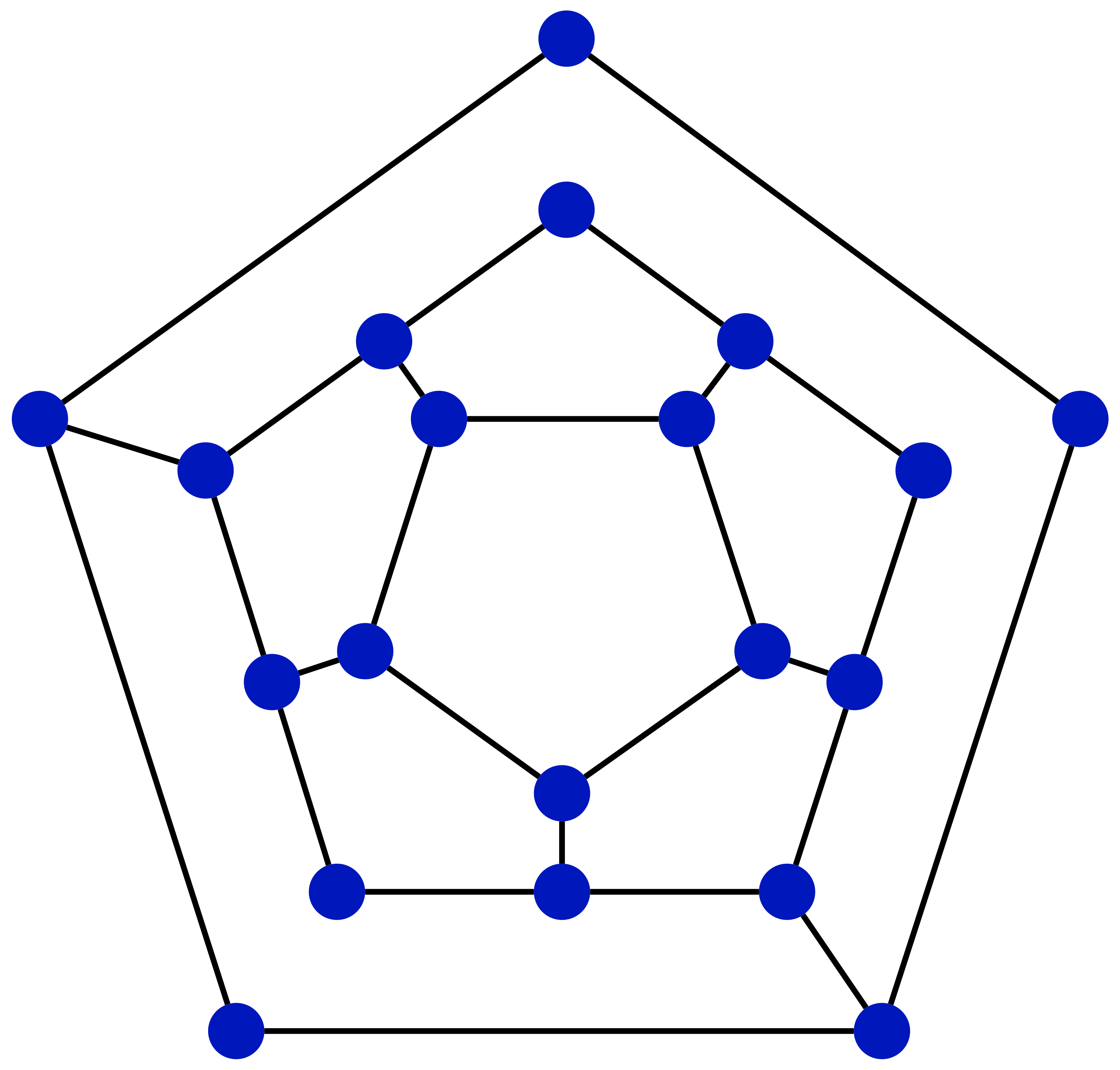


Question

Is the following graph Hamiltonian?

Same as asking if the graph has a cycle that visits each vertex exactly once.

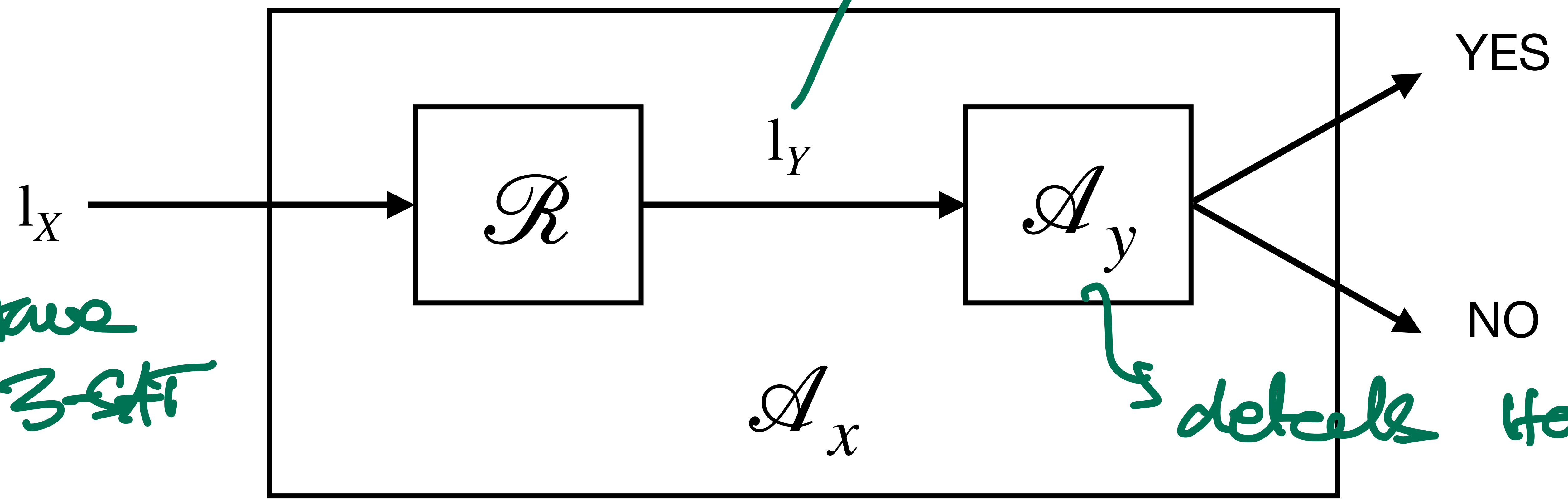
· Does not



Directed Hamiltonian cycle is NP-C

- Directed Hamiltonian Cycle is in NP: exercise
- Hardness: We will show
- 3-SAT $\leq_p$ Directed Hamiltonian Cycle

is a directed graph.



instance of 3-SAT

detects Hamiltonian cycle.

Reduction

Given 3-SAT formula φ create a graph G_φ such that

- G_φ has a Hamiltonian cycle if and only if φ is satisfiable

Reduction

Given 3-SAT formula φ create a graph G_φ such that

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ϕ: phi φ: psi

Reduction

Given 3-SAT formula φ create a graph G_φ such that

- G_φ has a Hamiltonian cycle if and only if φ is satisfiable
- G_φ should be constructible from φ by a polynomial time algorithm

Notation: φ has n variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.

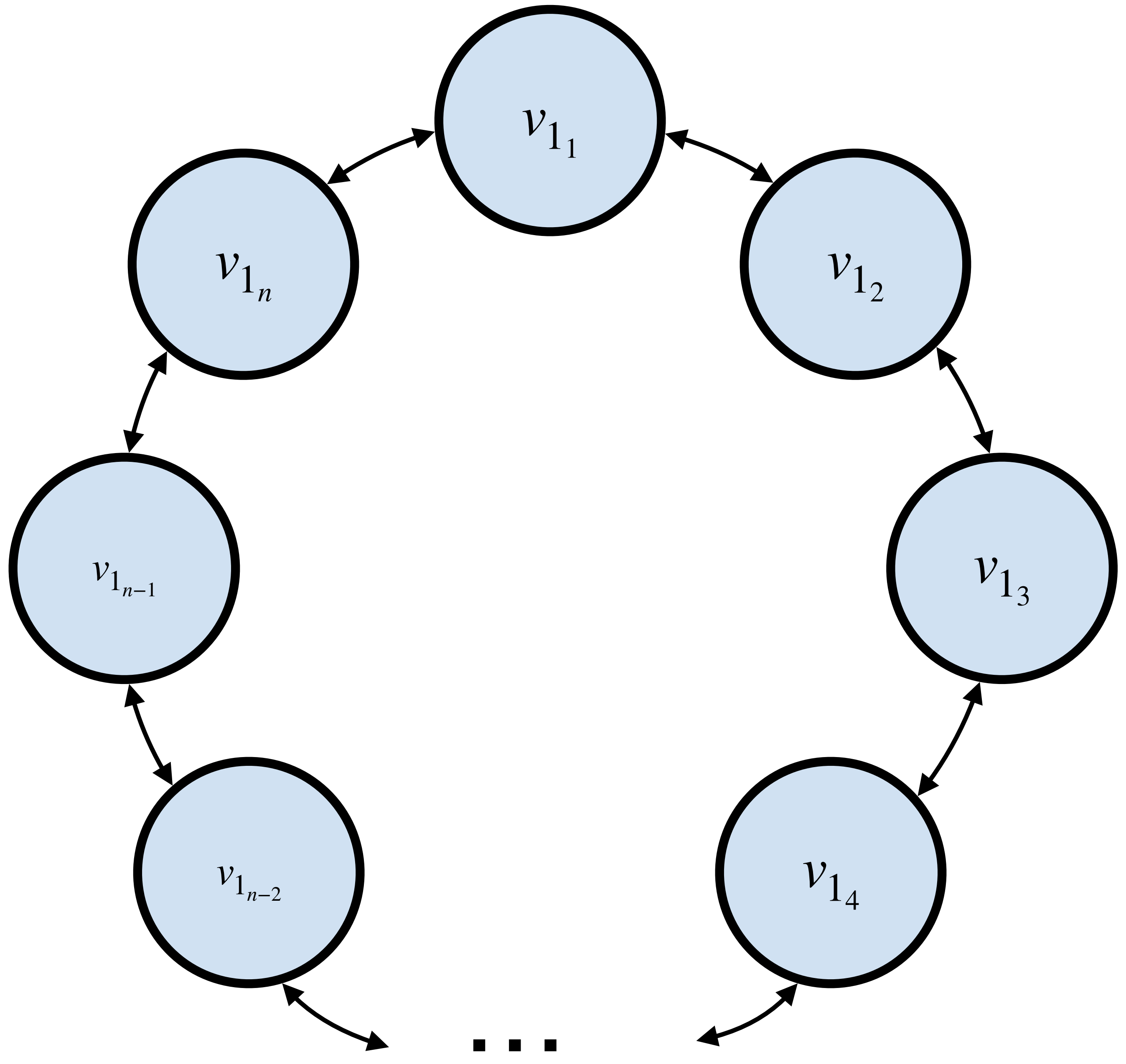
Reduction

Encoding idea I

Need to create a graph from any arbitrary boolean assignment. Consider the expression:

$$f(X_1) = 1$$

We create a cyclic graph that always has a Hamiltonian cyle.



Reduction

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Need to create a graph from any arbitrary boolean assignment. Consider the expression:

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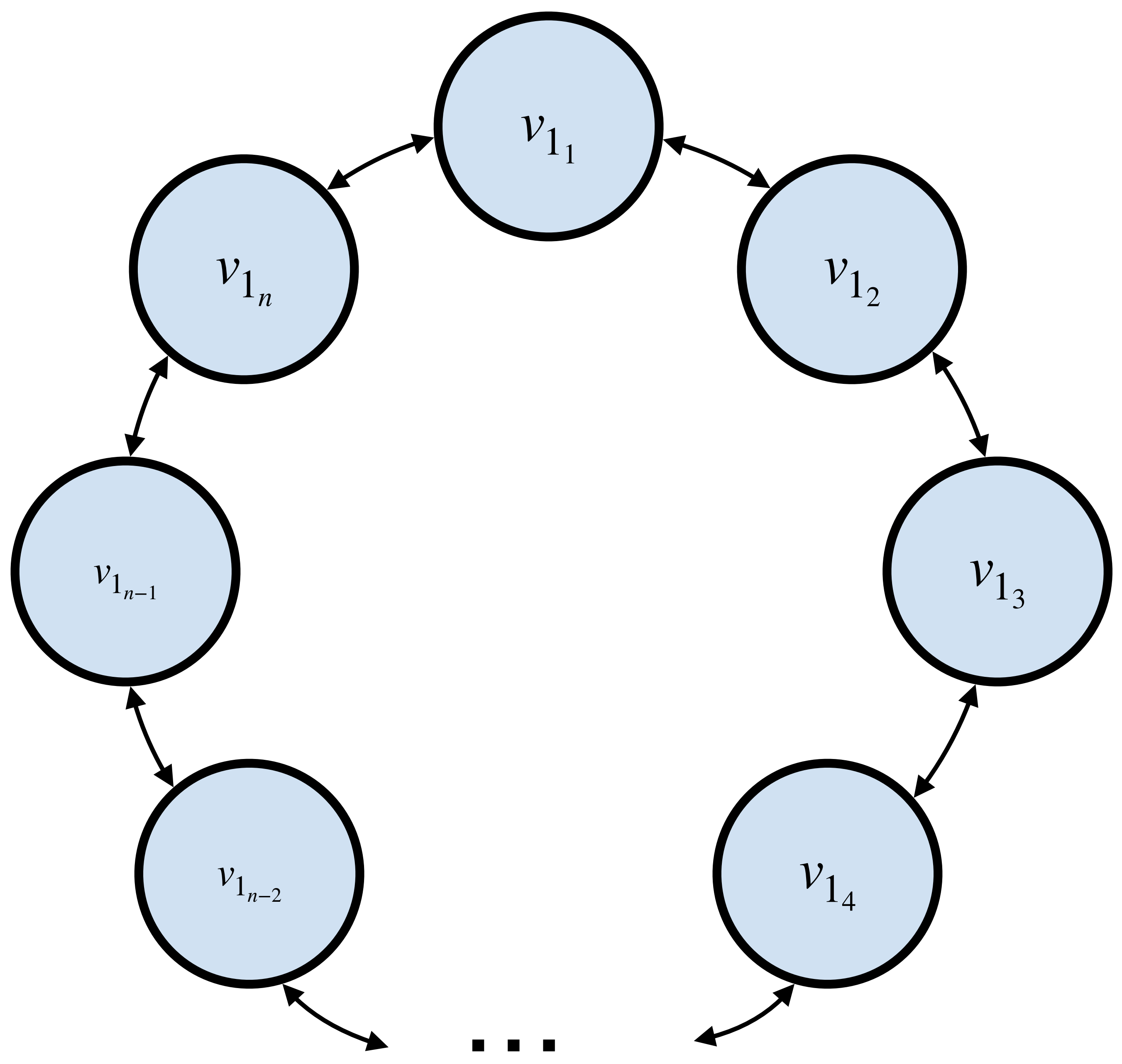
We create a cyclic graph that always has a Hamiltonian cyle.

But how do we encode the variable?



Reduction

Encoding idea I

Need to create a graph from any arbitrary boolean assignment. Consider the expression:

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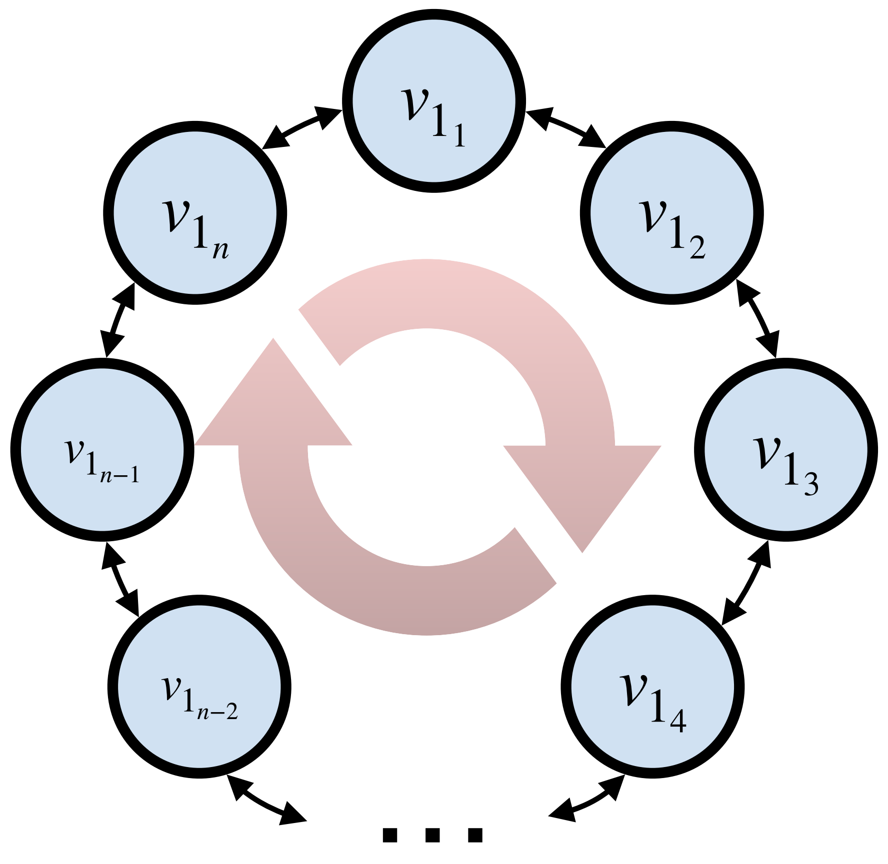
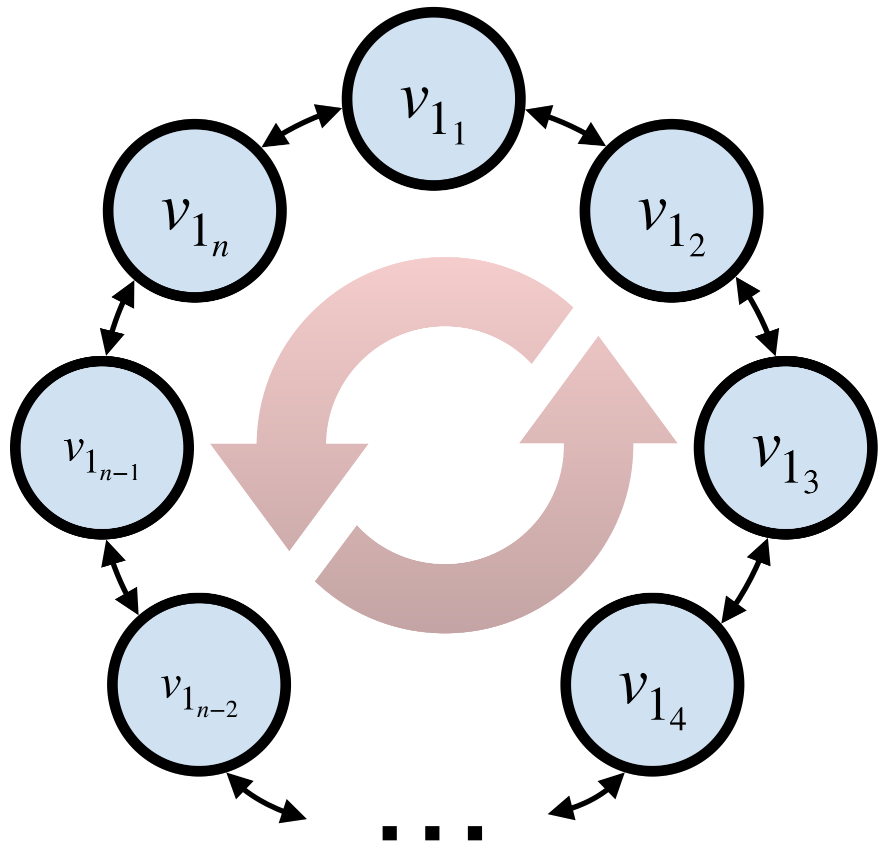
Maybe we can encode the variable X_1 in terms of the cycle direction.



If $X_1 = 1$



If $X_1 = 0$

Reduction

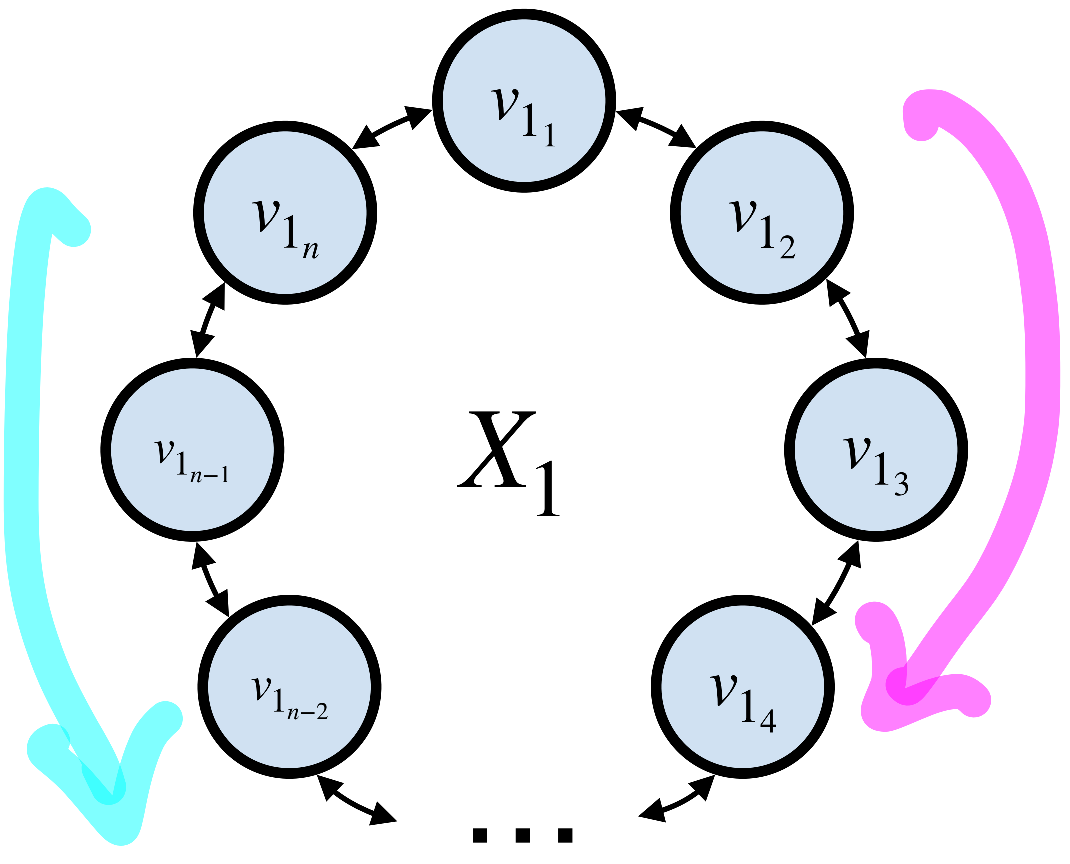
Encoding idea II

How: $f(x_1, x_2) \longleftrightarrow$ to a graph w/ a Hamiltonian cycle if 1 or no such cycle if 0.

How do we encode multiple variables?

$$f(X_1, X_2) = 1$$

Maybe two circles?



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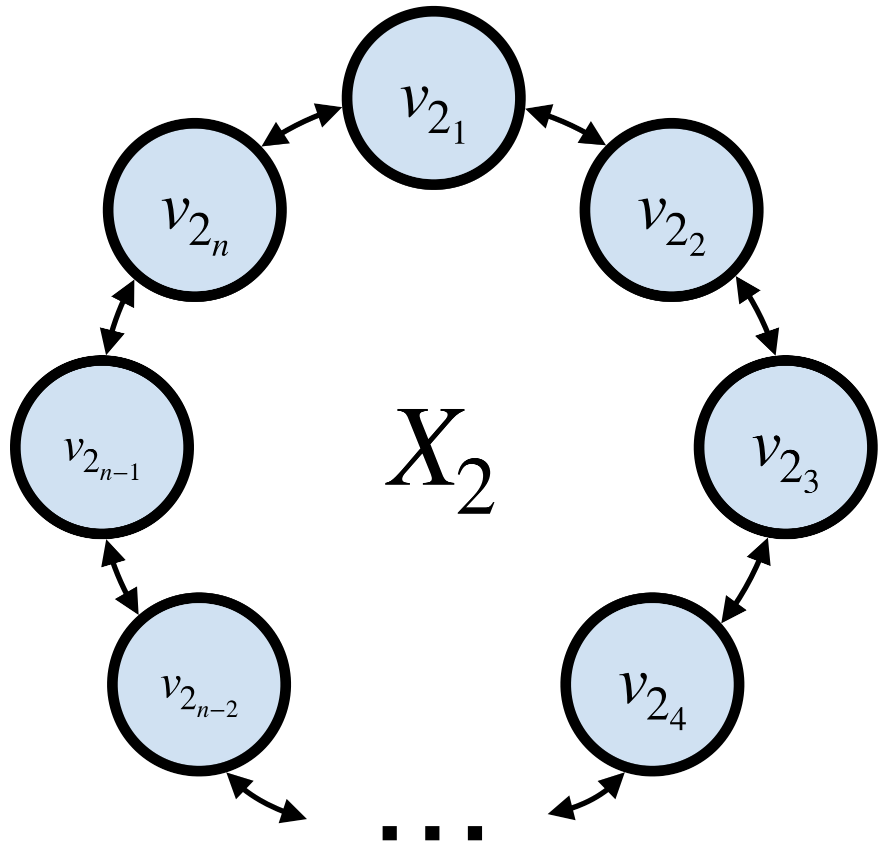


Similarly

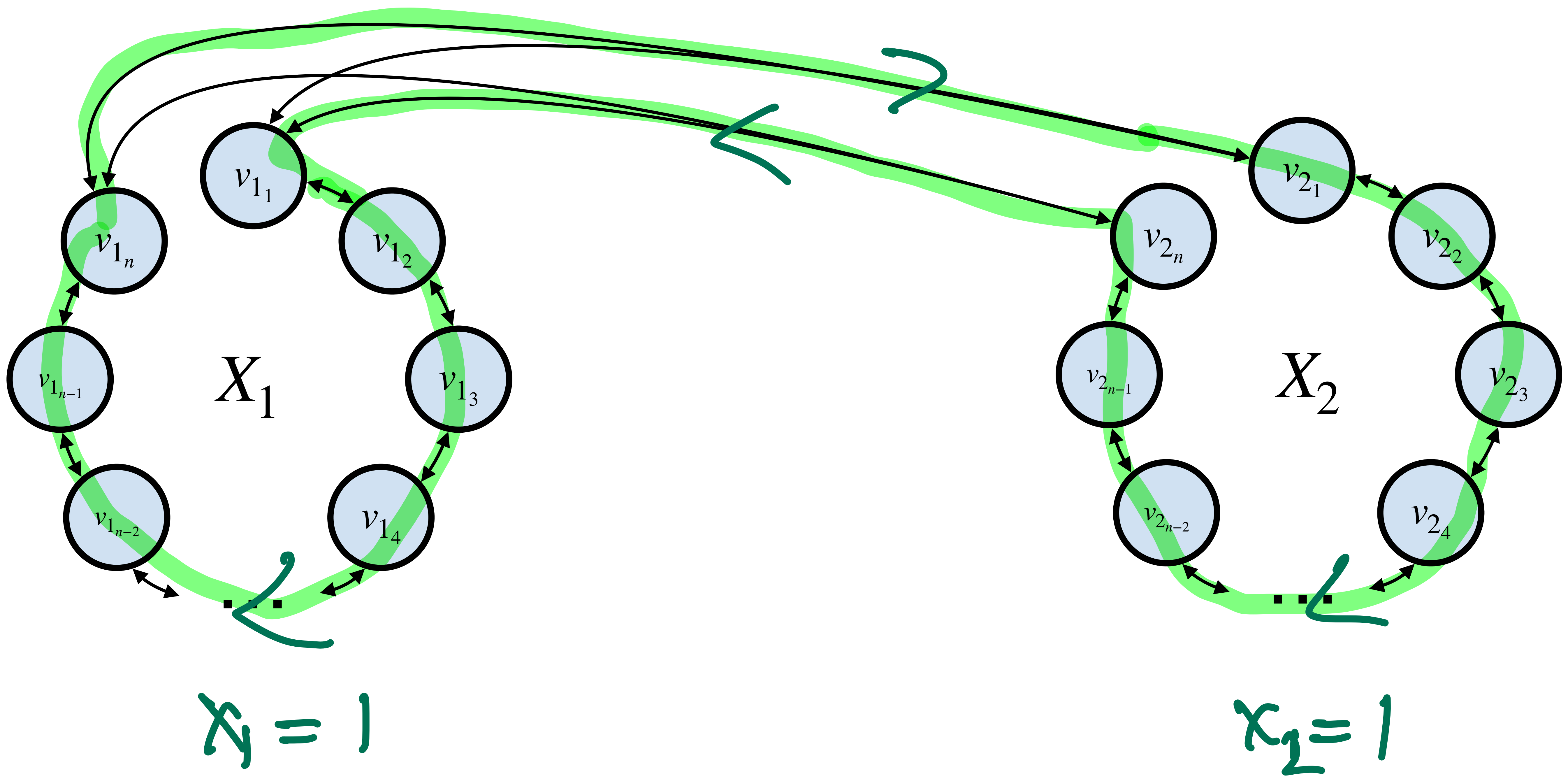
Reduction

Encoding idea II

How do we encode multiple variables?

$$f(X_1, X_2) = 1$$

Need to connect them so that we have a single hamiltonian path for each possible variable assignment.



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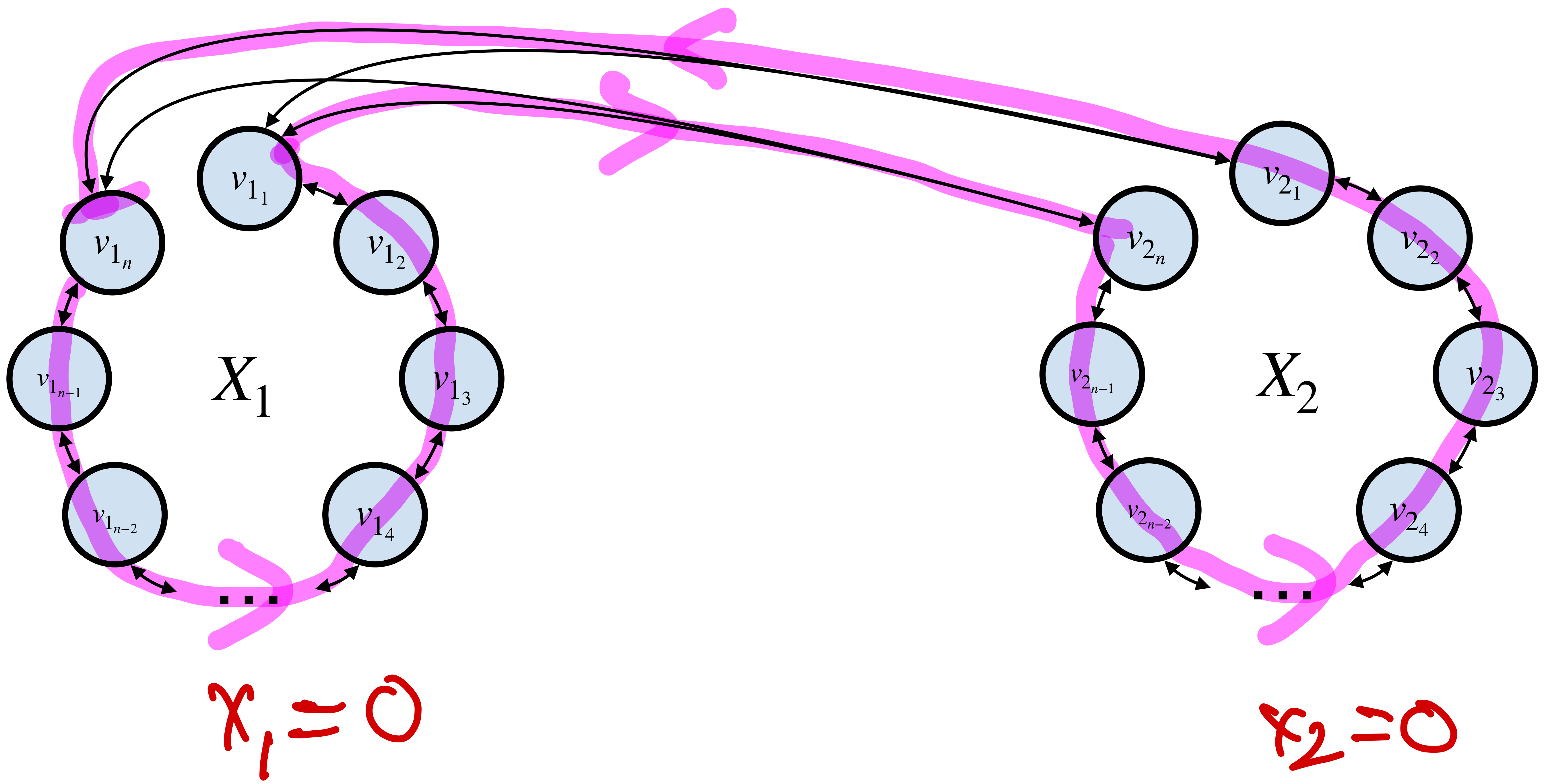

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How do we encode multiple variables?

$$f(X_1, X_2) = 1$$

Would be nice to have a single start/stop node.

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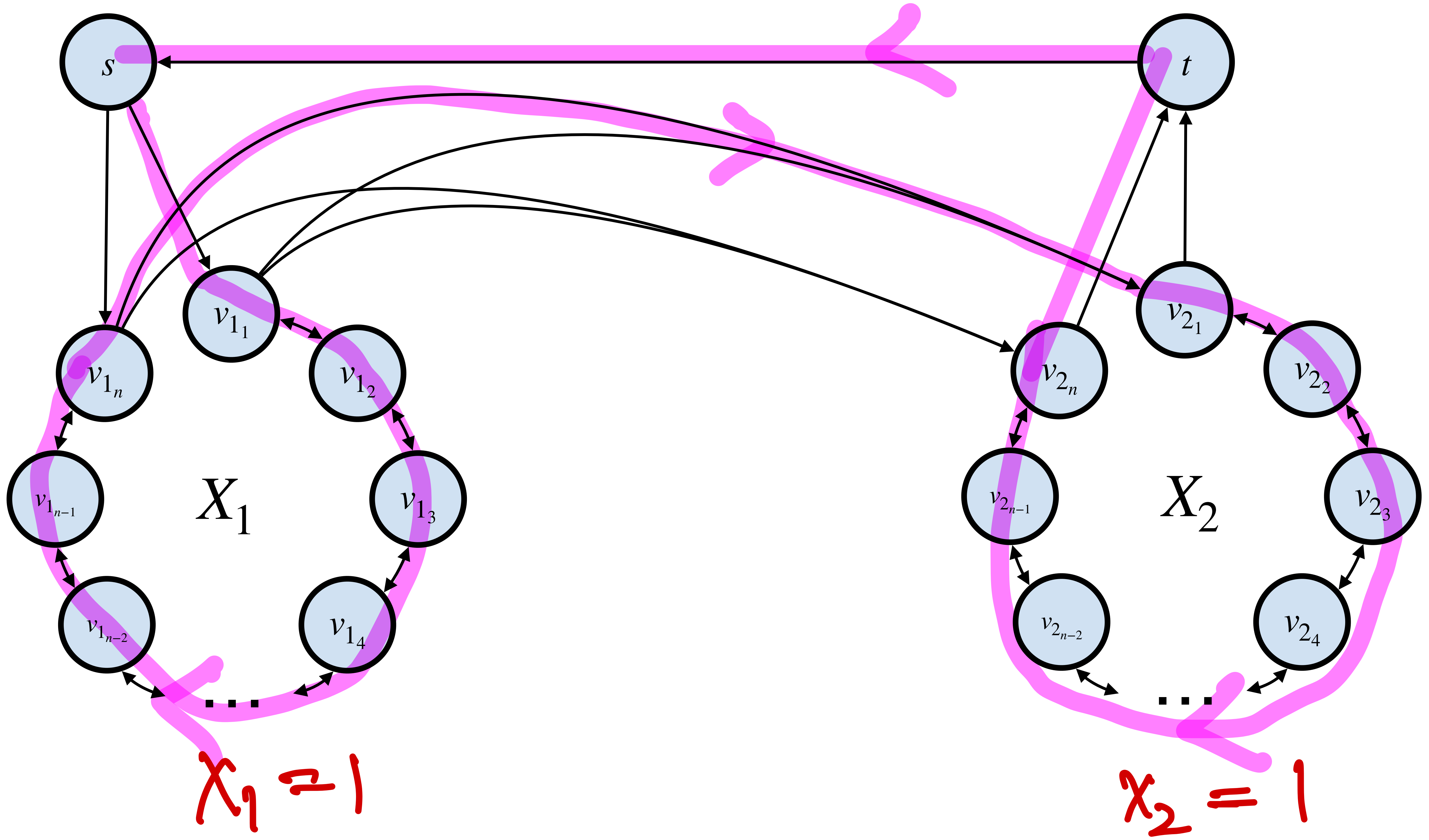

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Getting a bit messy. Let's reorganize:

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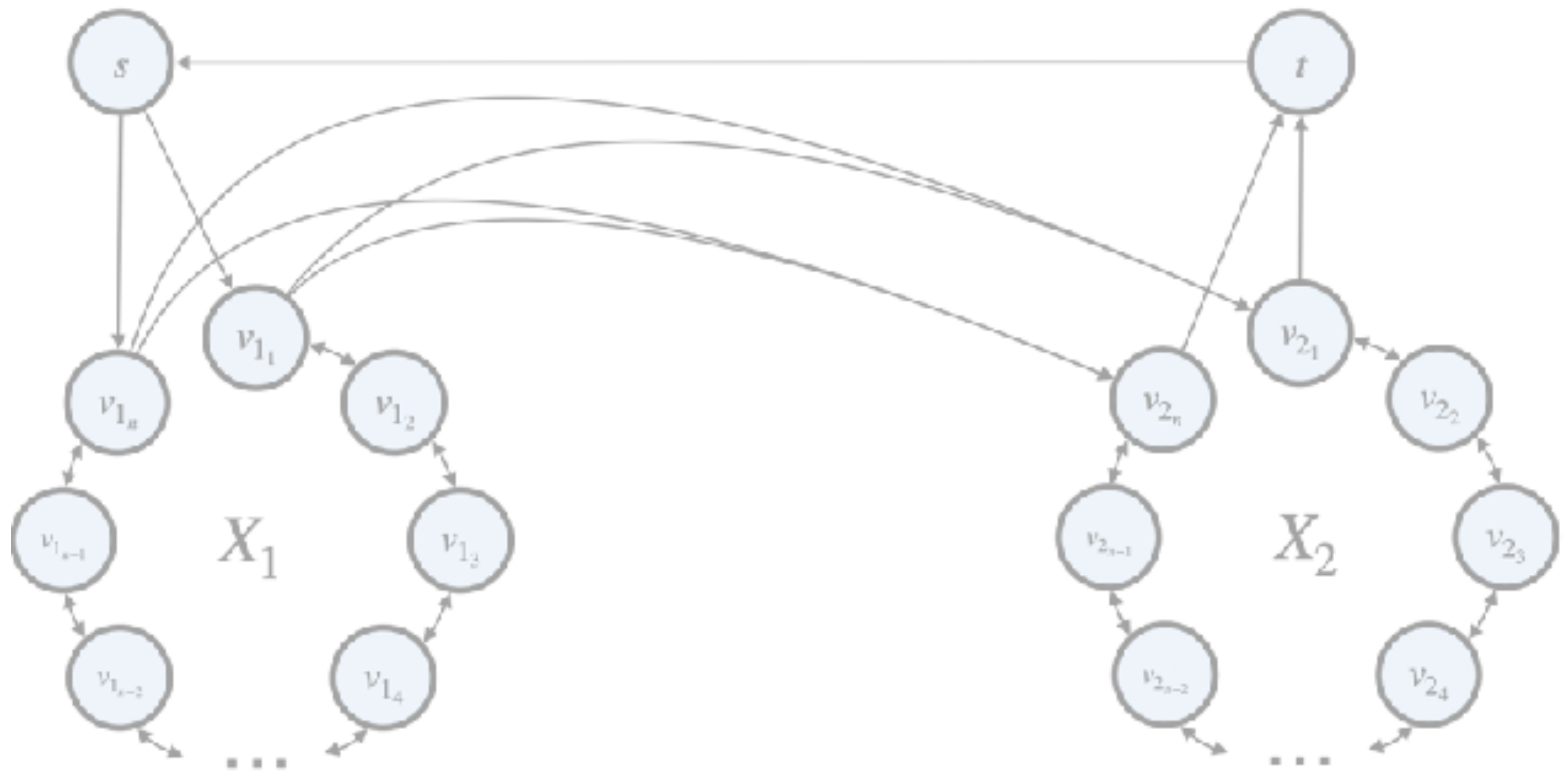
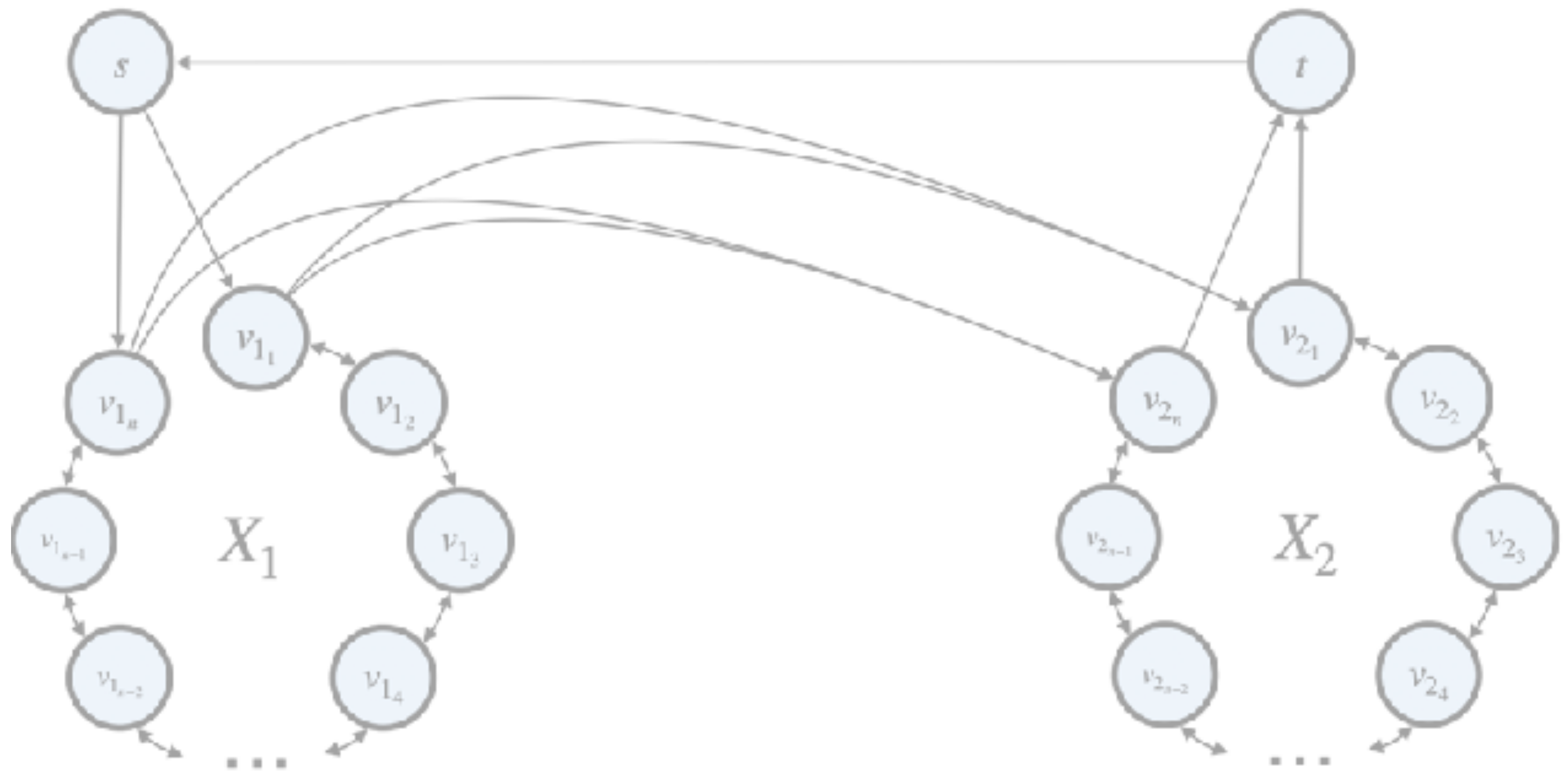


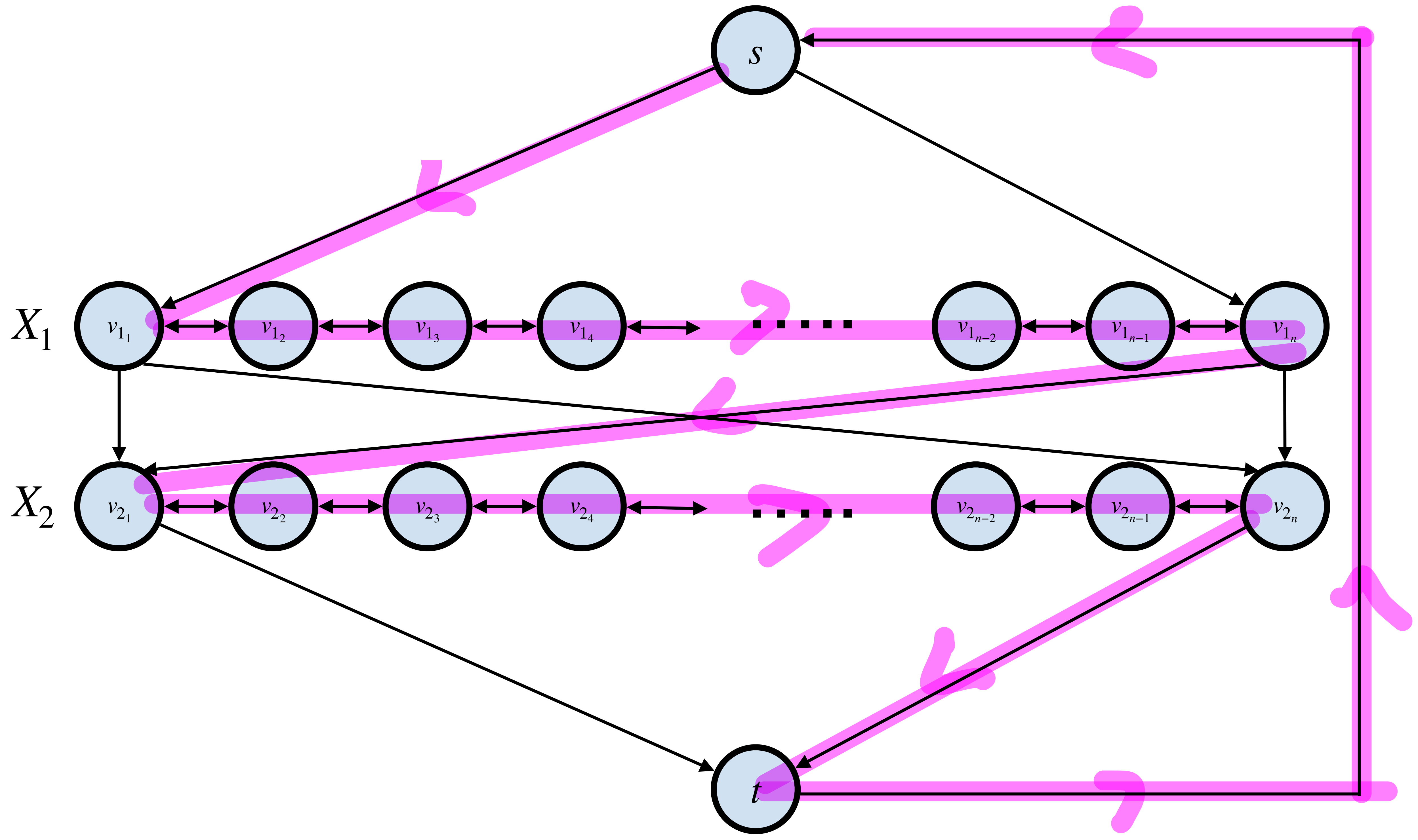
Reduction

Encoding idea II

1) CW → Left to right
2) CCW → Right to left

Getting a bit messy. Let's reorganize:





$x_1 = 1$, $x_2 = 1$

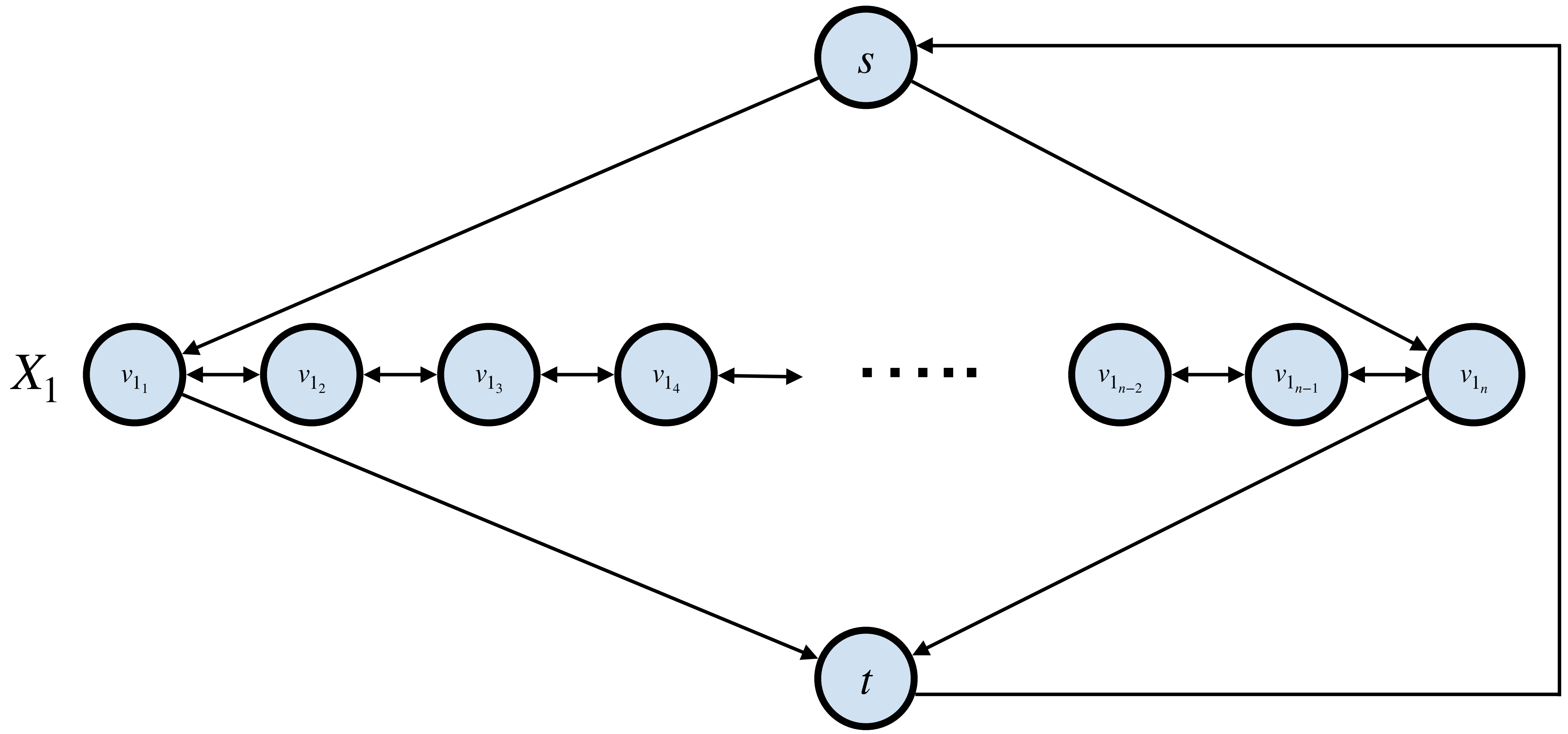
Reduction

Encoding idea III

$(- - -) \wedge (- - - -) \ldots$

How do we handle clauses ?

Lets go back to our one variable graph.

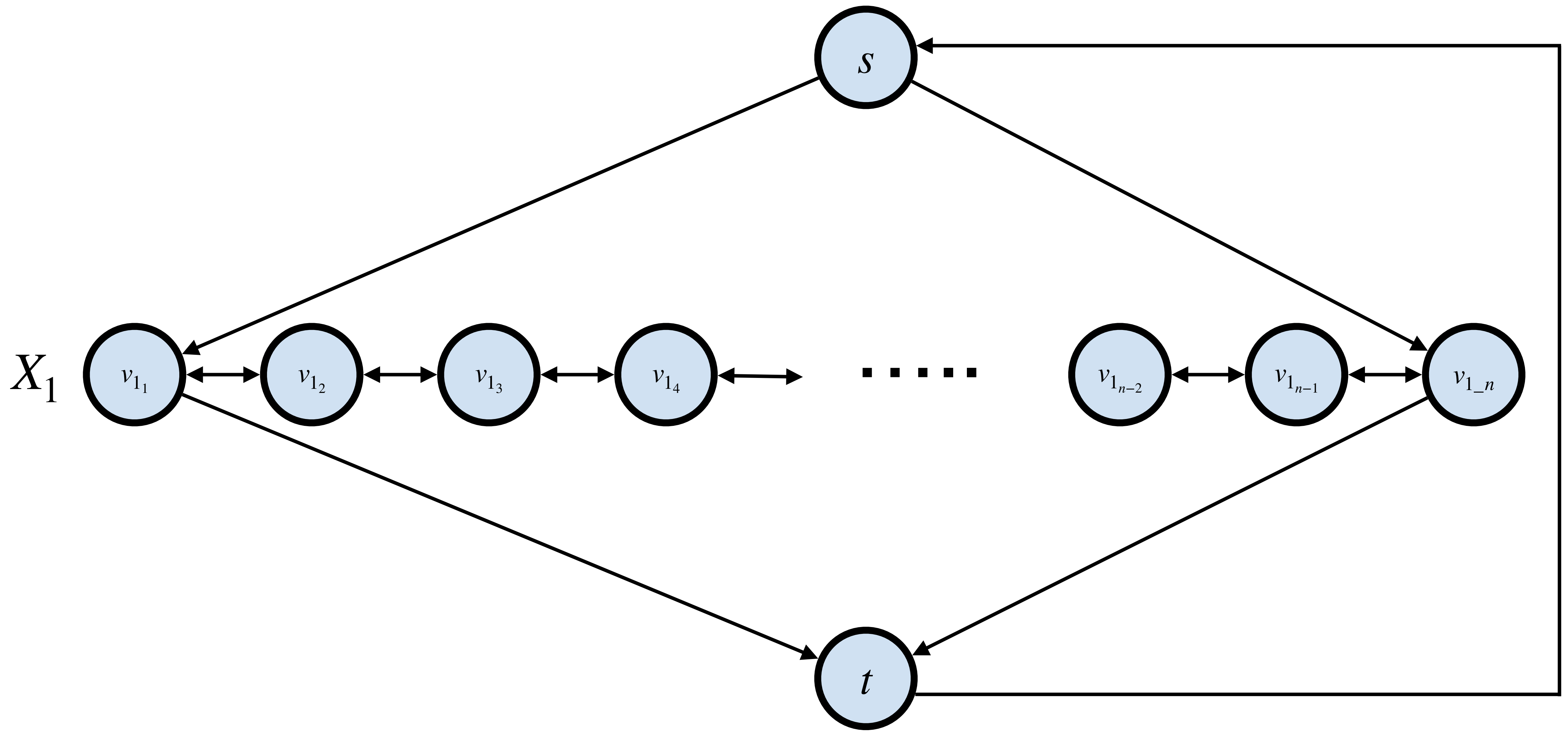


Reduction

Encoding idea III

How do we handle a clause ?

$$f(X_1) = X_1$$



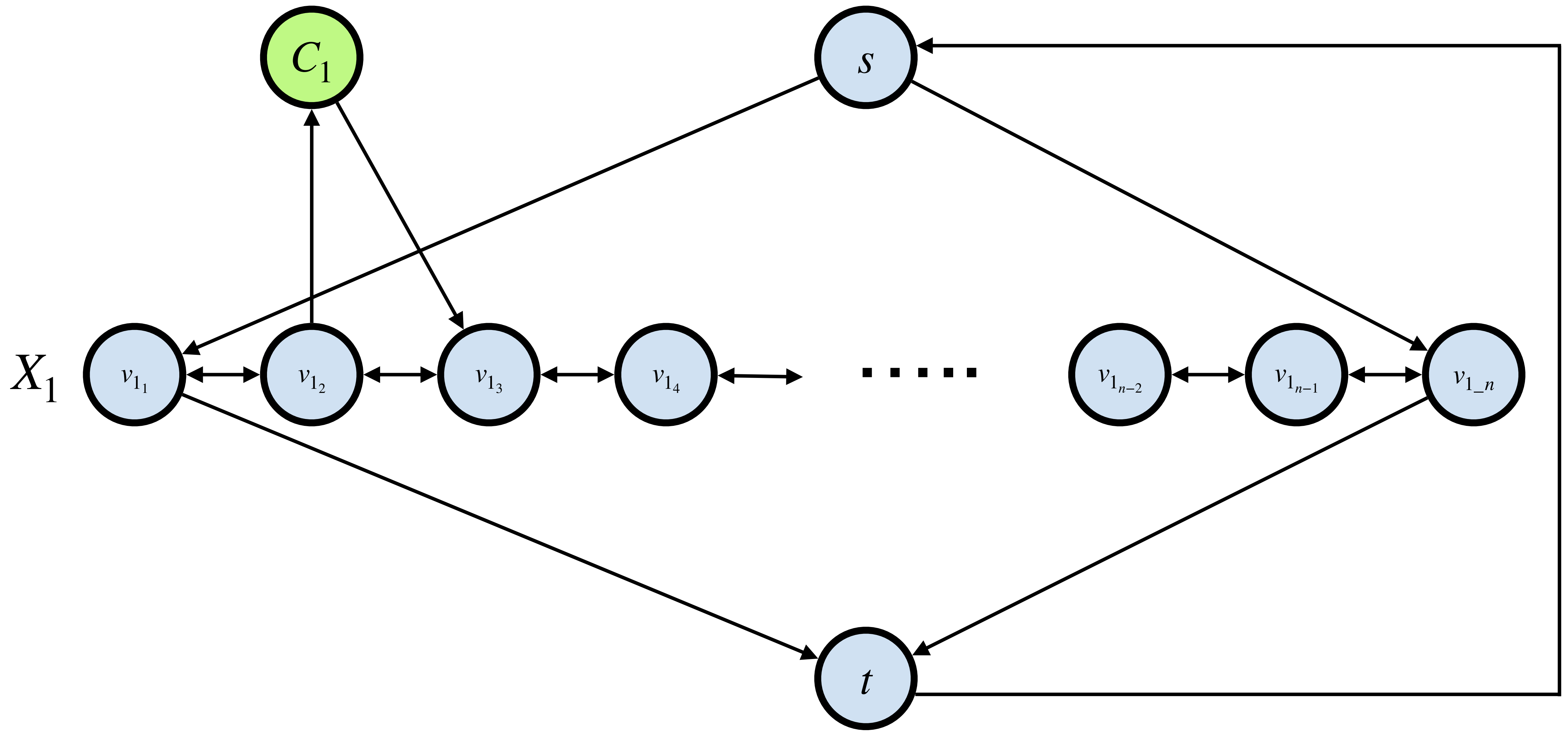
Reduction

Encoding idea III

How do we handle a clause ?

$$f(X_1) = X_1$$

Add node for clause.



Reduction

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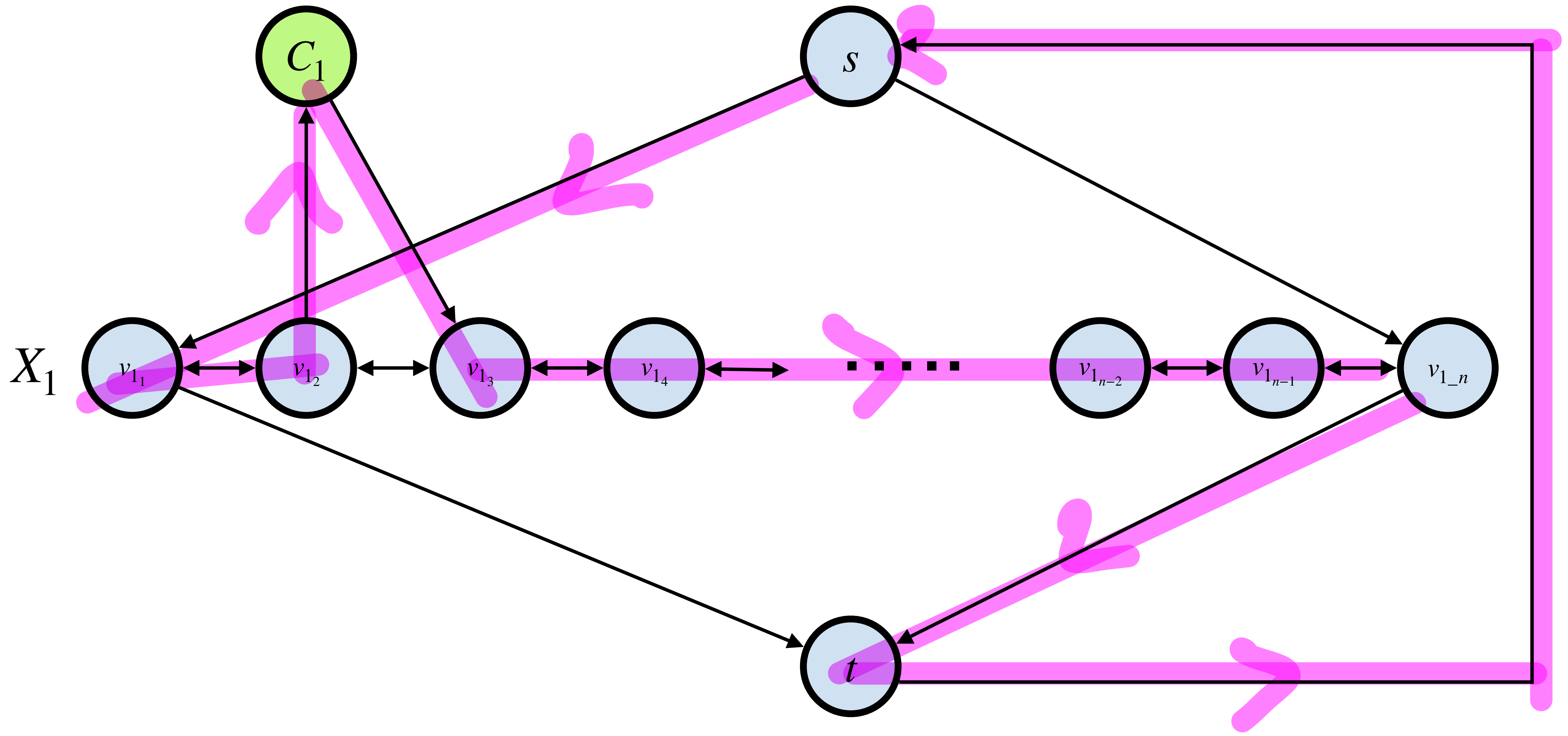
How do we handle a clause ?

$$f(X_1) = X_1$$

Add node for clause.

Enforces traversal in single direction.



Reduction

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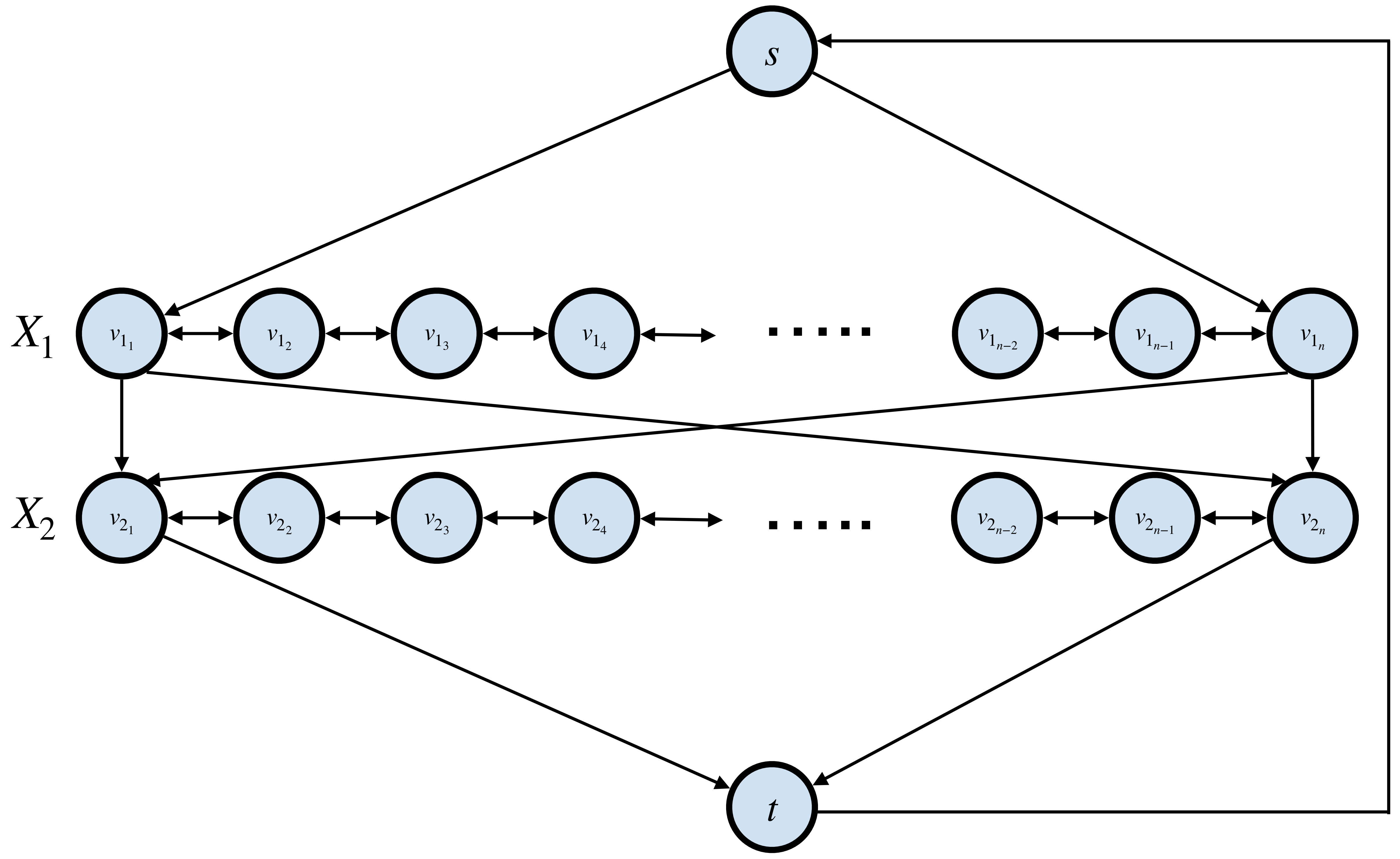
How do we handle a clause ?

What do we do if the clause has **two literals**?

$$f(X_1, X_2) = (X_1 \vee \overline{X_2})$$



Reduction

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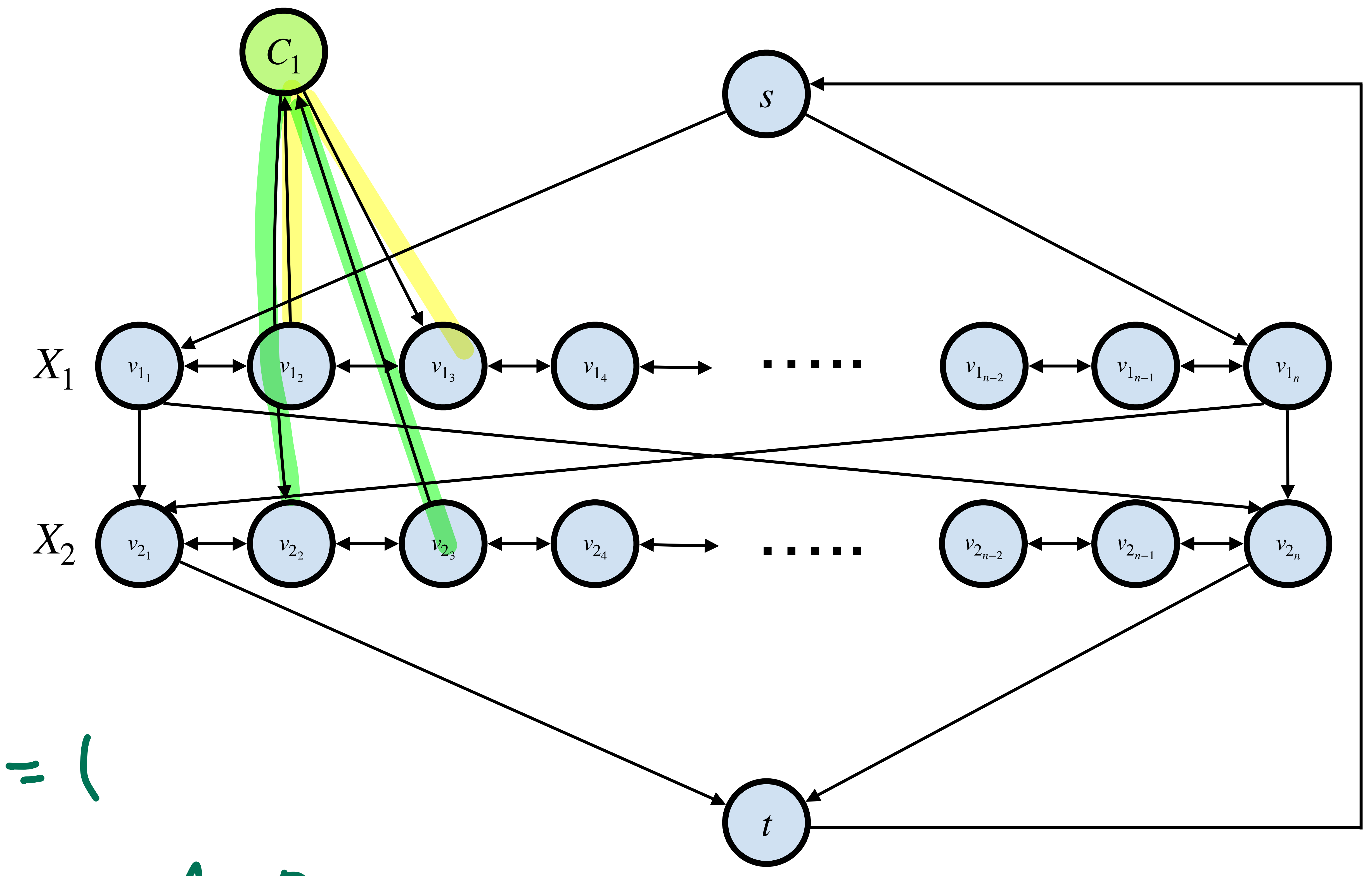
How do we handle a clause ?

What do we do if the clause has **two literals**?

$$f(X_1, X_2) = (X_1 \vee \overline{X_2})$$

Assume $X_1 = X_2 = 1$

X_2 now traveled $R \to L$



Reduction

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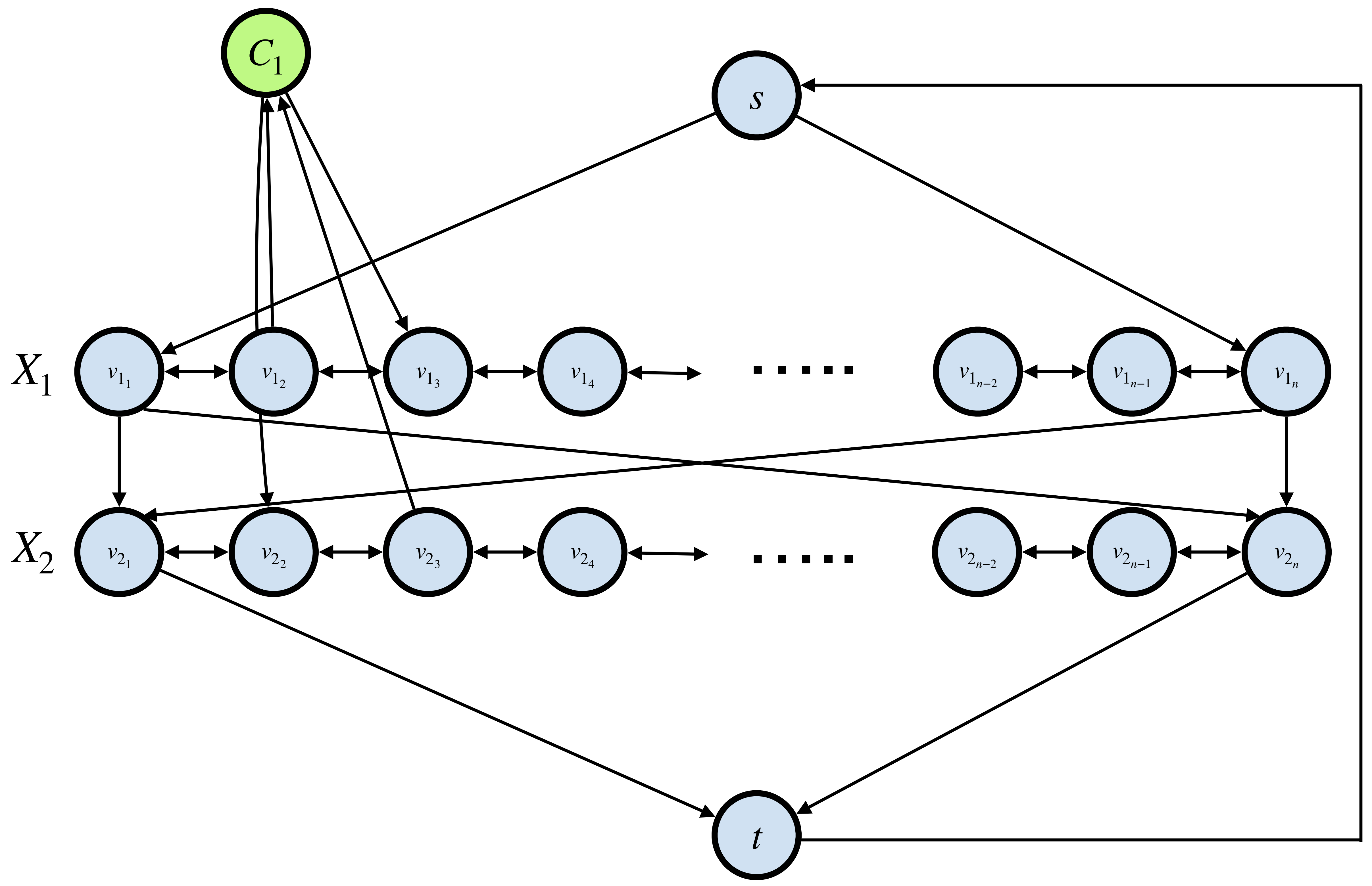
How do we handle clauses ?

What if the expression has **multiple clauses**?

$$f(X_1, X_2) = (X_1 \vee \overline{X_2}) \wedge (\overline{X_1} \vee X_2)$$



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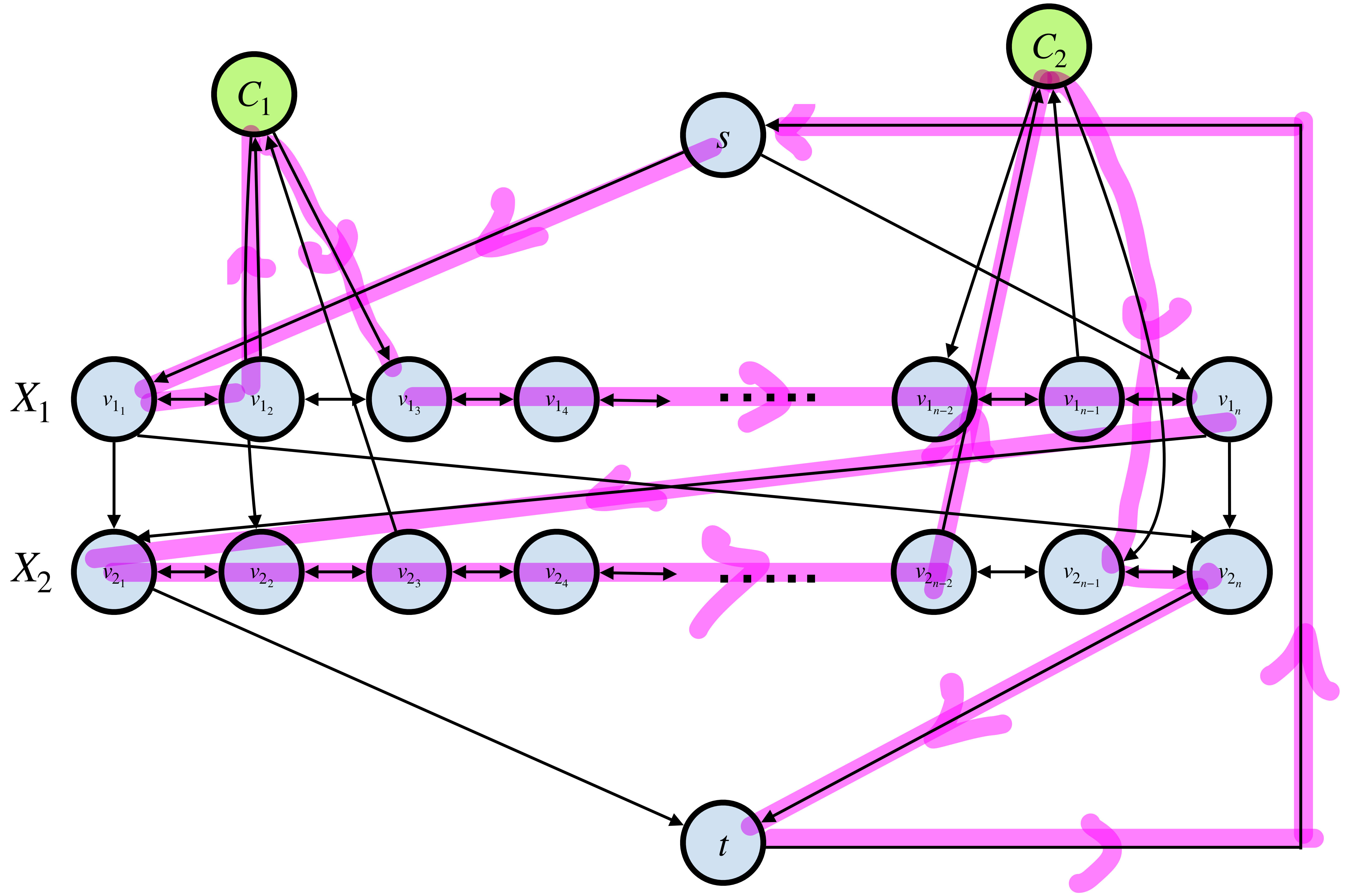
$$f(X_1, X_2) = (X_1 \vee \overline{X_2}) \wedge (\overline{X_1} \vee X_2)$$

$(1 \vee 0) \wedge (0 \vee 1)$

$$\boxed{\begin{array}{l} X_1 = 1 \\ X_2 = 1 \end{array}}$$



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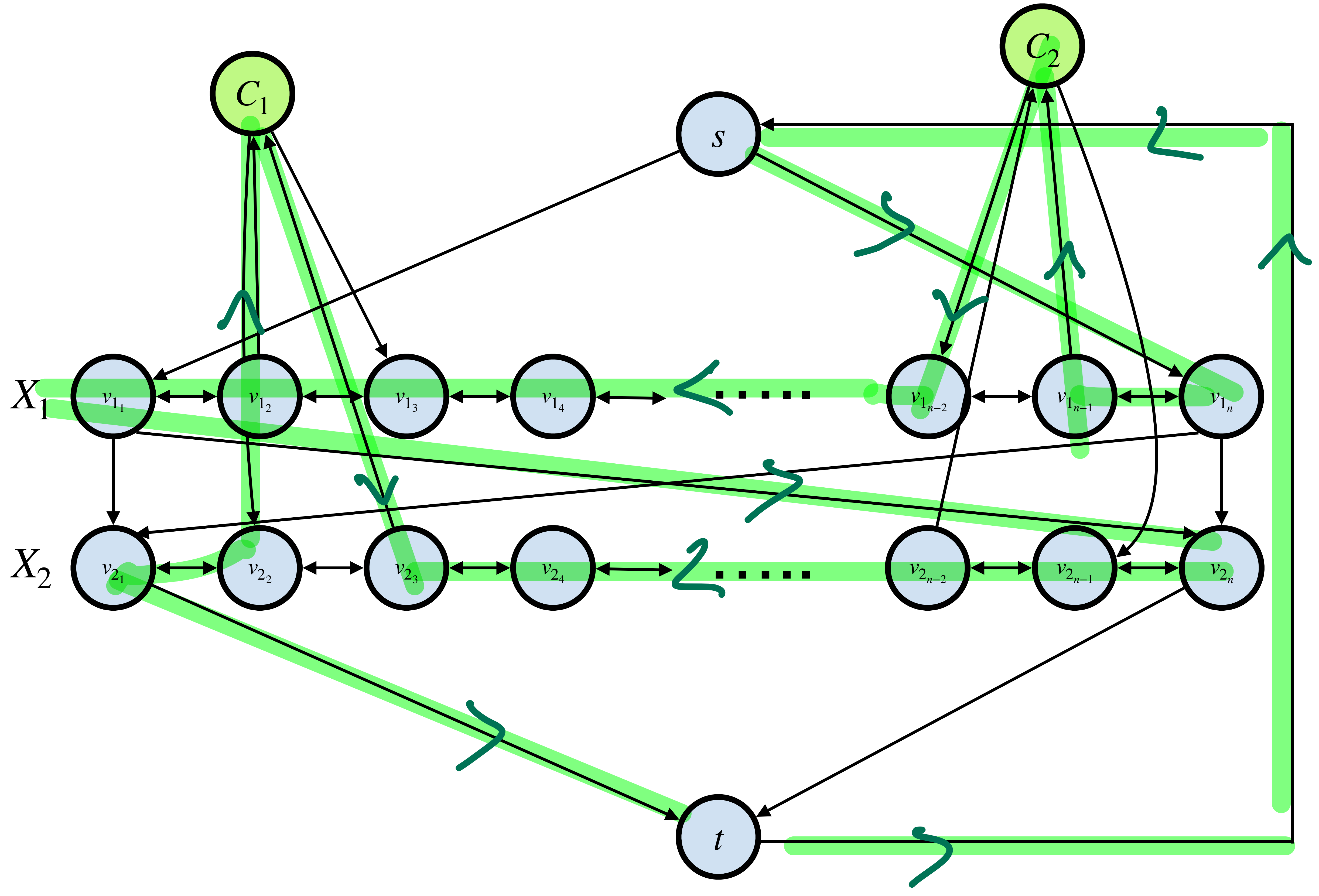
What if the expression has **multiple clauses**?

$$f(X_1, X_2) = (X_1 \vee \overline{X_2}) \wedge (\overline{X_1} \vee X_2)$$

$(0 \vee 1) \wedge (1 \vee 0)$

$x_1 = 0$

$x_2 = 0$



The Reduction

Review I

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$\longrightarrow$ iff $x_i = 1$

- Traverse path i from left to right if and only if x_i is set to true

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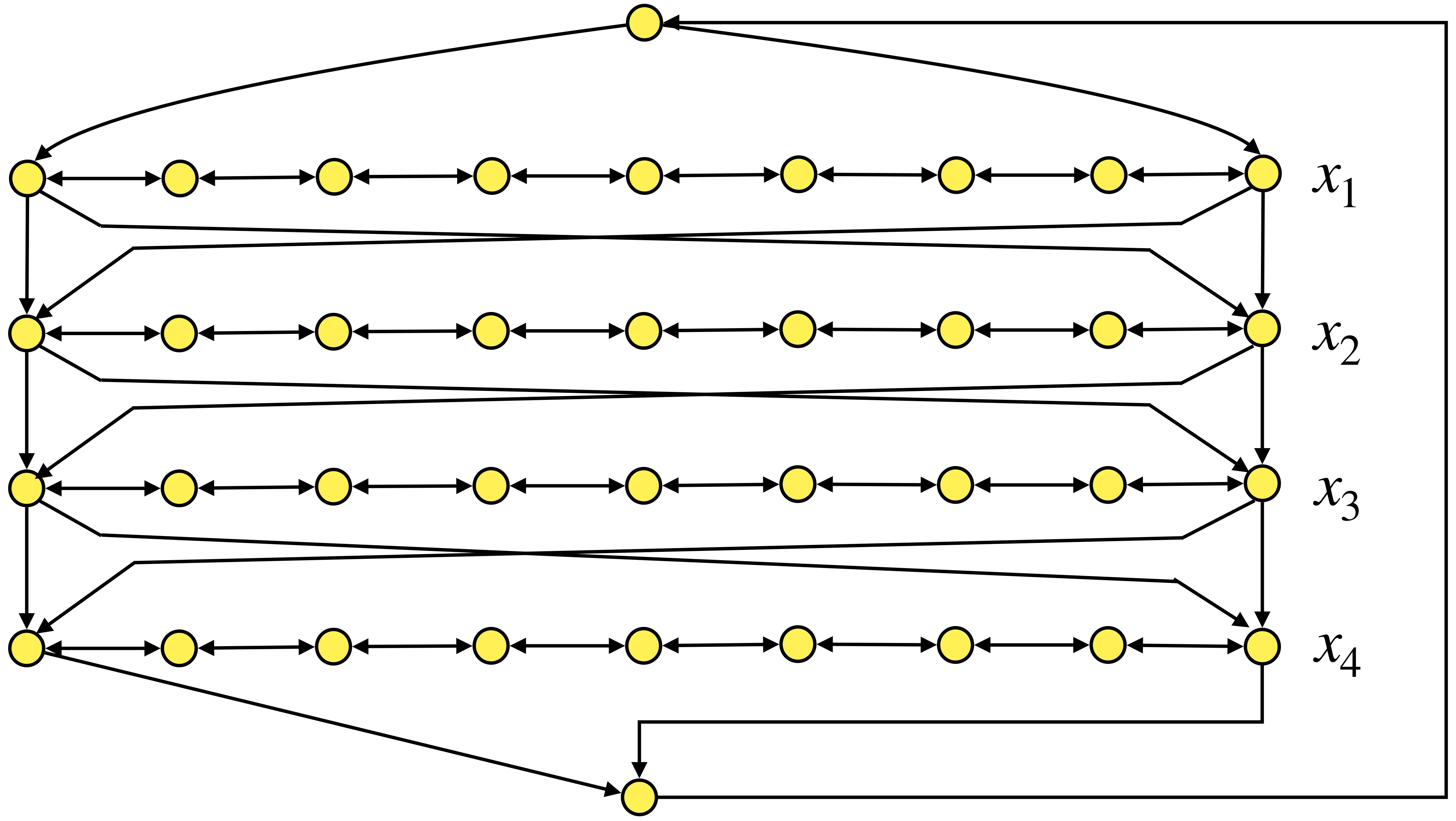
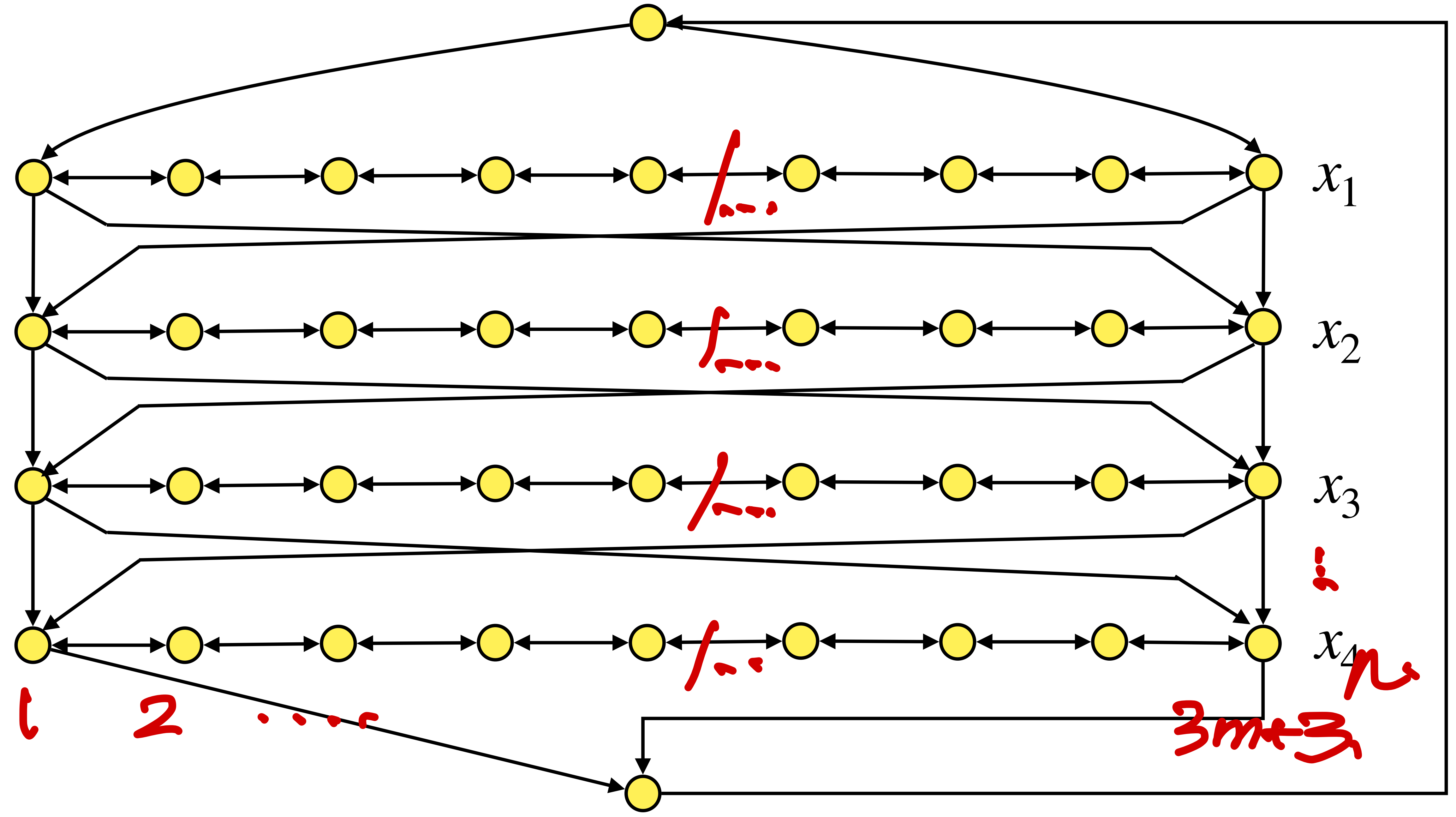


The Reduction

Review I

- Traverse path i from left to right if and only if x_i is set to true

- Each path has $3(m+1)$ nodes where m is number of clauses in φ; nodes numbered from left to right $(1 \text{ to } 3m+3)$



The Reduction

Review II

- Add vertex c_j for clause C_j .
- Vertex c_j has edge from vertex 3_j and to vertex $3_j + 1$ on path i if x_i appears in clause C_j , and
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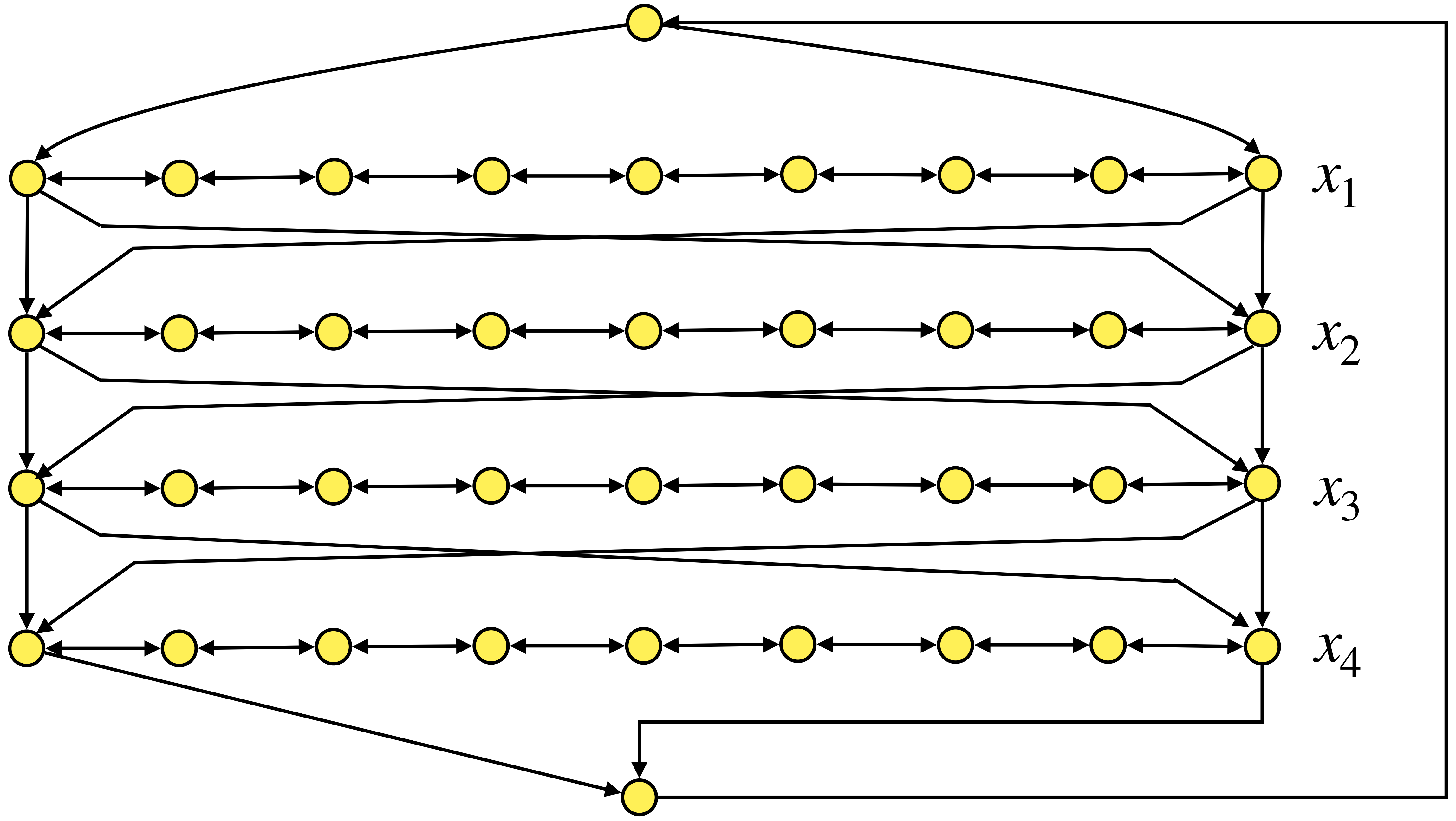

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$m = 2$ (# of clauses)

$3(m+1) = 9$

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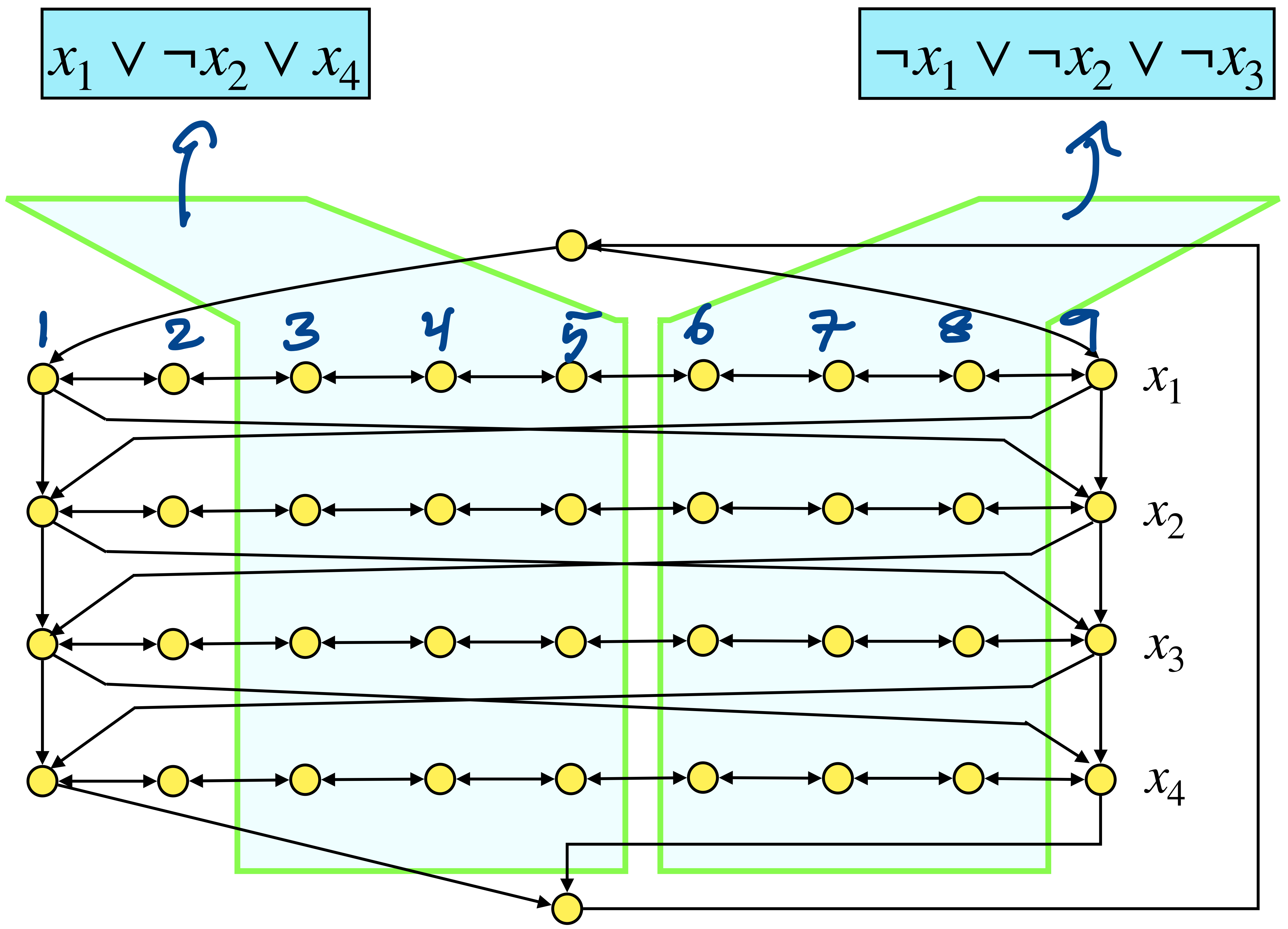
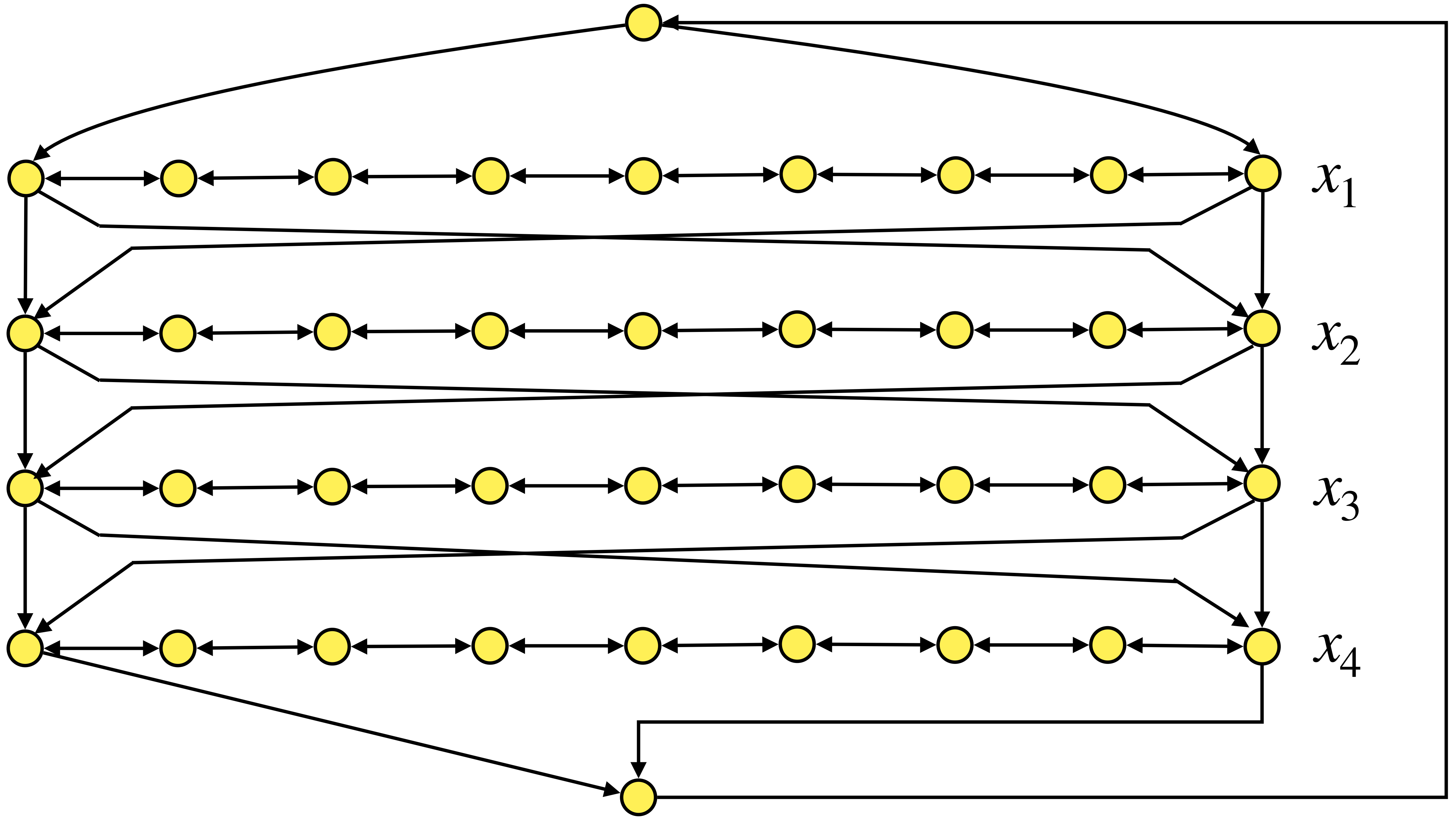


The Reduction

Review II

$\boxed{x_1 \vee \neg x_2 \vee x_4}$



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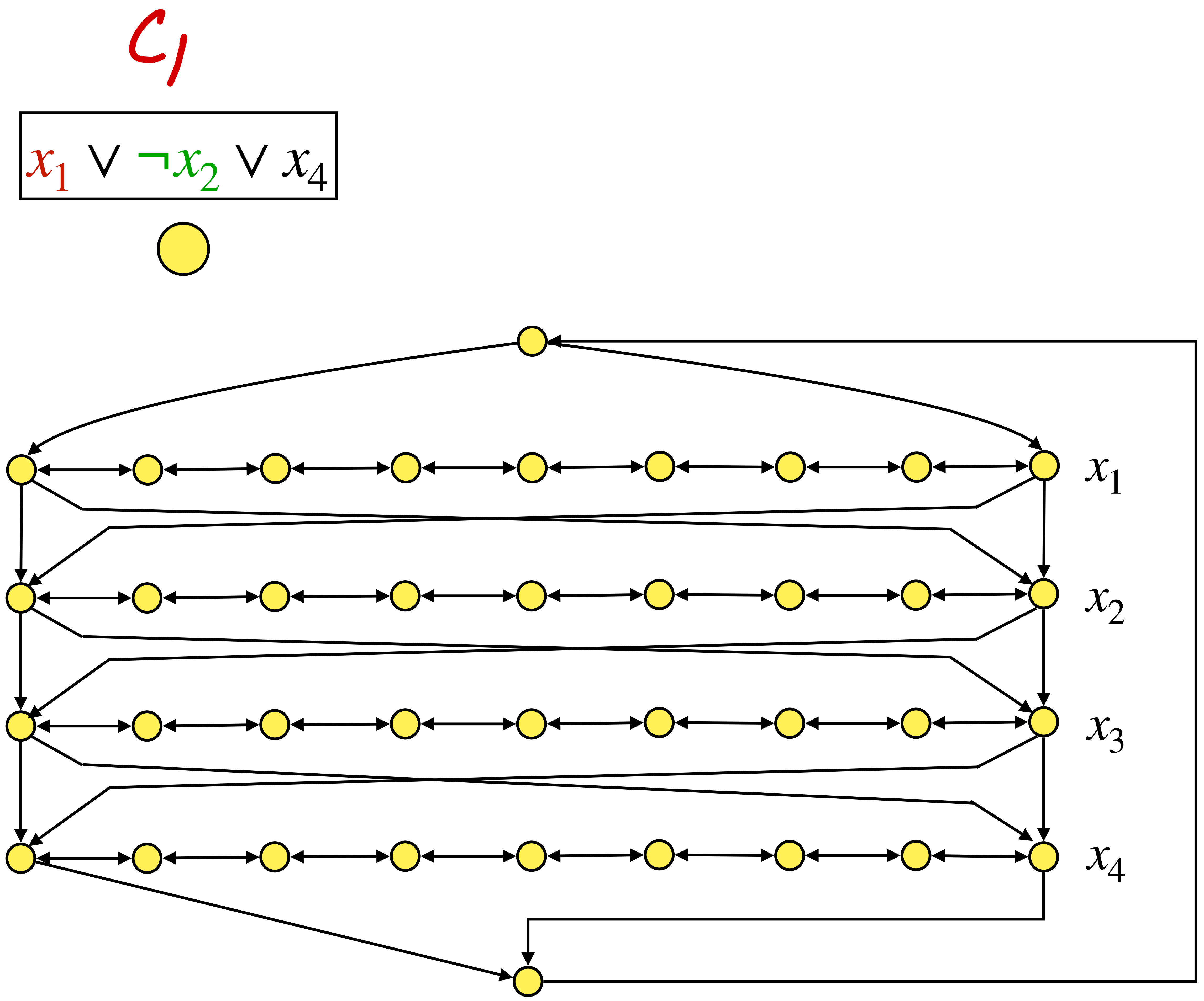

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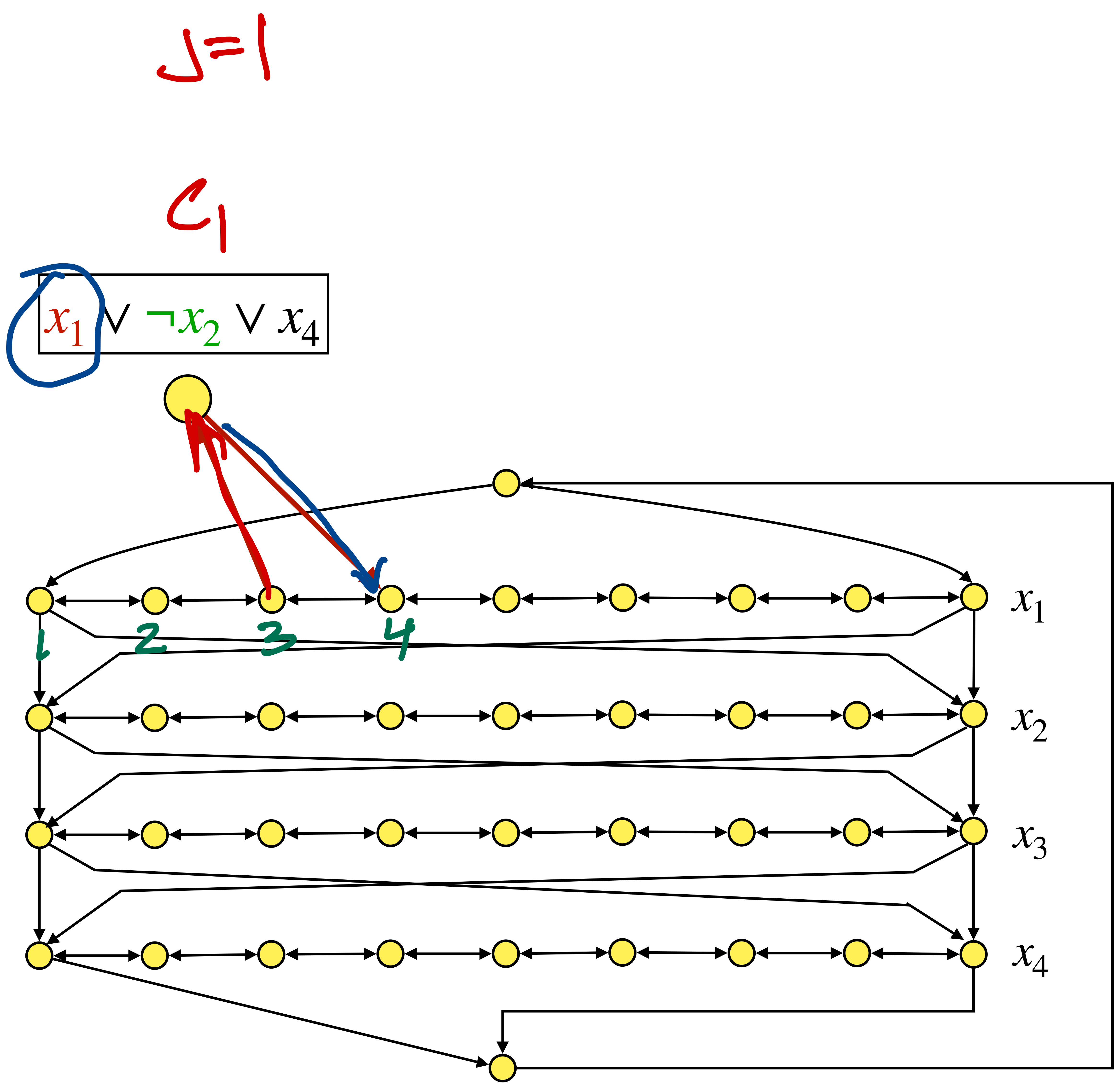

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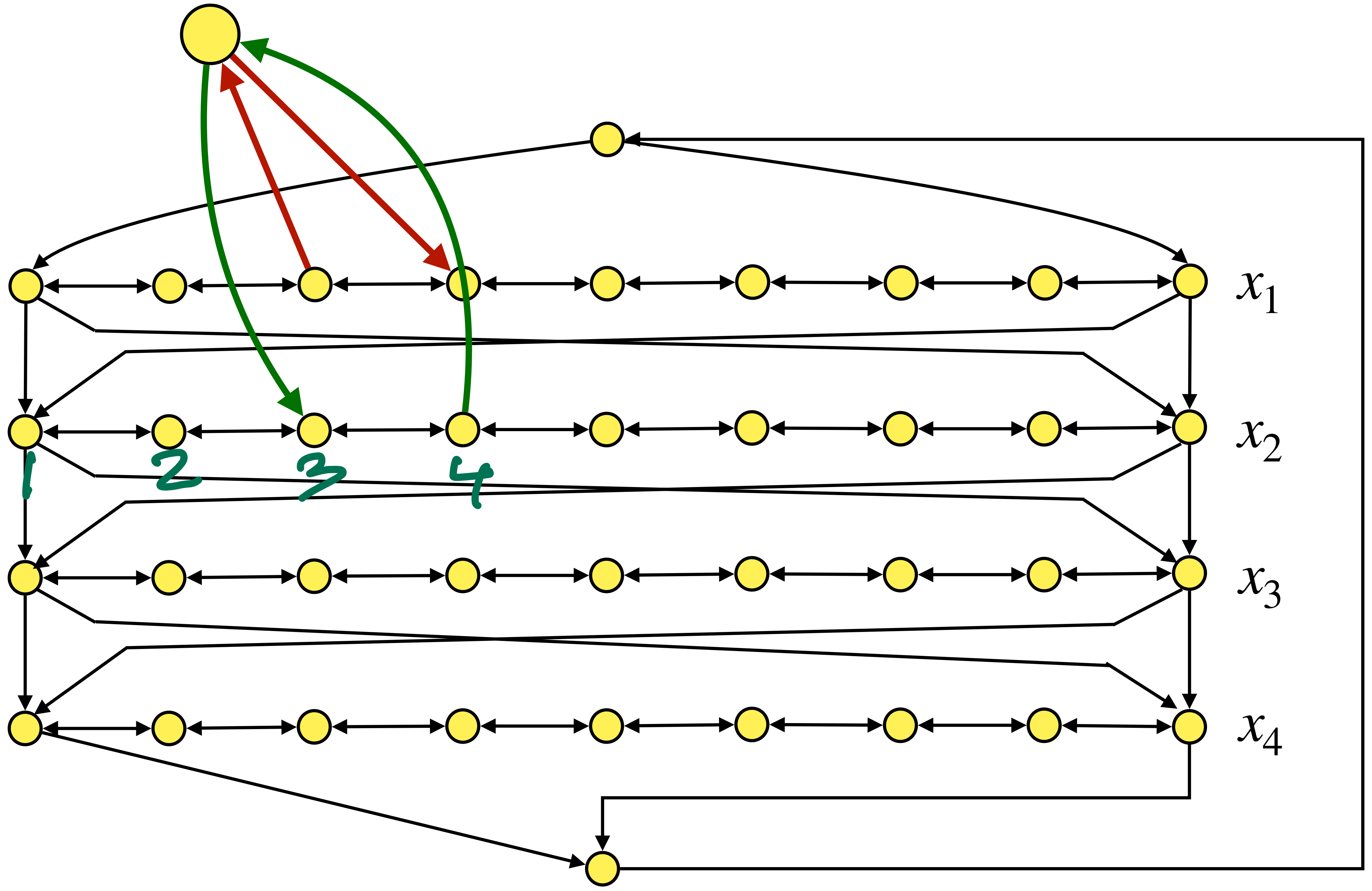

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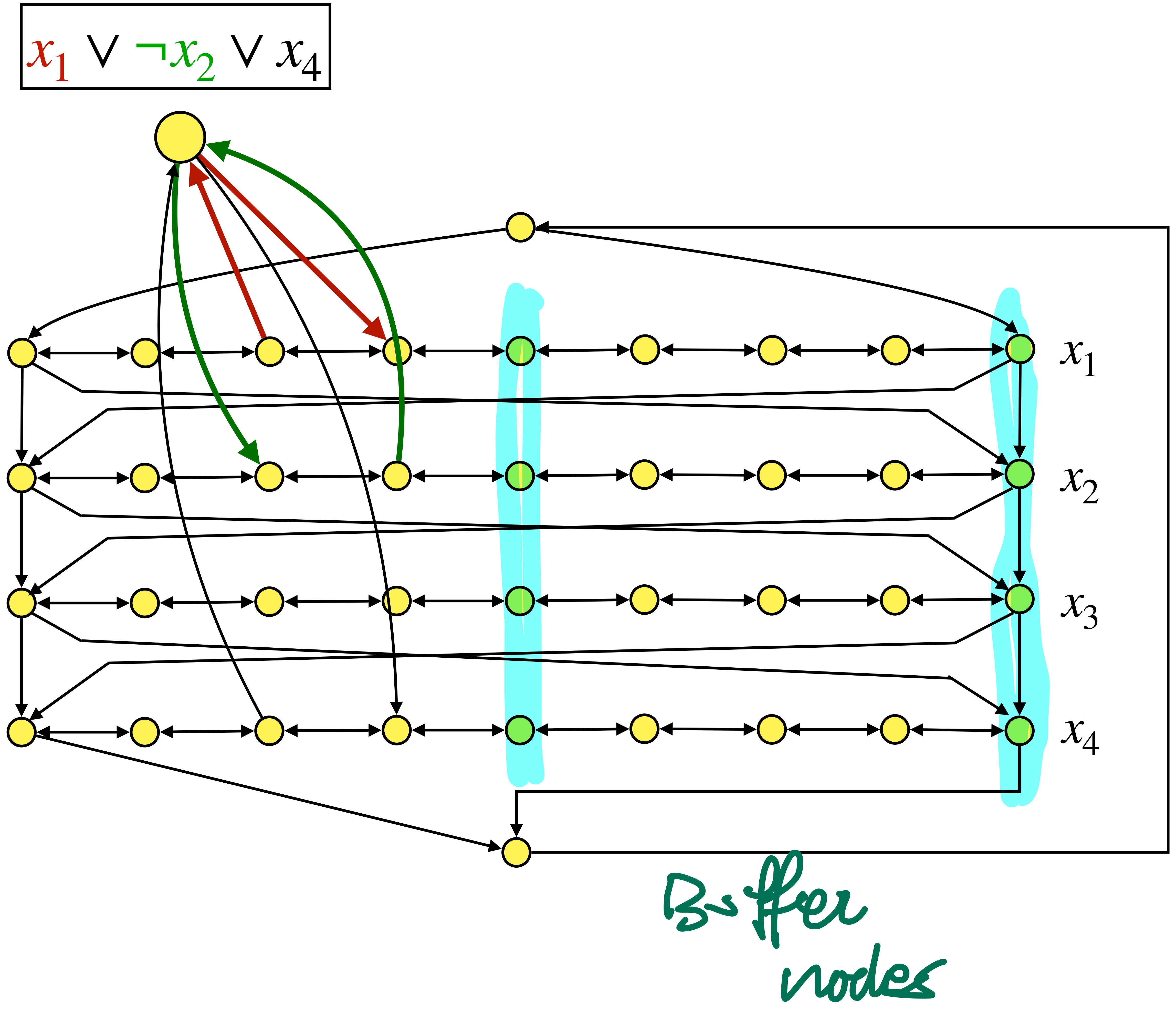

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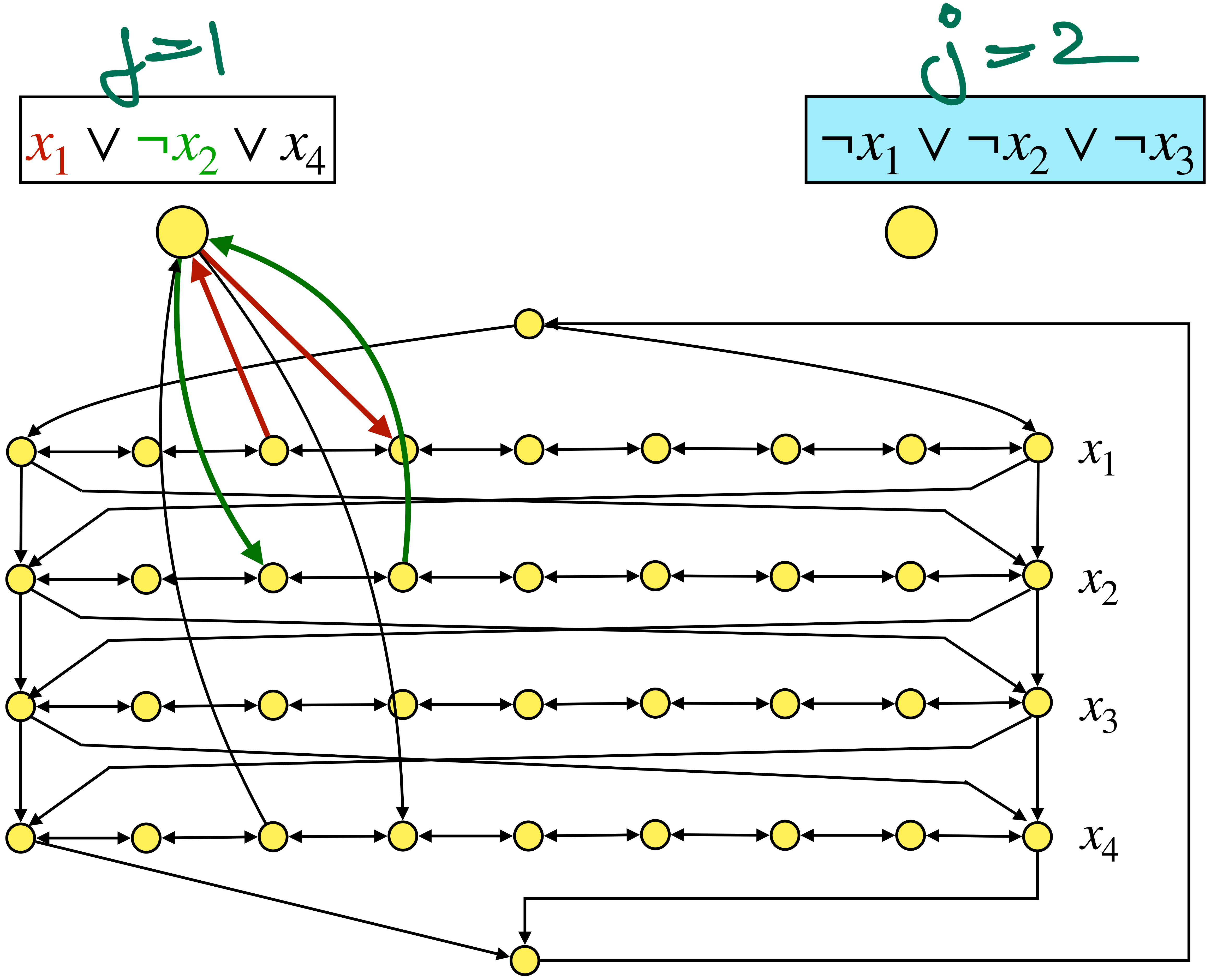

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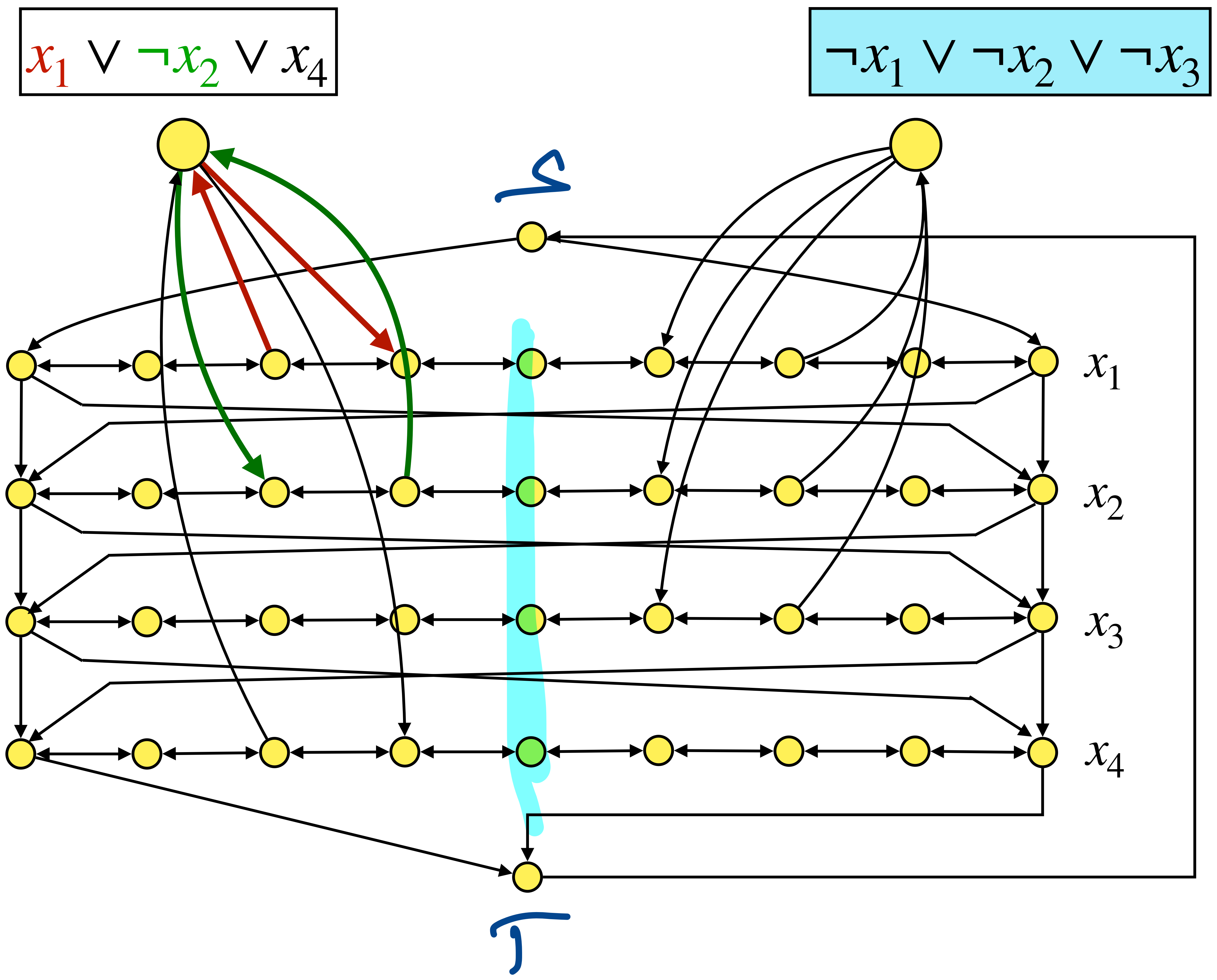

Correctness proof

- **Theorem:** φ has a satisfying assignment ***iff*** G_φ has a Hamiltonian cycle.
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- ***Only if***: If φ has a satisfying assignment then G_φ has a Hamilton cycle.
  - By construction (we just did it)
- ***If***: If G_φ has a Hamilton cycle then φ has a satisfying assignment.
  - Far more involved … we will skip (see Kani's archived slides).

Hamiltonian cycle in undirected graphs

Problem

Input: Given undirected graph $G = (V, E)$

Goal: Does G have a Hamiltonian cycle?

Hamiltonian cycle in undirected graphs

Problem

Input: Given undirected graph $G = (V, E)$

Goal: Does G have a Hamiltonian cycle?

That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?

NP-Completeness

Theorem: *Hamiltonian cycle* problem for undirected graphs is NP-complete.

Proof

NP-Completeness

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Proof

- The problem is in **NP**; proof left as exercise.

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NP-Completeness

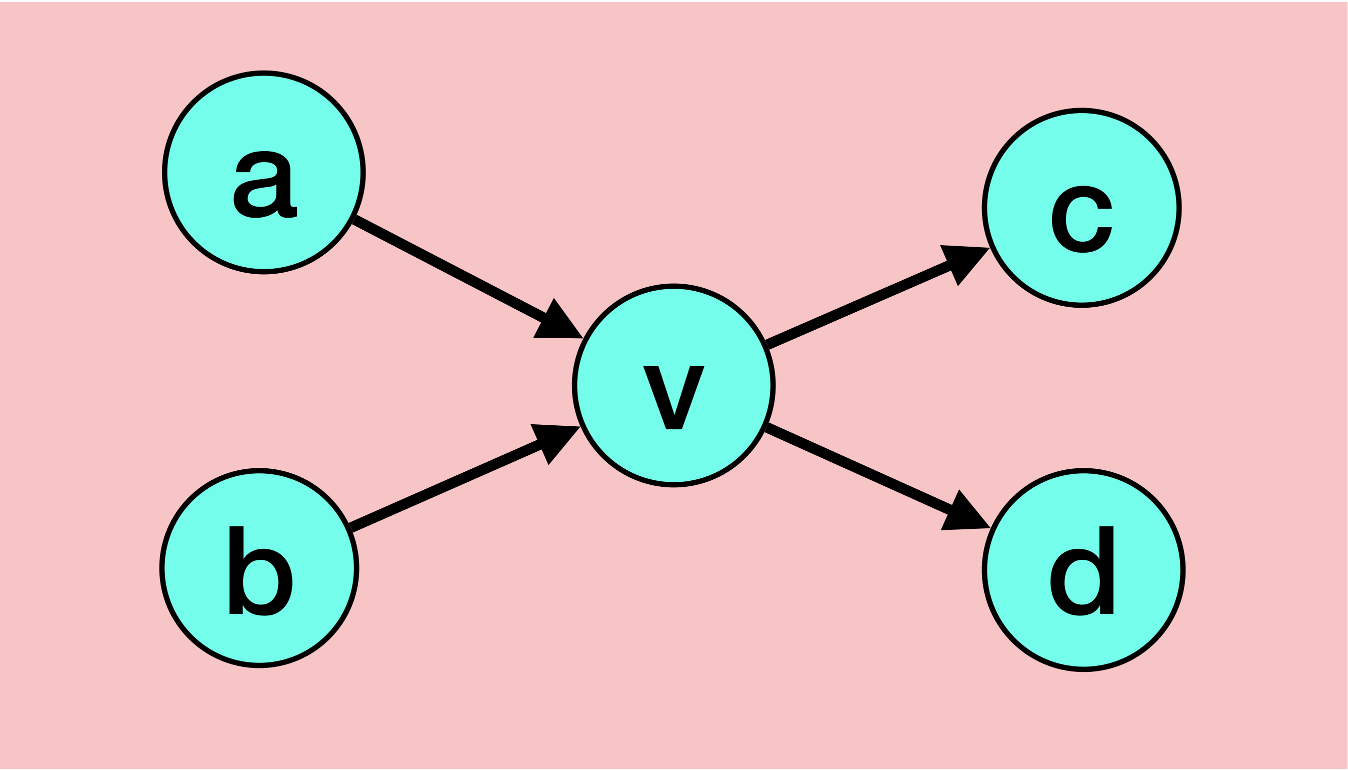
Theorem: *Hamiltonian cycle* problem for undirected graphs is NP-complete.

Proof

- The problem is in **NP**; proof left as exercise.
- Hardness proved by reducing directed Hamiltonian cycle to this problem
  - Need to go from *directed graph to undirected graph*

Reduction sketch

Goal: Given directed graph G, need to construct undirected graph G' such that G has Hamiltonian cycle iff G' has Hamiltonian cycle.



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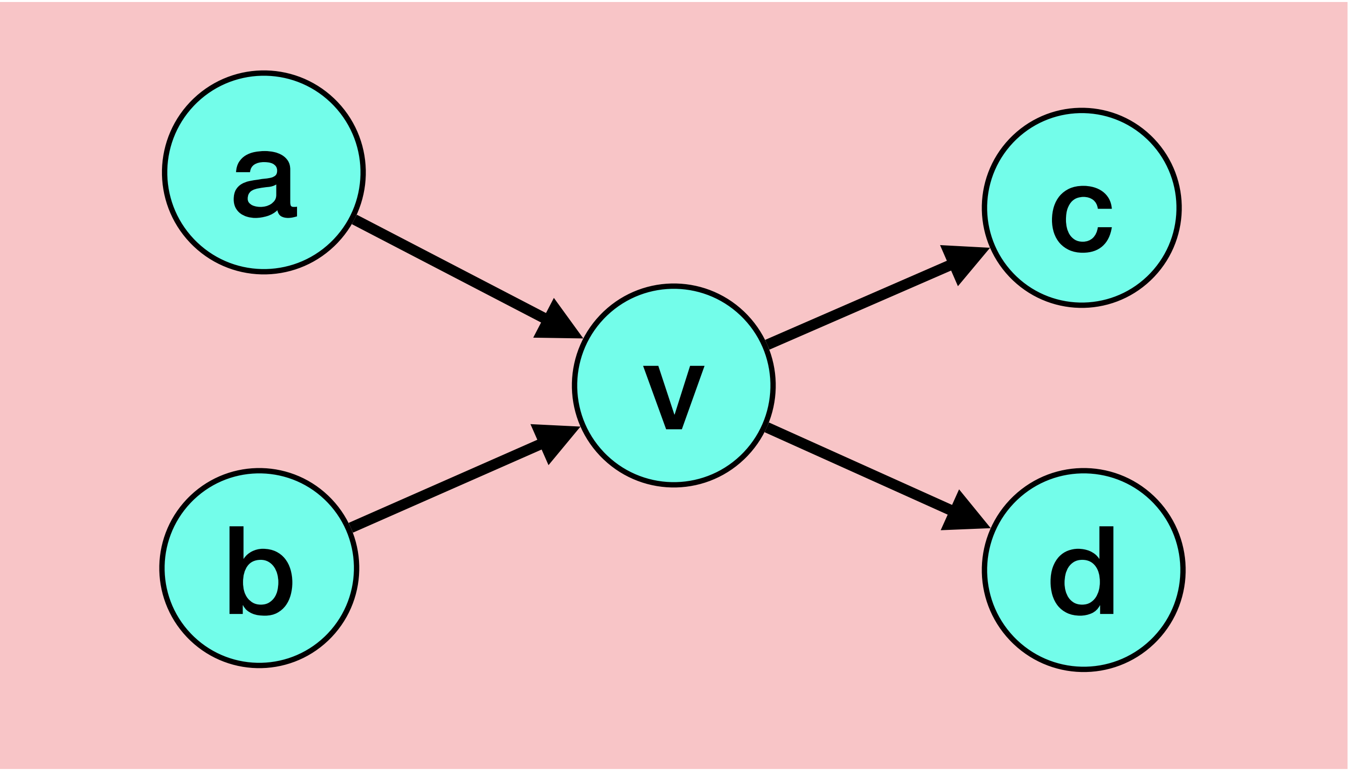
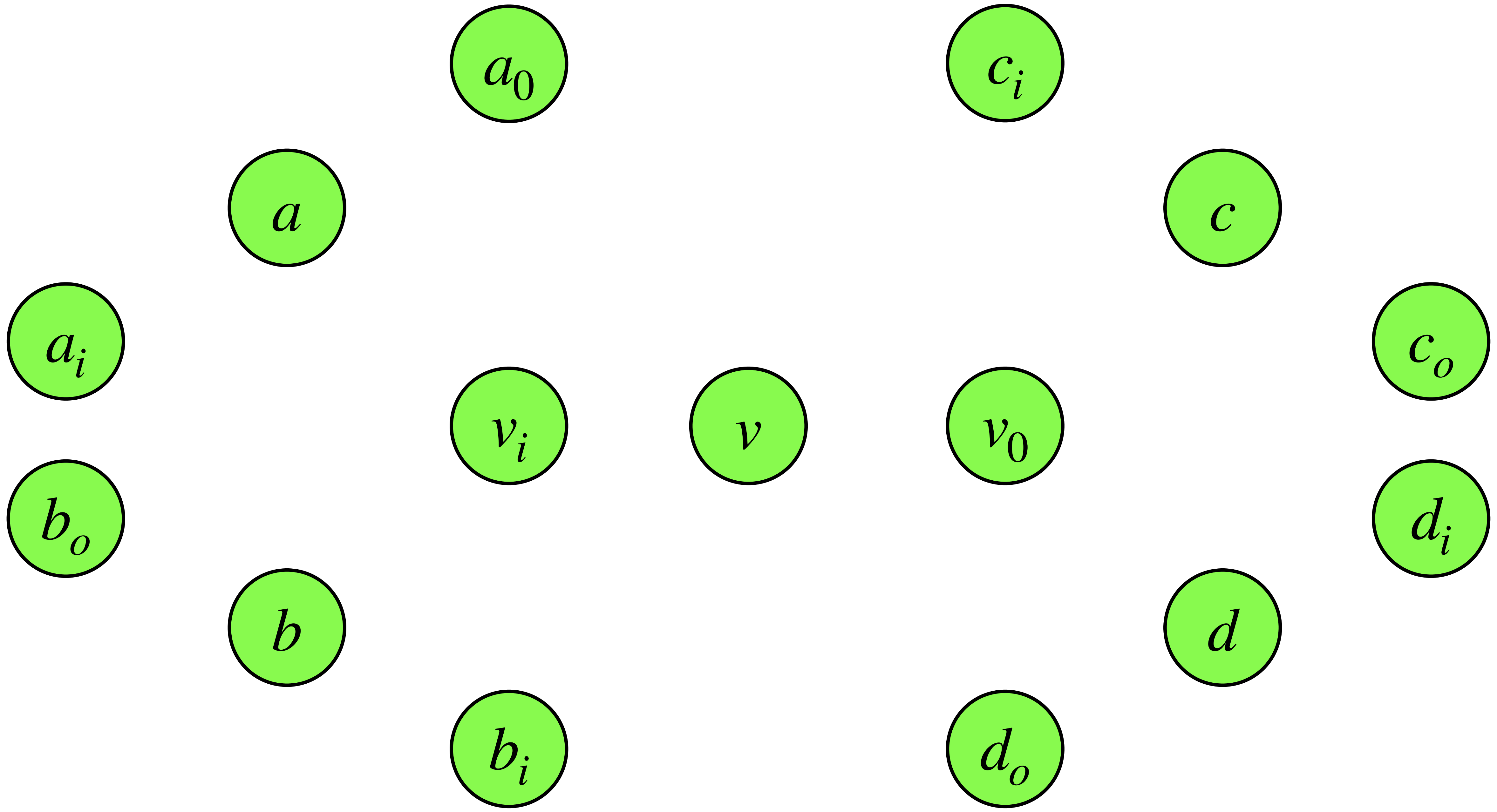
- Replace each vertex v by 3 vertices: v_i, v, and v_o





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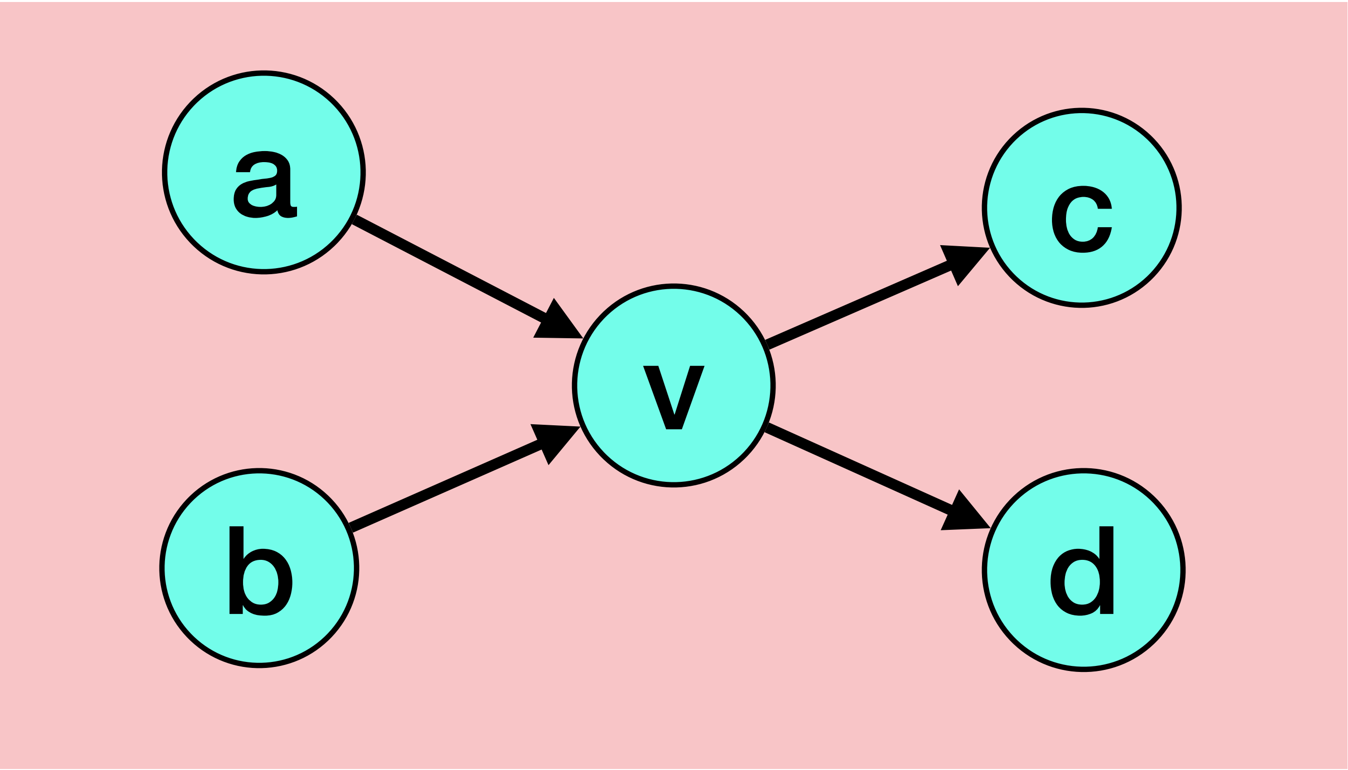

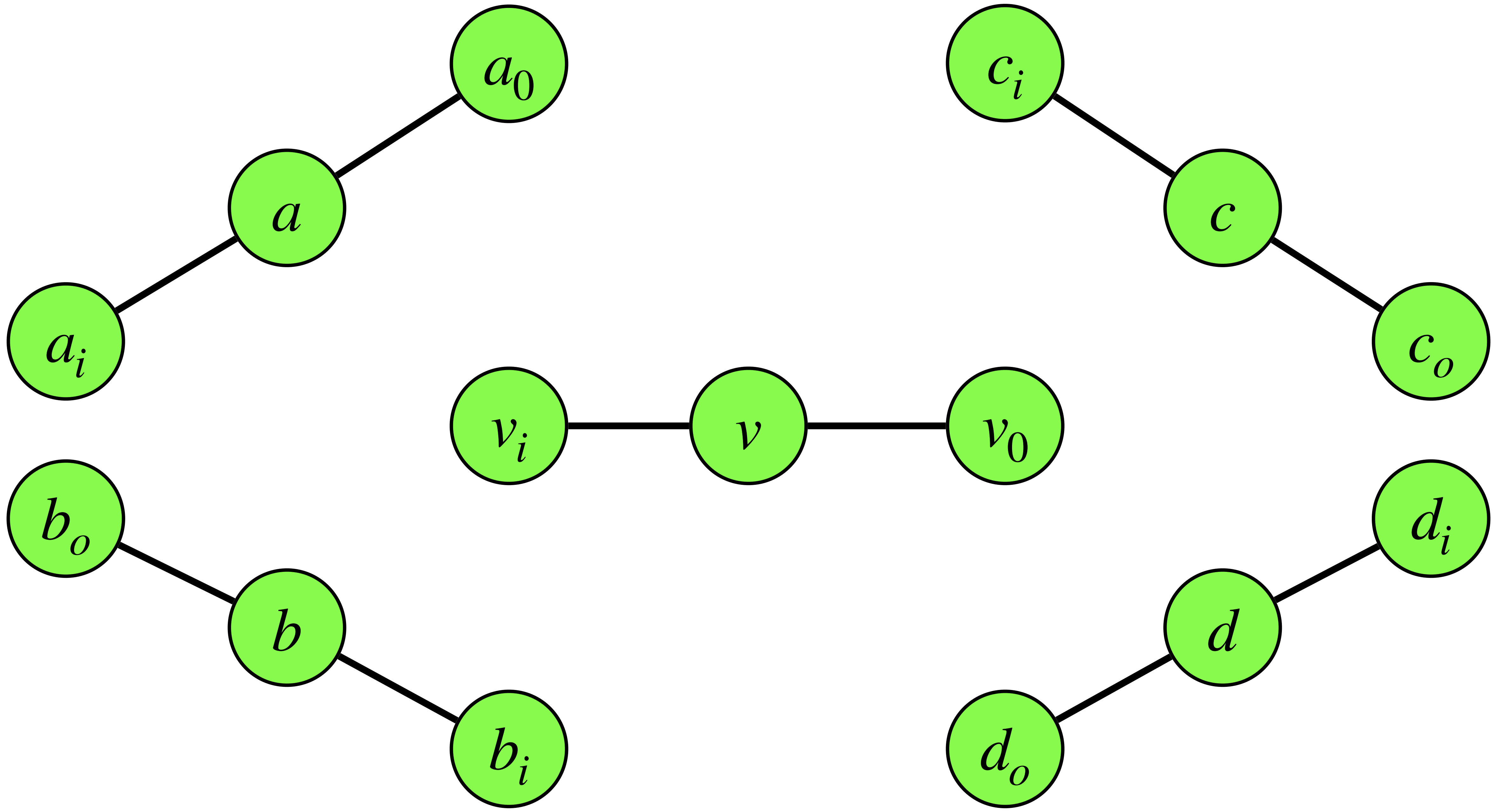

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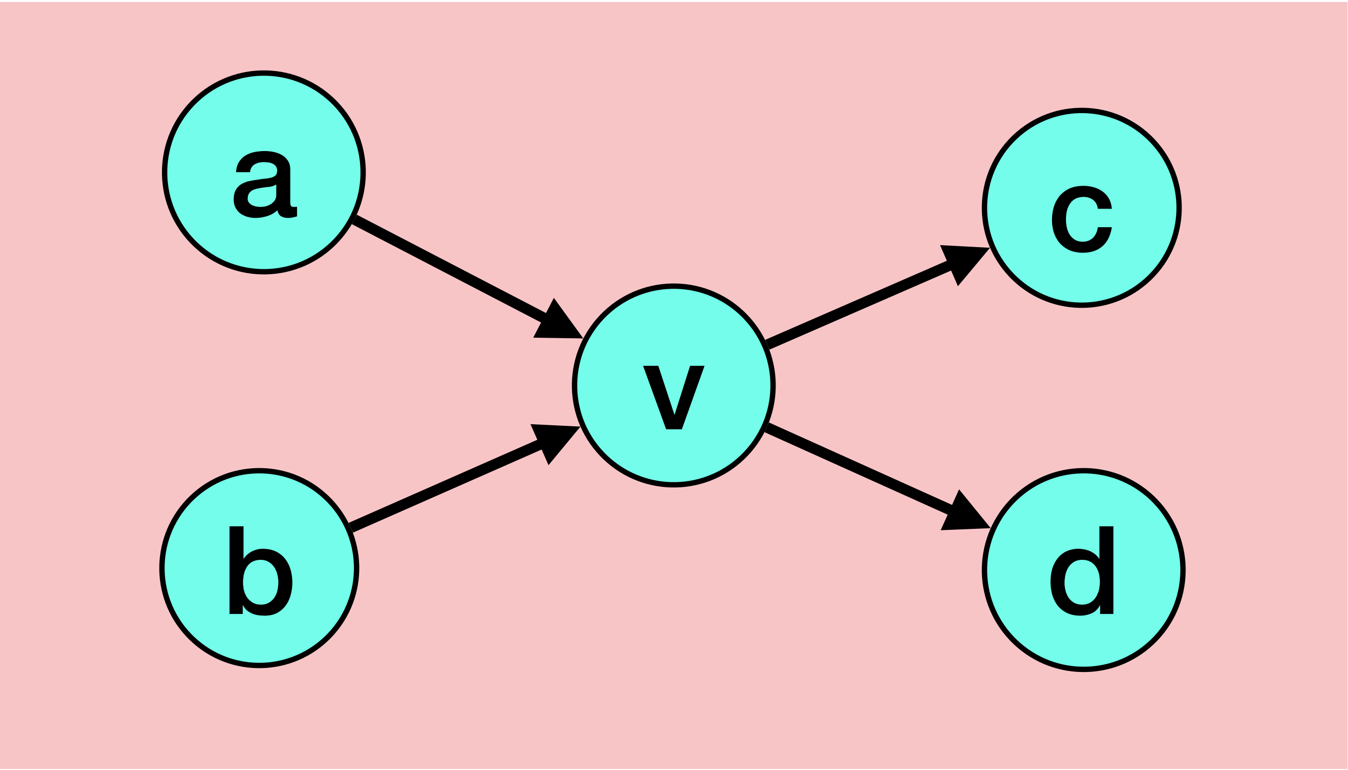
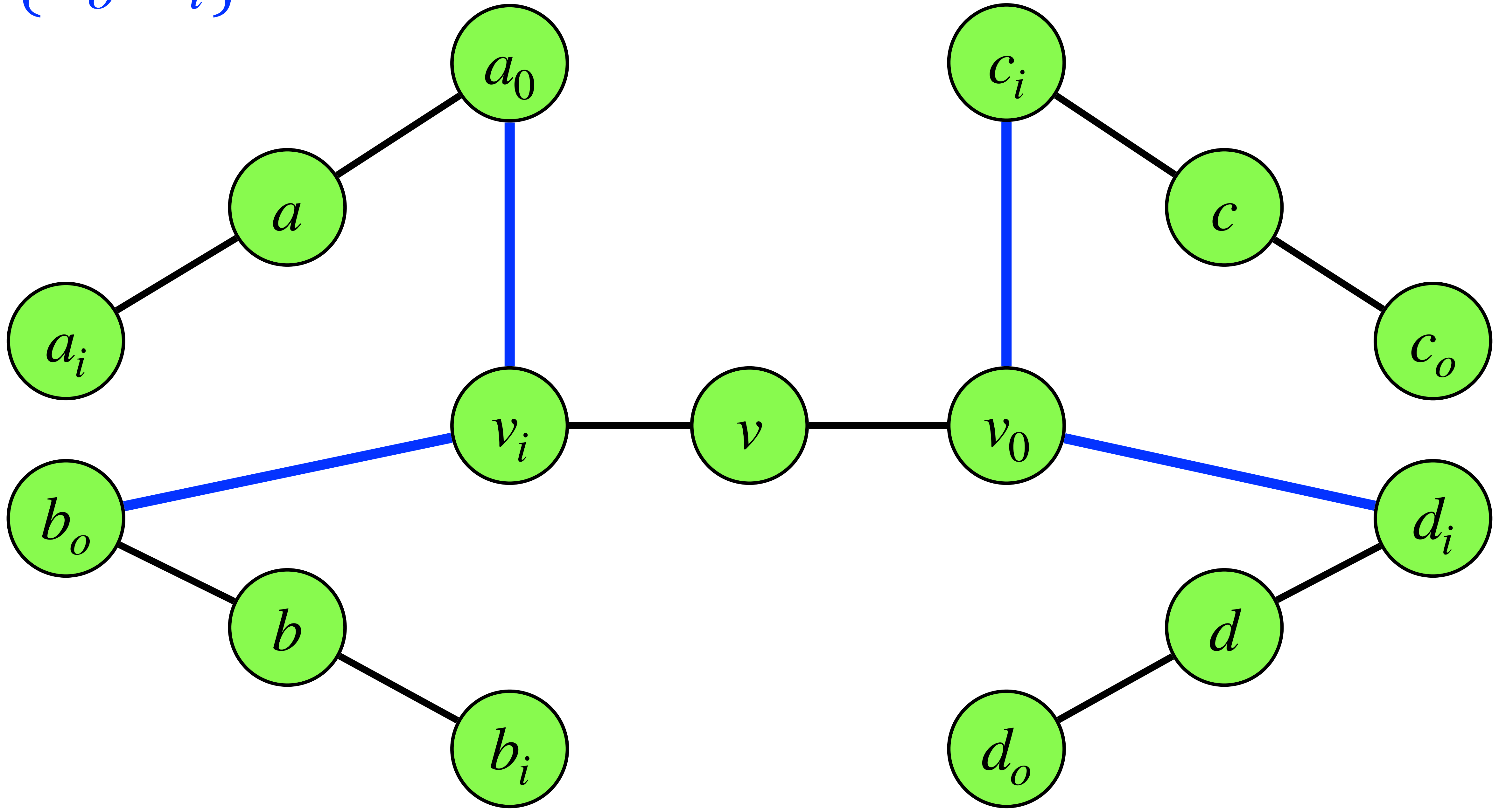
- Replace each vertex v by 3 vertices: v_i, v, and v_o
- A directed edge (a, b) is replaced by edge $\{a_o, b_i\}$





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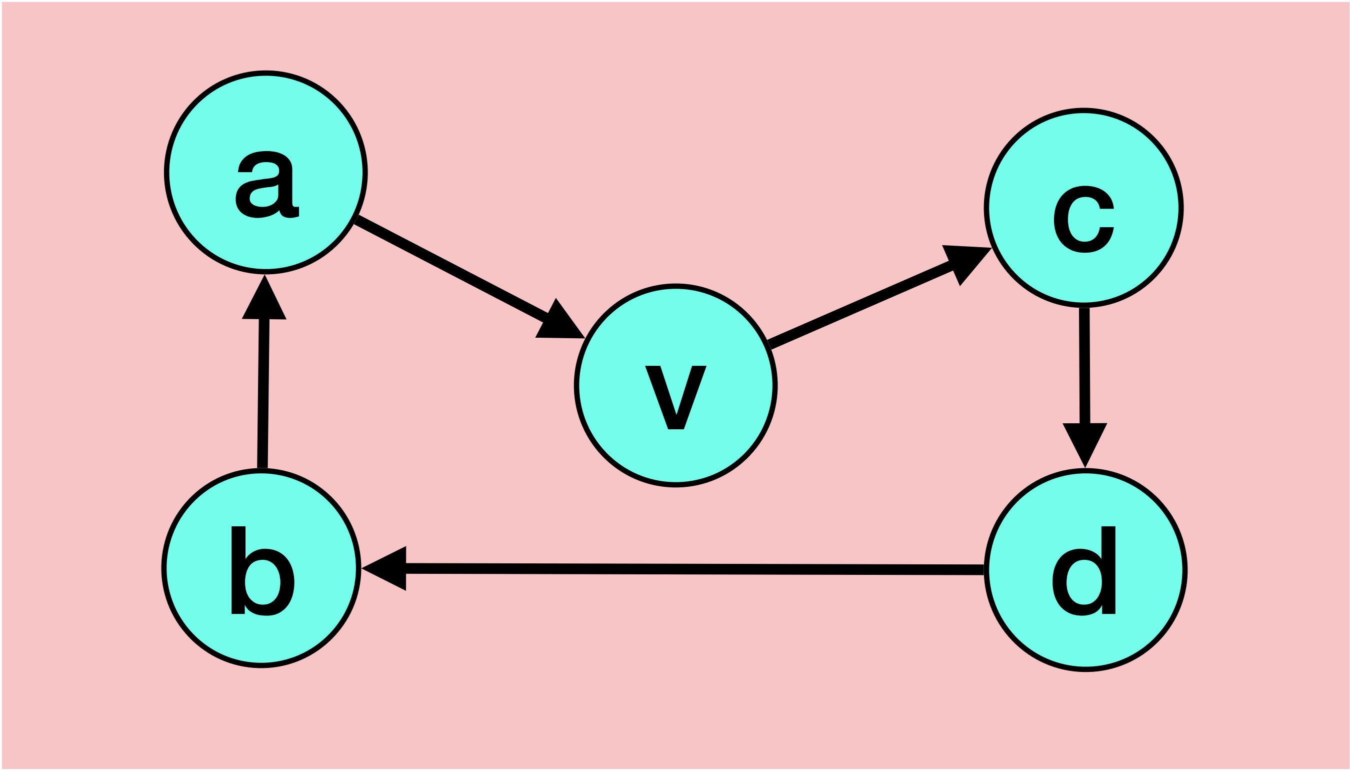
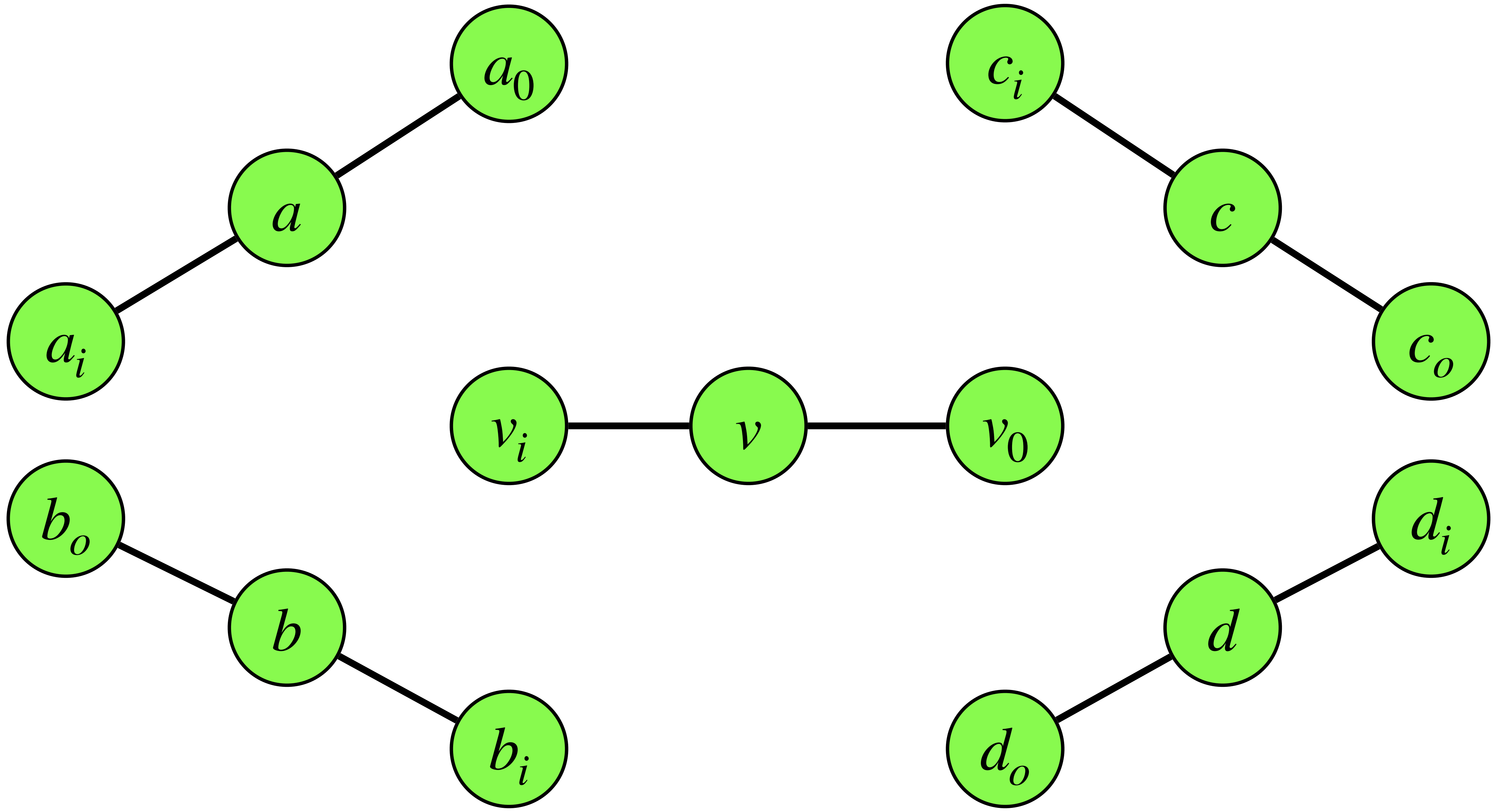
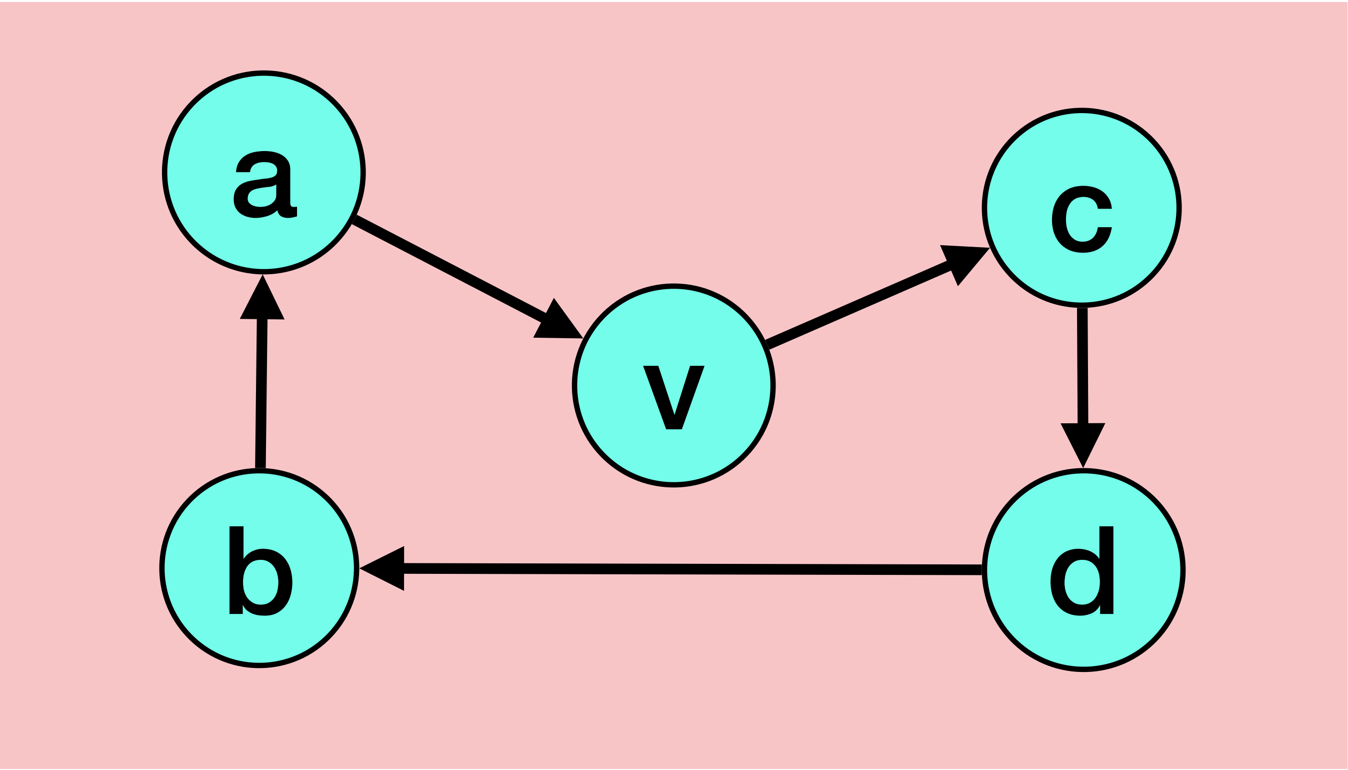


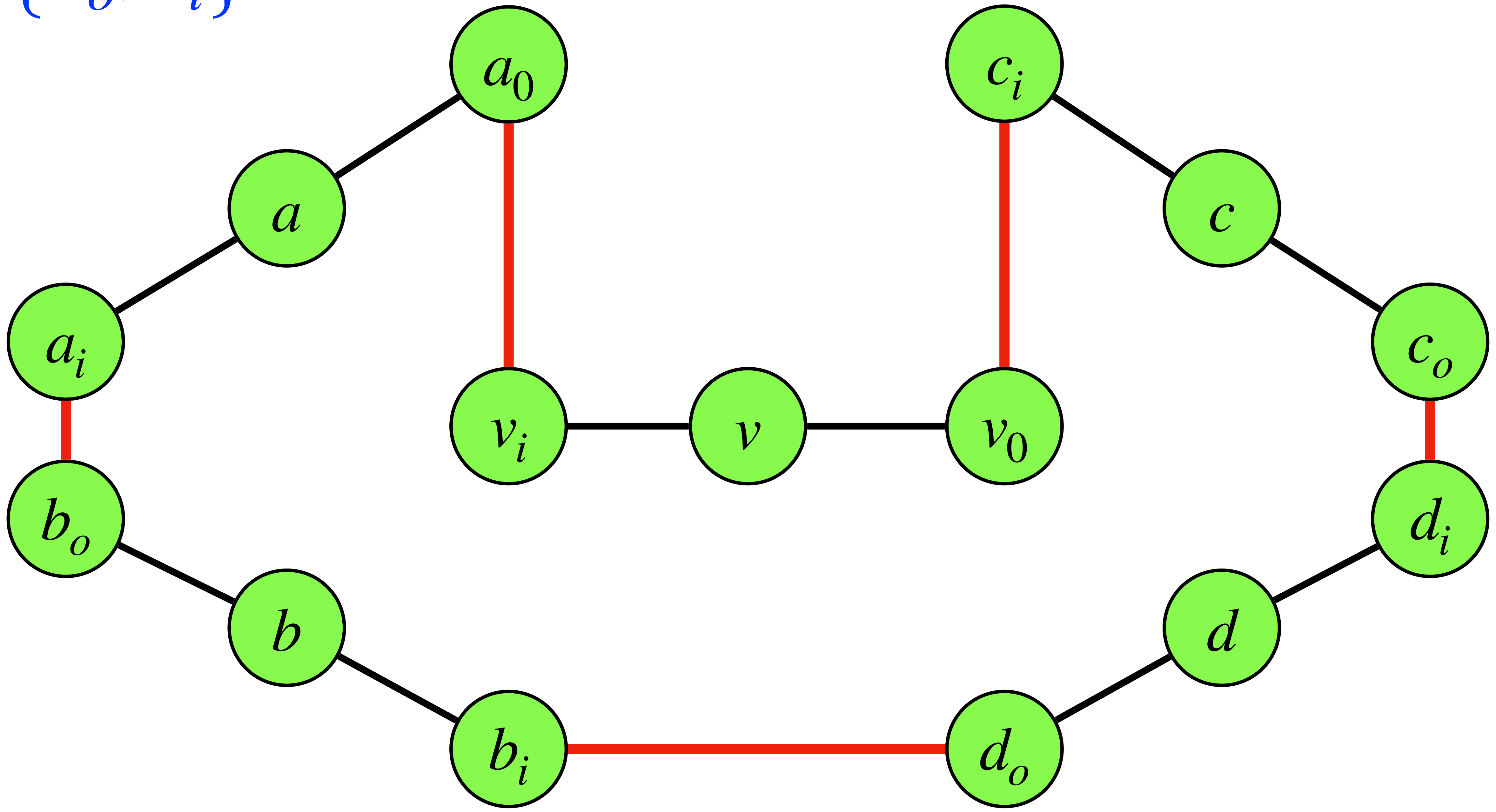


Reduction sketch

Goal: Given directed graph G, need to construct undirected graph G' such that G has Hamiltonian cycle iff G' has Hamiltonian cycle.

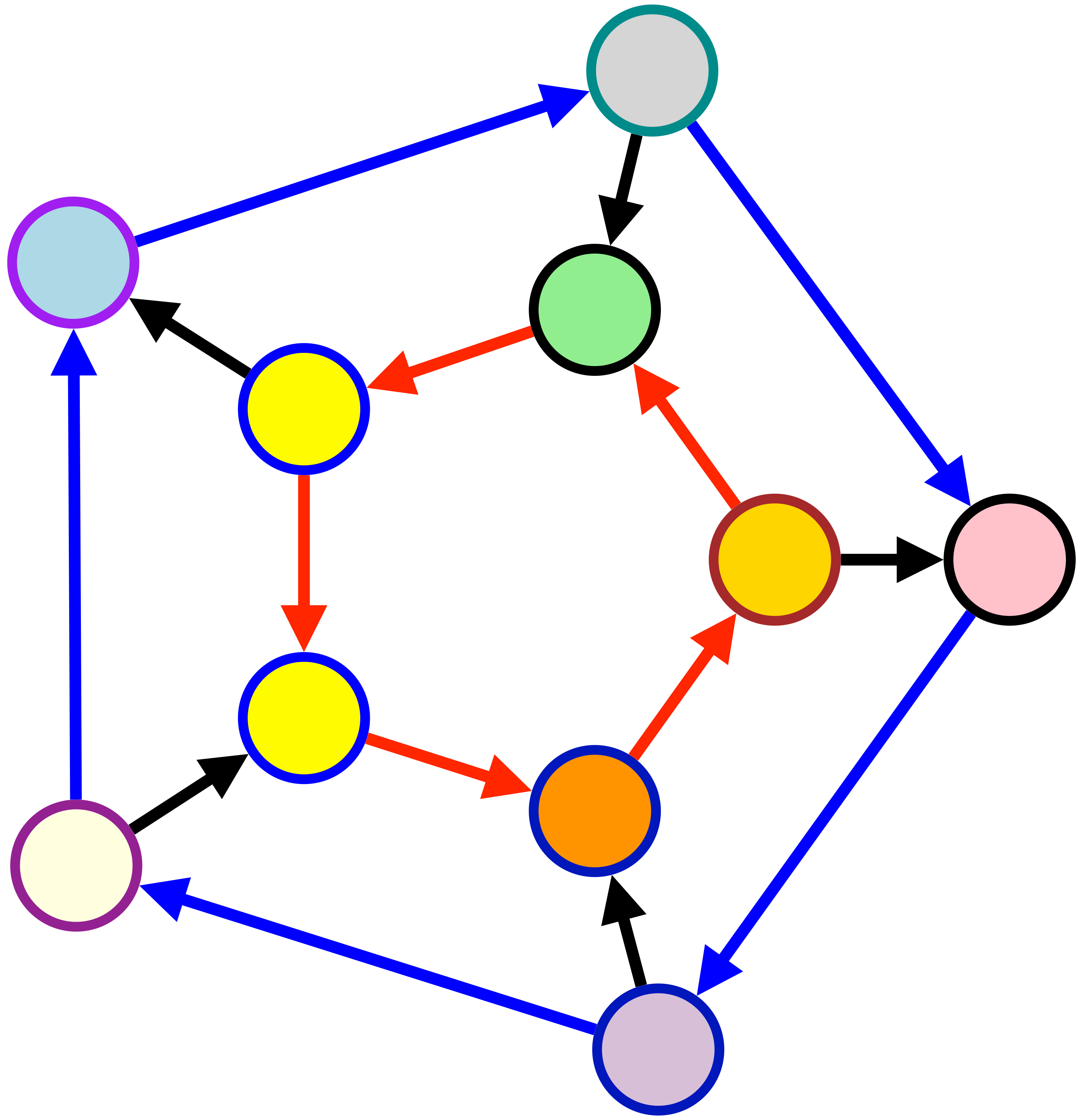
- Replace each vertex v by 3 vertices: v_i, v, and v_o
- A directed edge (a, b) is replaced by edge $\{a_o, b_i\}$





Hamiltonian cycle reduction

Directed to undirected



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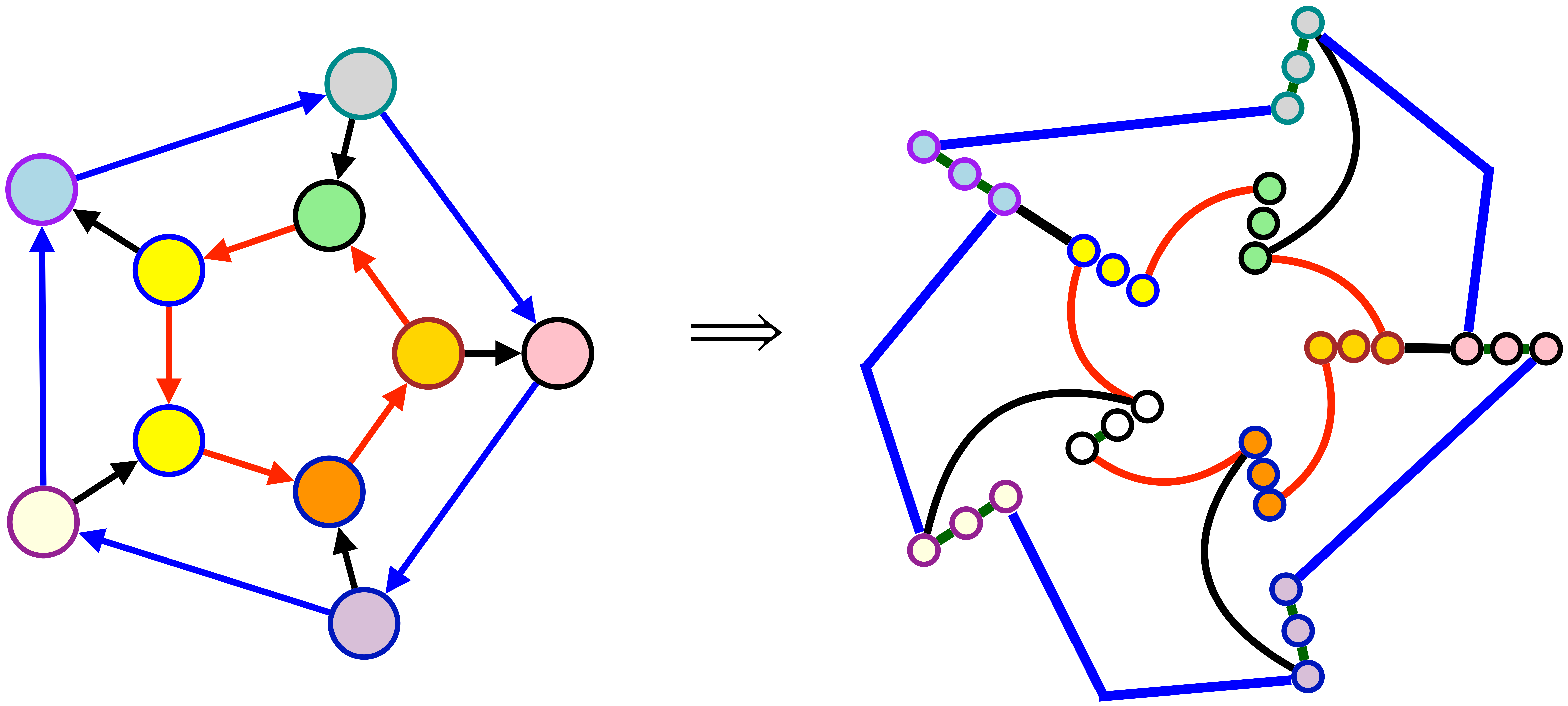

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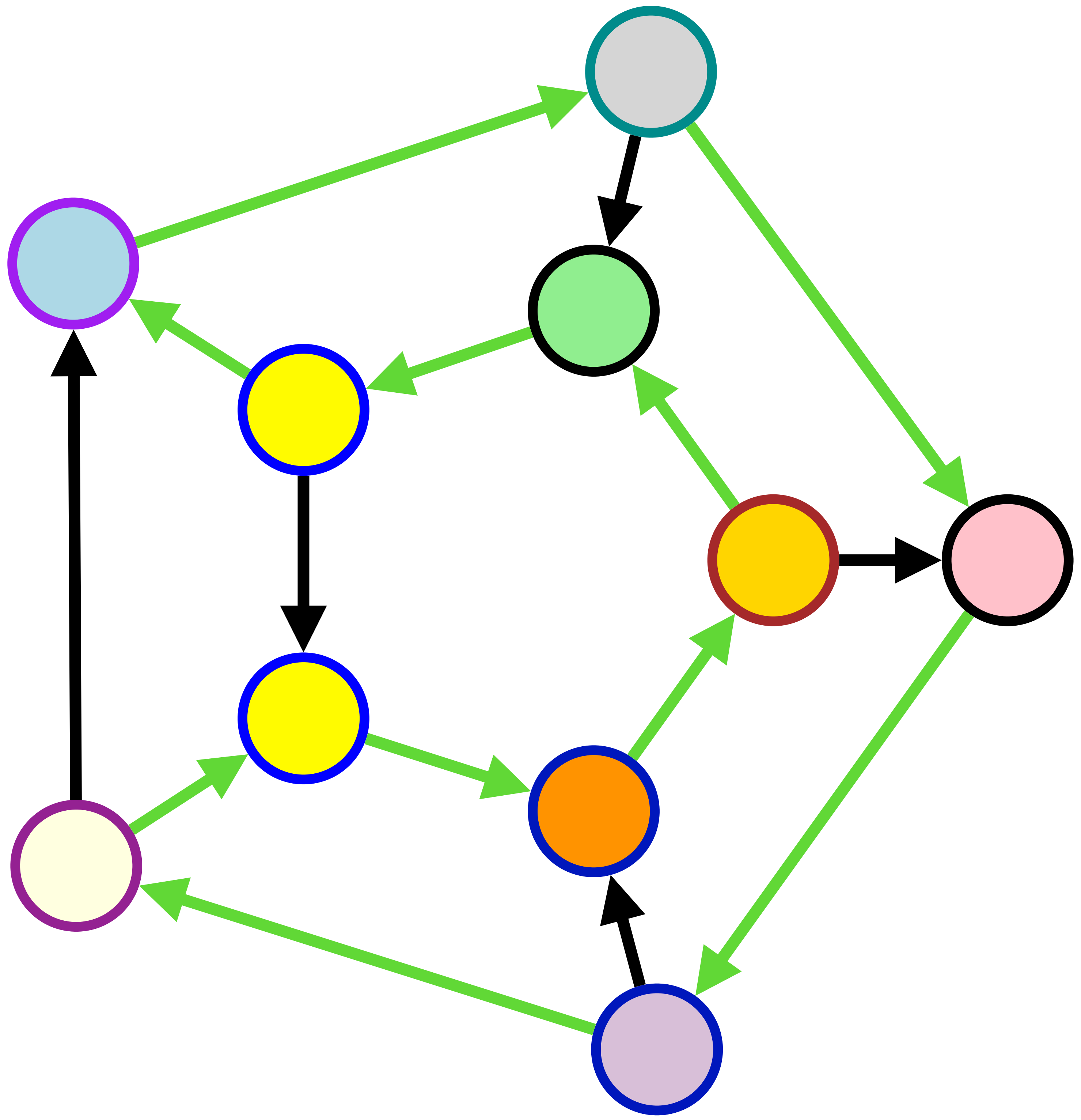
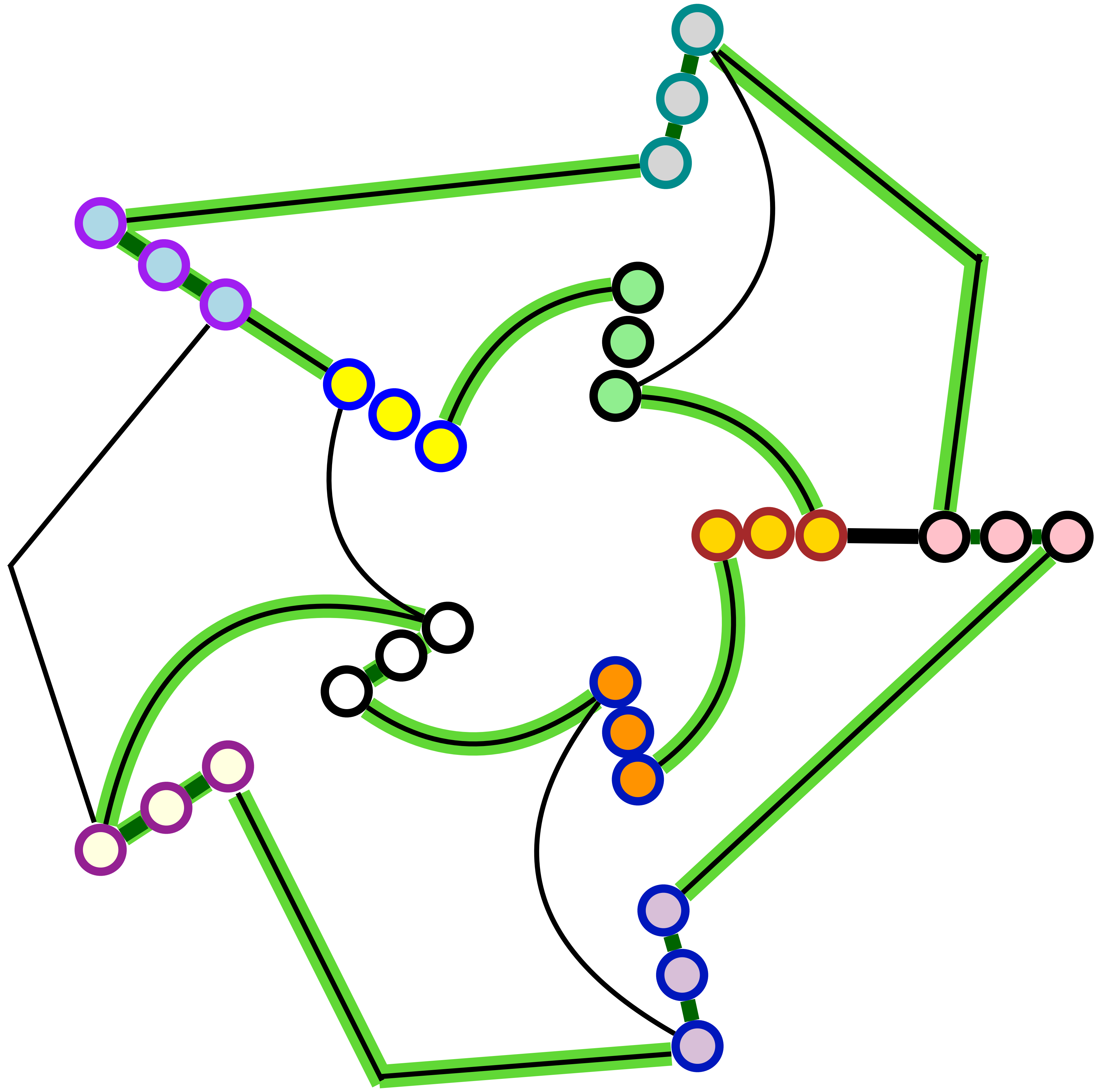


$\Longrightarrow$



Hamiltonian paths

Input: Given a graph $G = (V, E)$ with n vertices

Goal: Does G have a Hamiltonian path?

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Theorem: *Directed Hamiltonian Path* and *Undirected Hamiltonian Path* are NP-Complete.

Modify the reduction from *3-SAT* to *Hamiltonian Cycle* or do a reduction from *Hamiltonian Cycle* (homework?)

NP-completeness of graph coloring

Generic graph coloring

Instance: $G = (V, E)$: Undirected graph, integer k.

Question: Can the vertices of the graph be colored using k colors so that vertices connected by an edge **do not** get the same color?

NP-completeness of graph coloring

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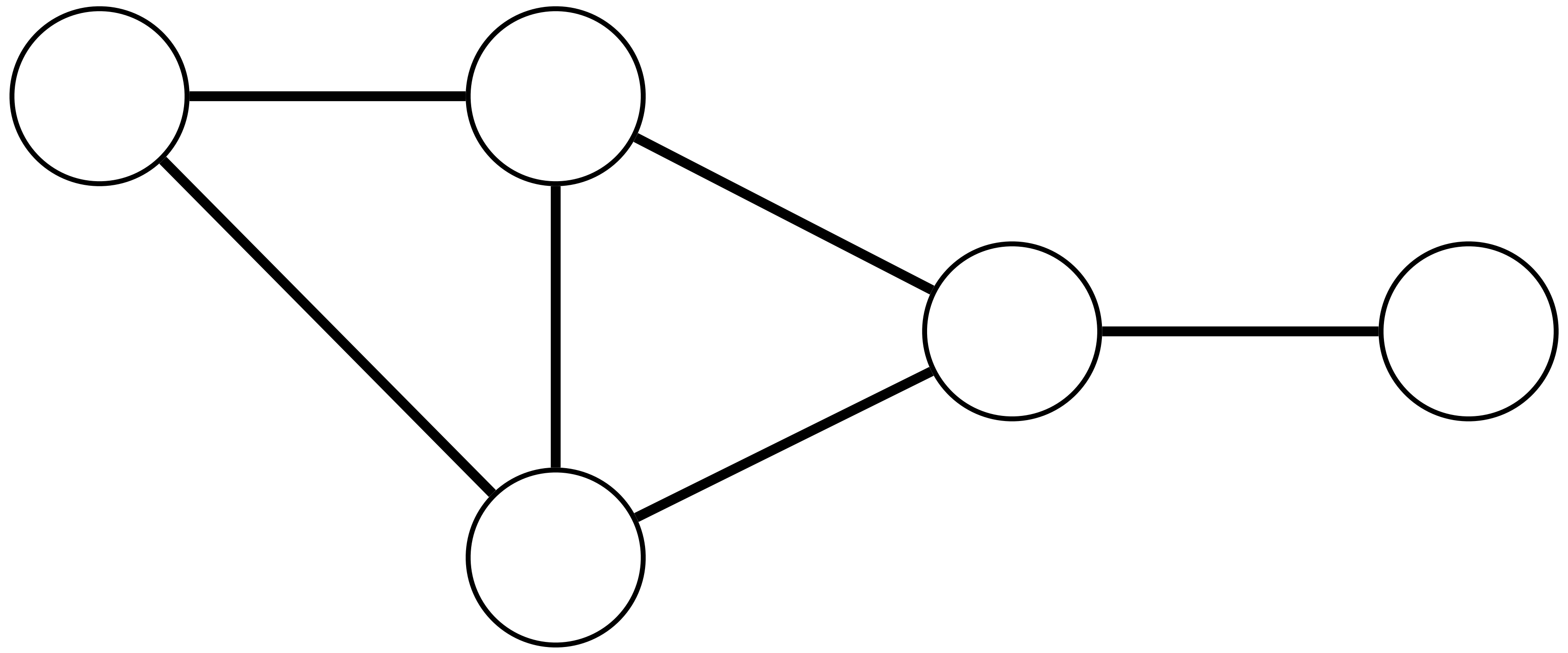

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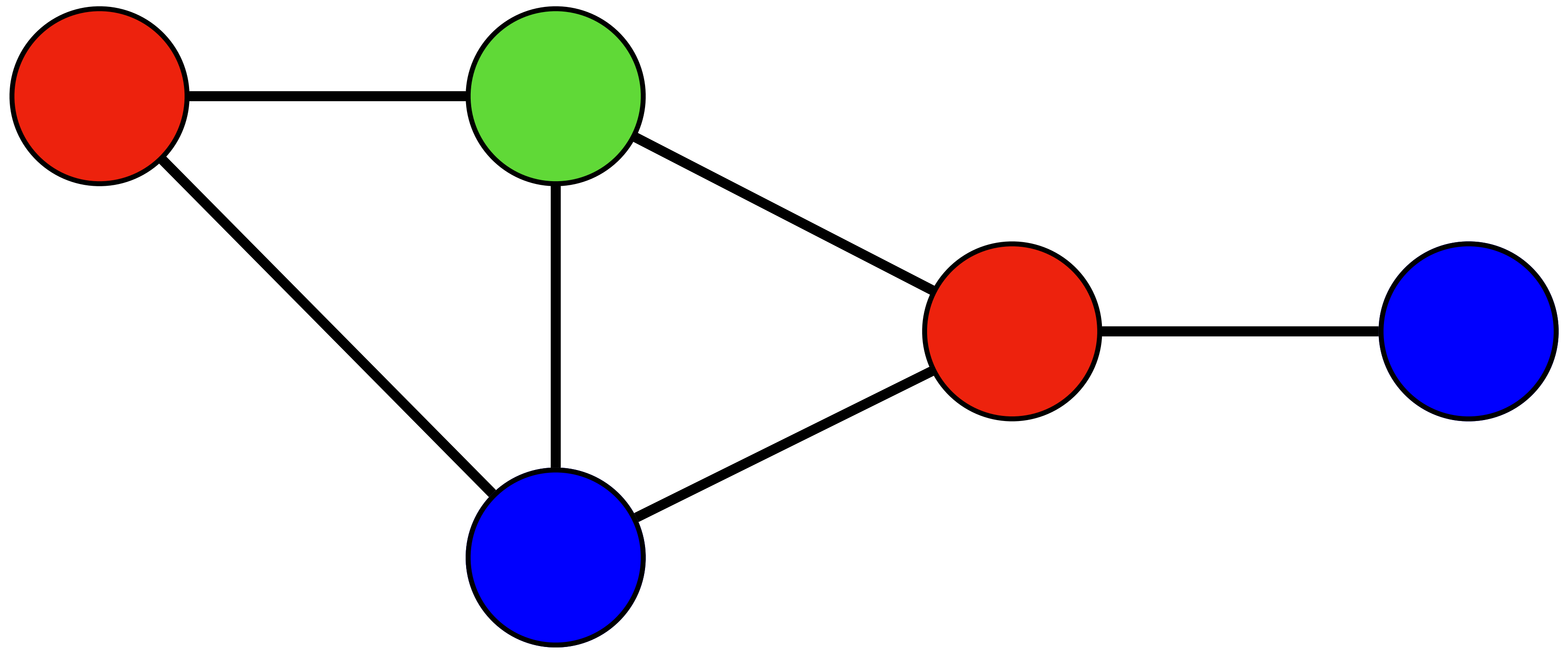

Graph coloring

Graph 2-Coloring

$G=(\mathcal{U},\mathcal{V},E)$ s.t $\mathcal{U}\cap\mathcal{V}=\emptyset$

and $e=(u,v)\in E$ s.t

$u\in\mathcal{U}$, $v\in\mathcal{V}$

Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G. Thus, G can be partitioned into k independent sets ***if and only if*** G is k-colorable.

Graph 2-Coloring can be decided in polynomial time.

- G is 2-colorable ***iff*** G is bipartite! There is a linear time algorithm to check if G is bipartite using Breadth-First-Search.

[Chatin] https://dl.acm.org/doi/10.1145/989393.989403

Problems related to graph coloring

Graph coloring and register allocation

Register Allocation: Assign variables to (at most) k registers such that variables needed at the same time are not assigned to the same register

Interference Graph: Vertices are variables, and there is an edge between two vertices, if the two variables are “live” at the same time.

Observations

- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with k colors
- Moreover, 3-COLOR $\leq_P k$ - Register Allocation, for any $k \geq 3$

Problems related to graph coloring

Frequency assignments in cellular networks

Cellular telephone systems that use ***Frequency Division Multiple Access*** (FDMA) (example: GSM in Europe and Asia and AT&T in USA)

- Breakup a FCC given frequency range $[a, b]$ into disjoint bands of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
- Each cell phone tower (simplifying) gets one band

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Problem: given k bands and some region with n towers, is there a way to assign the bands to avoid interference?

Can reduce to k-coloring by creating interference/conflict graph on towers.

Showing hardness of 3-COLORING

3-Coloring is NP-Complete

- **3-Coloring** is in **NP**
  - Non-deterministically guess a 3-coloring for each node
  - Check if for each edge (u, v), the color of u is different from that of v
- Hardness: We will show 3-SAT $\leq_P$ 3-Coloring.

Reduction Idea I - Simple 3-color gadget

We want to create a ***gadget***[1] that:

- Is 3 colorable if at least one of the literals is true
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Assume green=true and red=false, essentially need to create an OR-gate with graph coloring.

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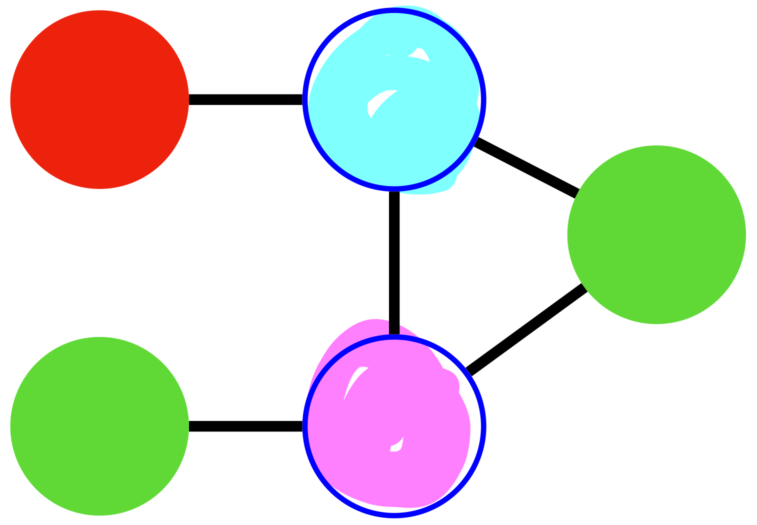
Fix output to be green

Reduction Idea I - Simple 3-color gadget

3rd color is blue.

$$f(X_1, X_2) = (X_1 \vee X_2)$$



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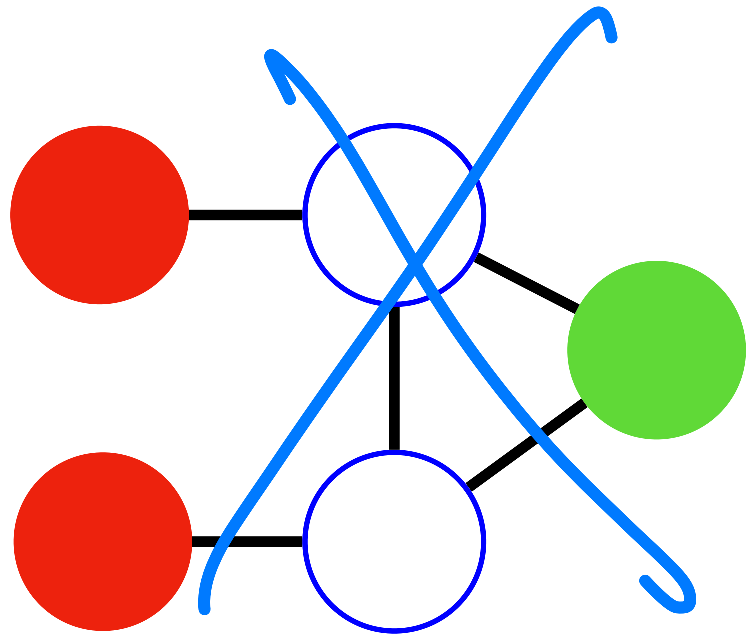

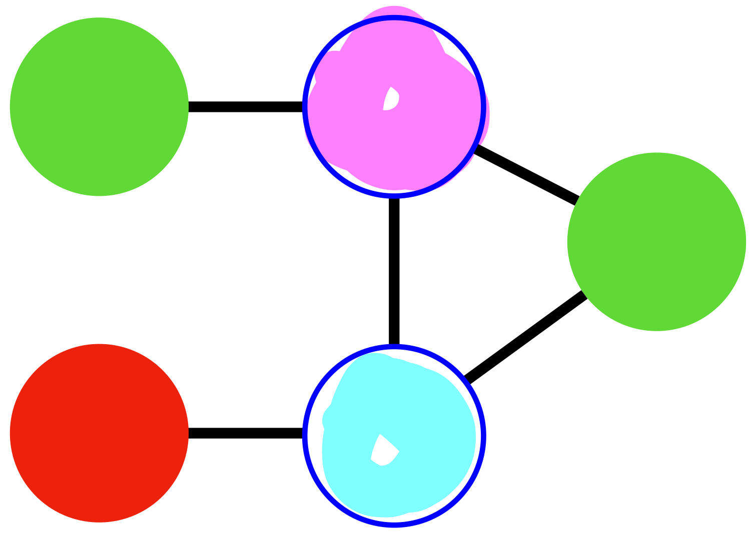

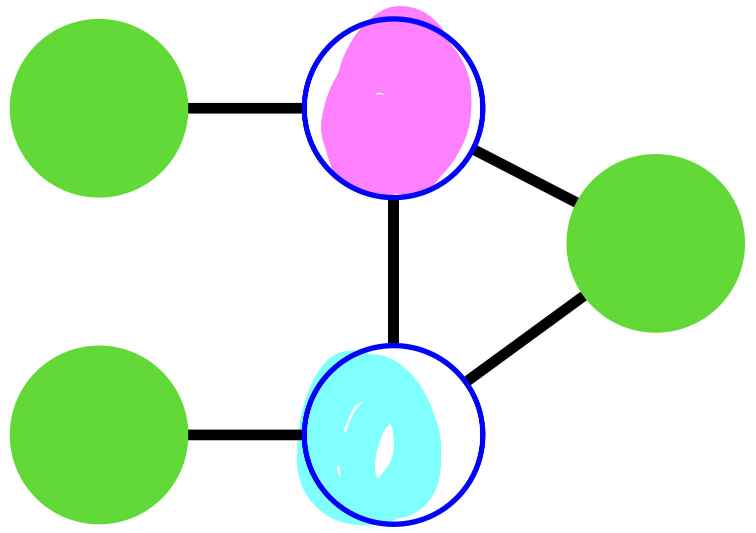

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We want to create a gadget that:

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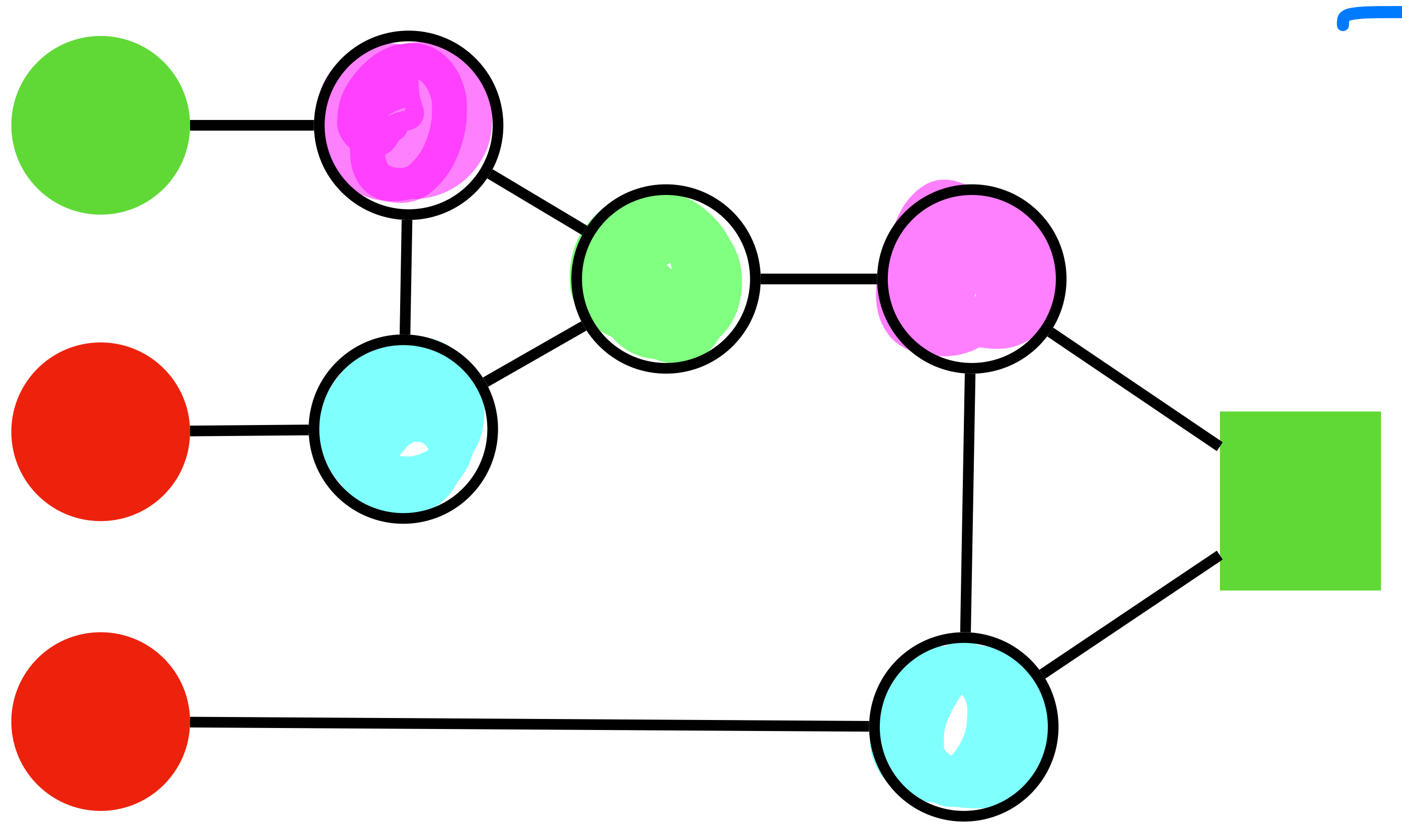
How do we do the same thing for 3 variables?:

$$f(X_1, X_2, X_3) = (X_1 \vee X_2 \vee X_3)$$

Assume green=true and red=false.

3-color this gadget I

You are given three colors: red, green and blue. Can the following graph be three colored in a valid way (assuming that some of the nodes are already colored as indicated).



acts as an OR.

A. Yes

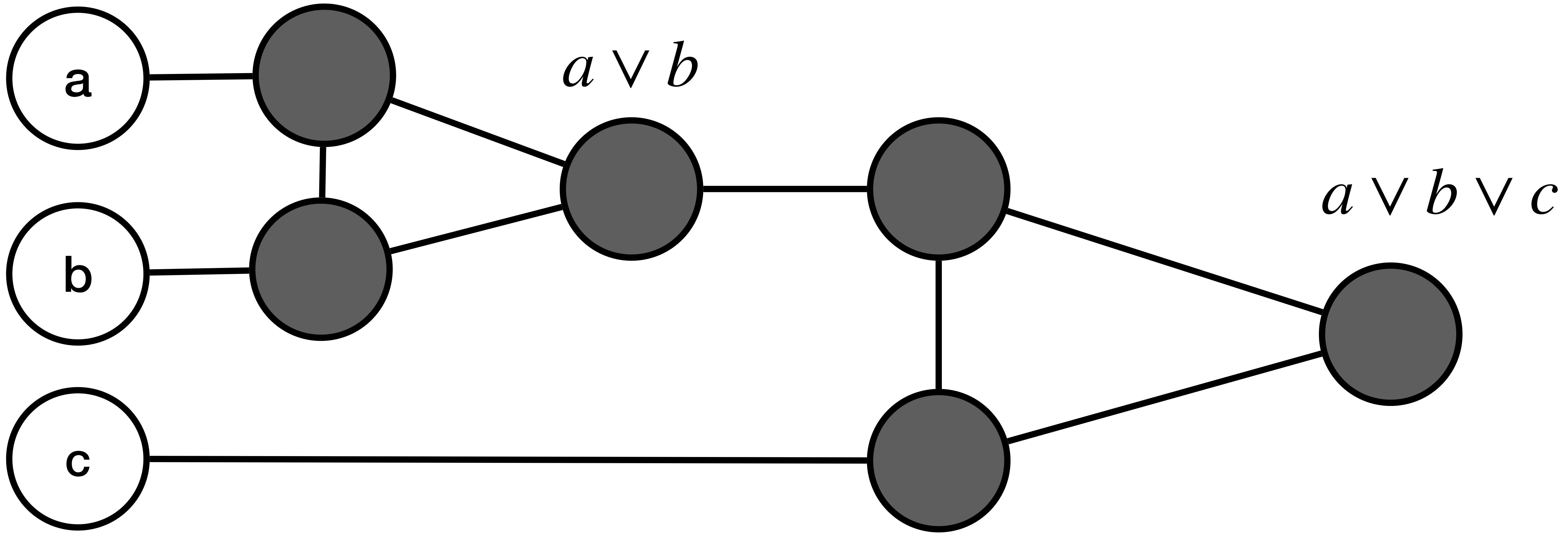
B. No

Clause Satisfiability gadget

For each clause $C_j = (a \vee b \vee c)$, create a small gadget graph

- gadget graph connects to nodes corresponding to a, b, c

- needs to implement OR



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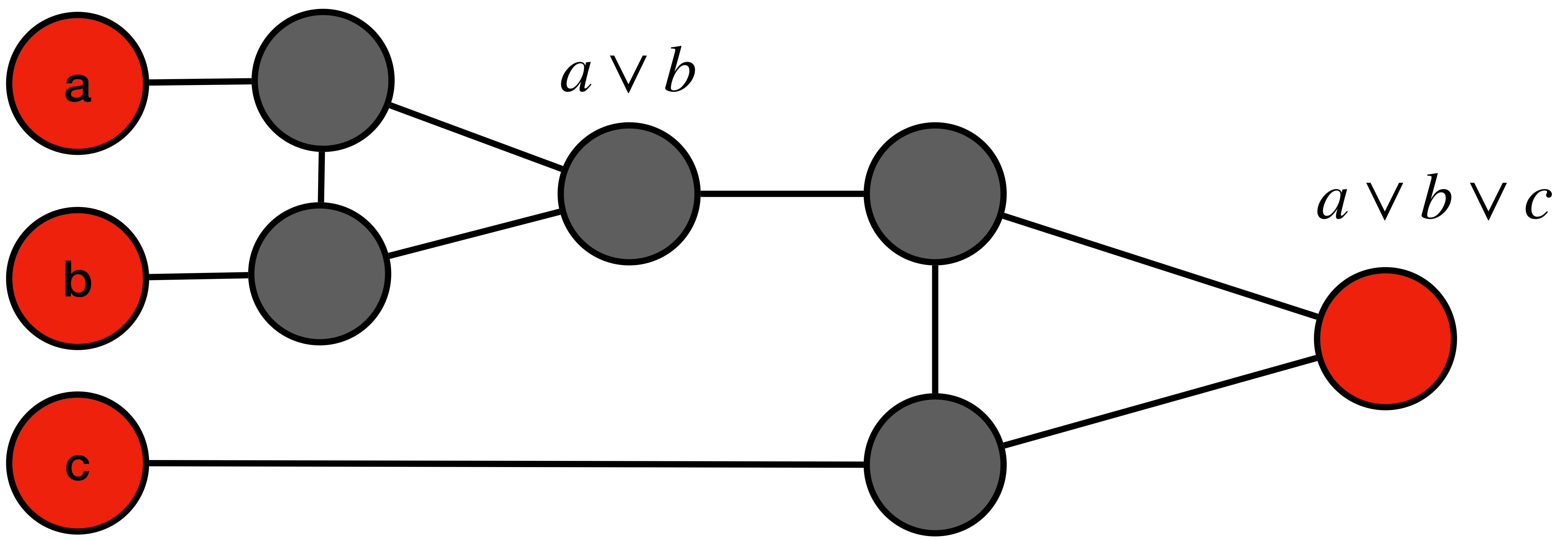

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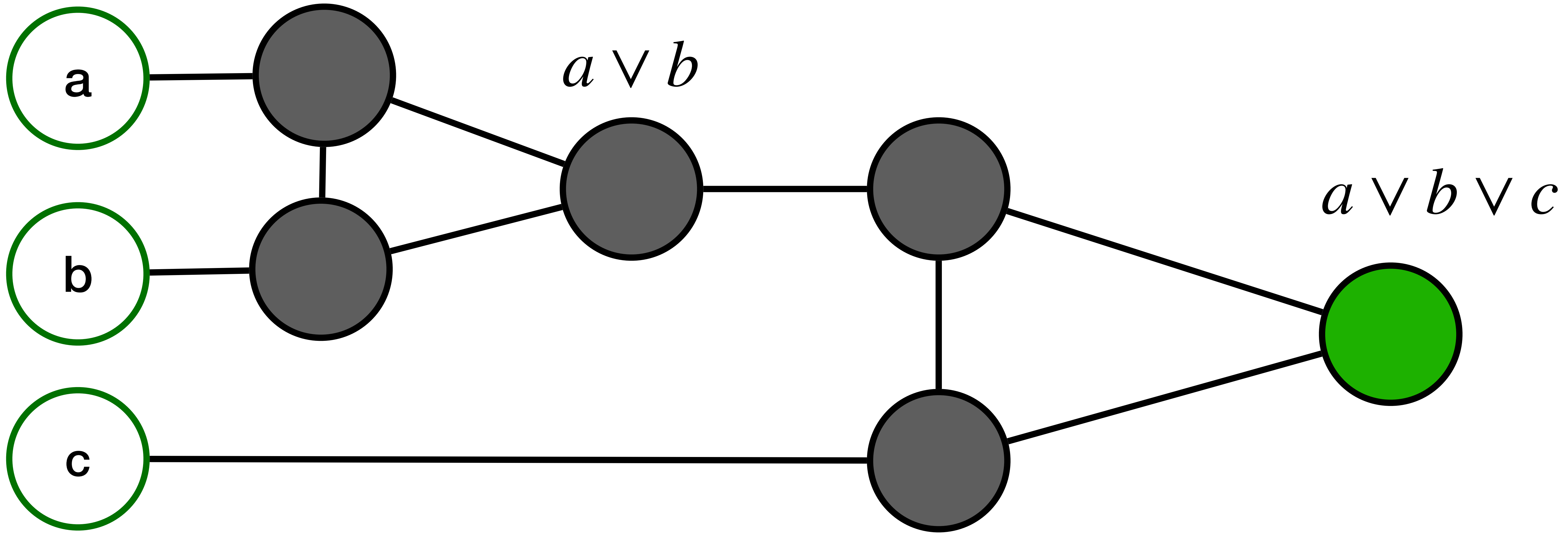
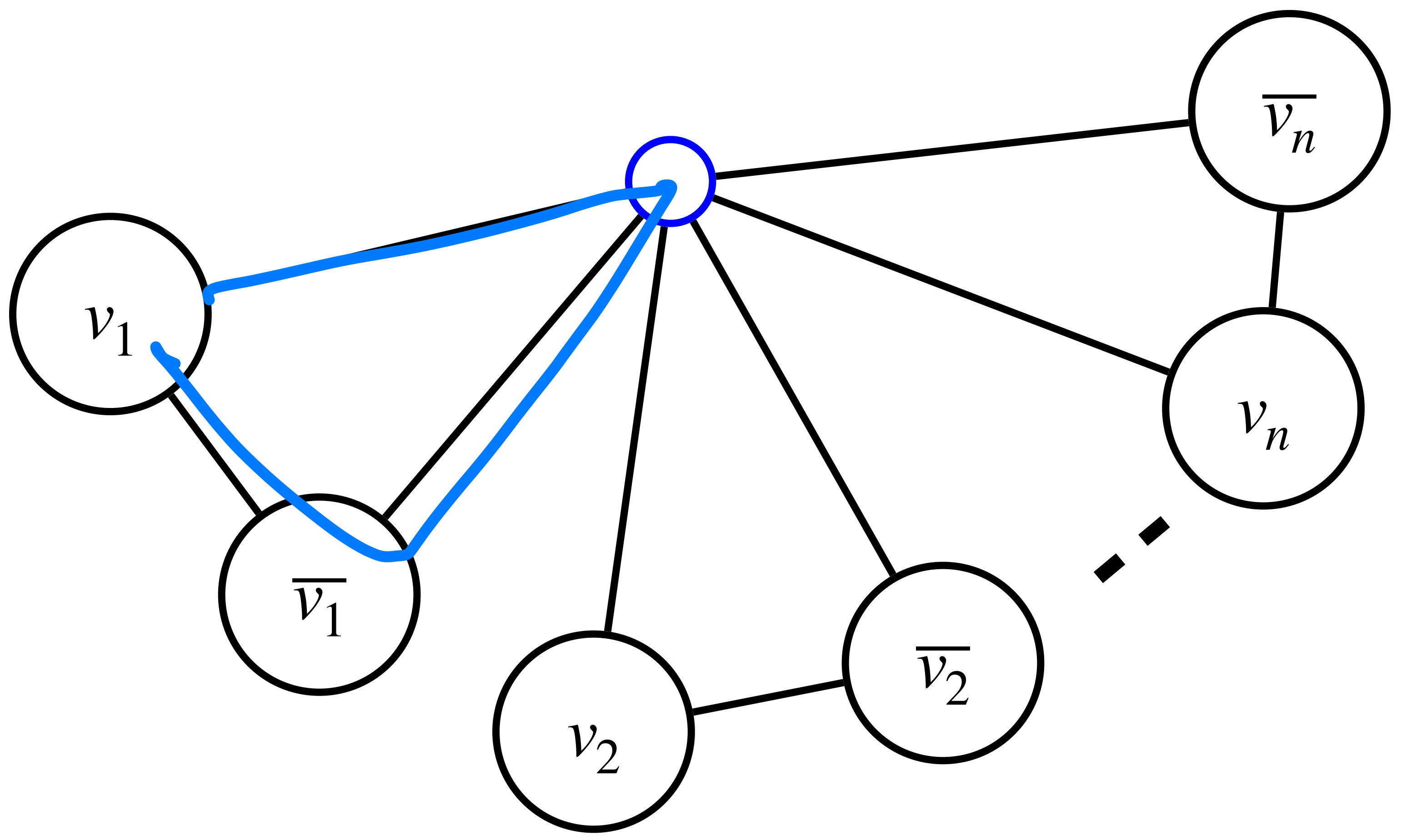


Reduction Idea II

Literal assignment I

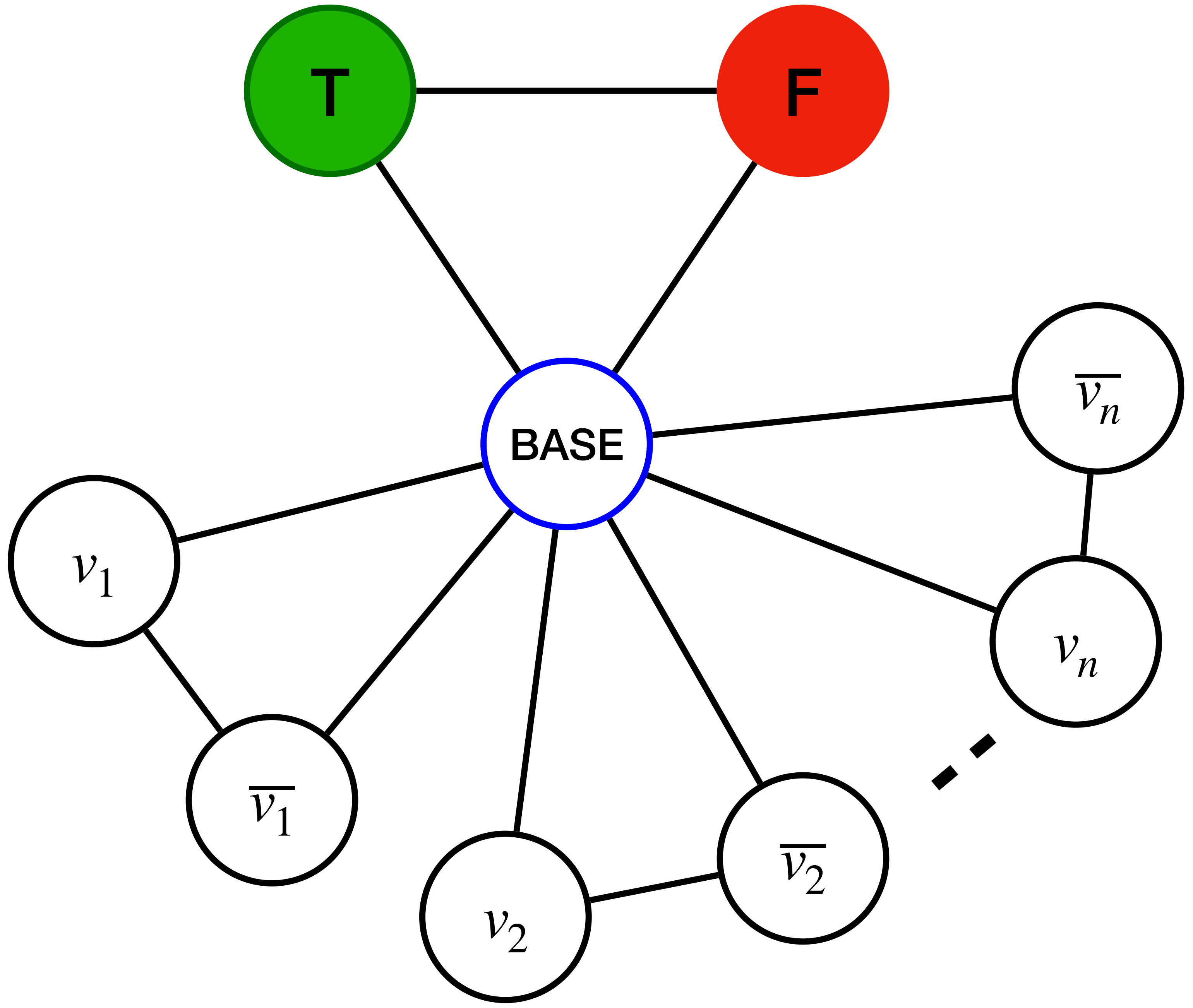
Next we need a gadget that assigns literals. Our previously constructed gadget assumes:

- All literals are either red or green.
- Need to limit graph so only x_1 or $\bar{x}_1$ is green. Other must be red.



Reduction idea

Start with **3SAT** formula (i.e., 3CNF formula) φ with n variables $X_1, \ldots, X_n$ and $m$clauses $C_1, \ldots, C_m$. Create graph G_φ such that G_φ is 3-colorable iff φ is satisfiable



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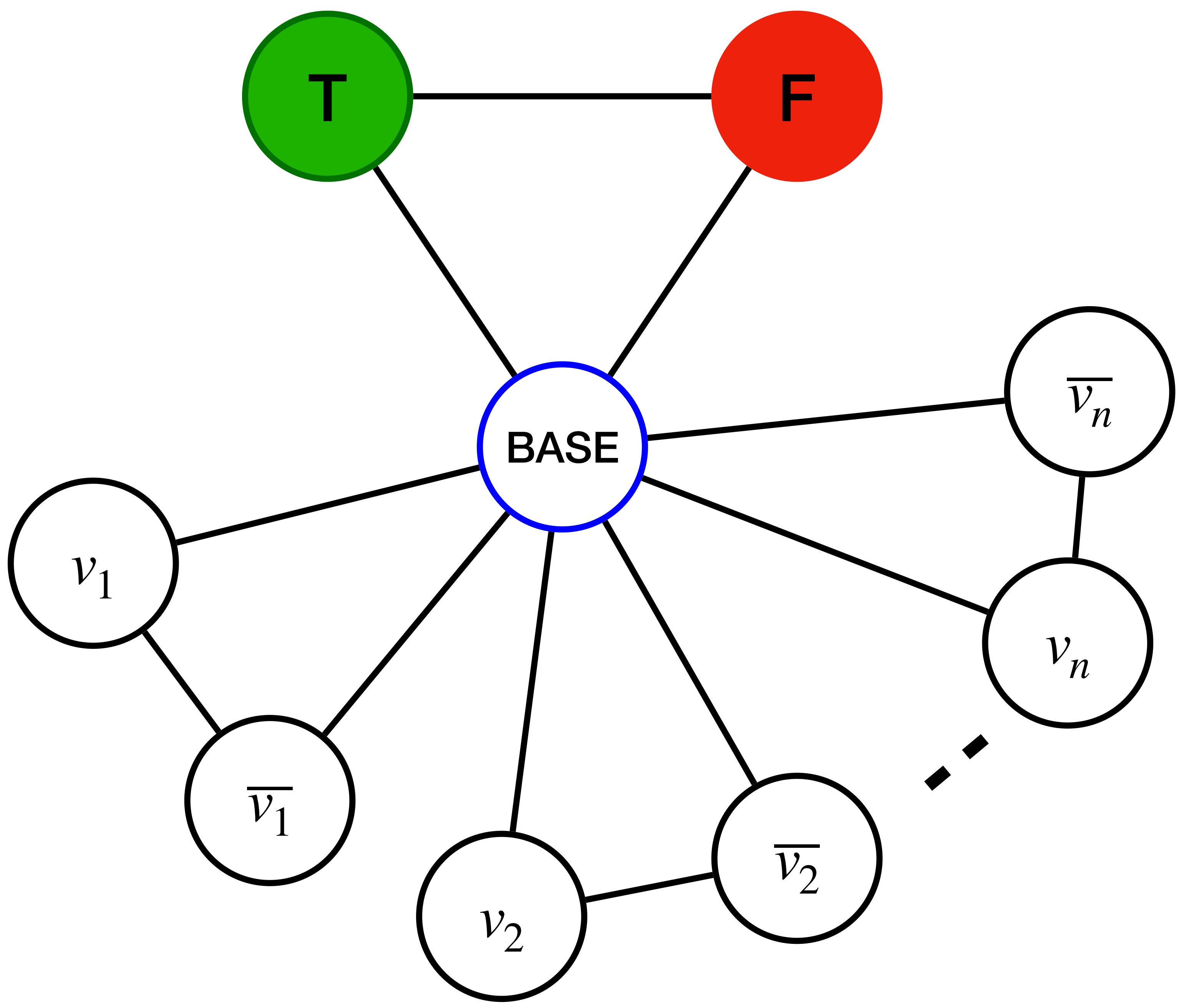

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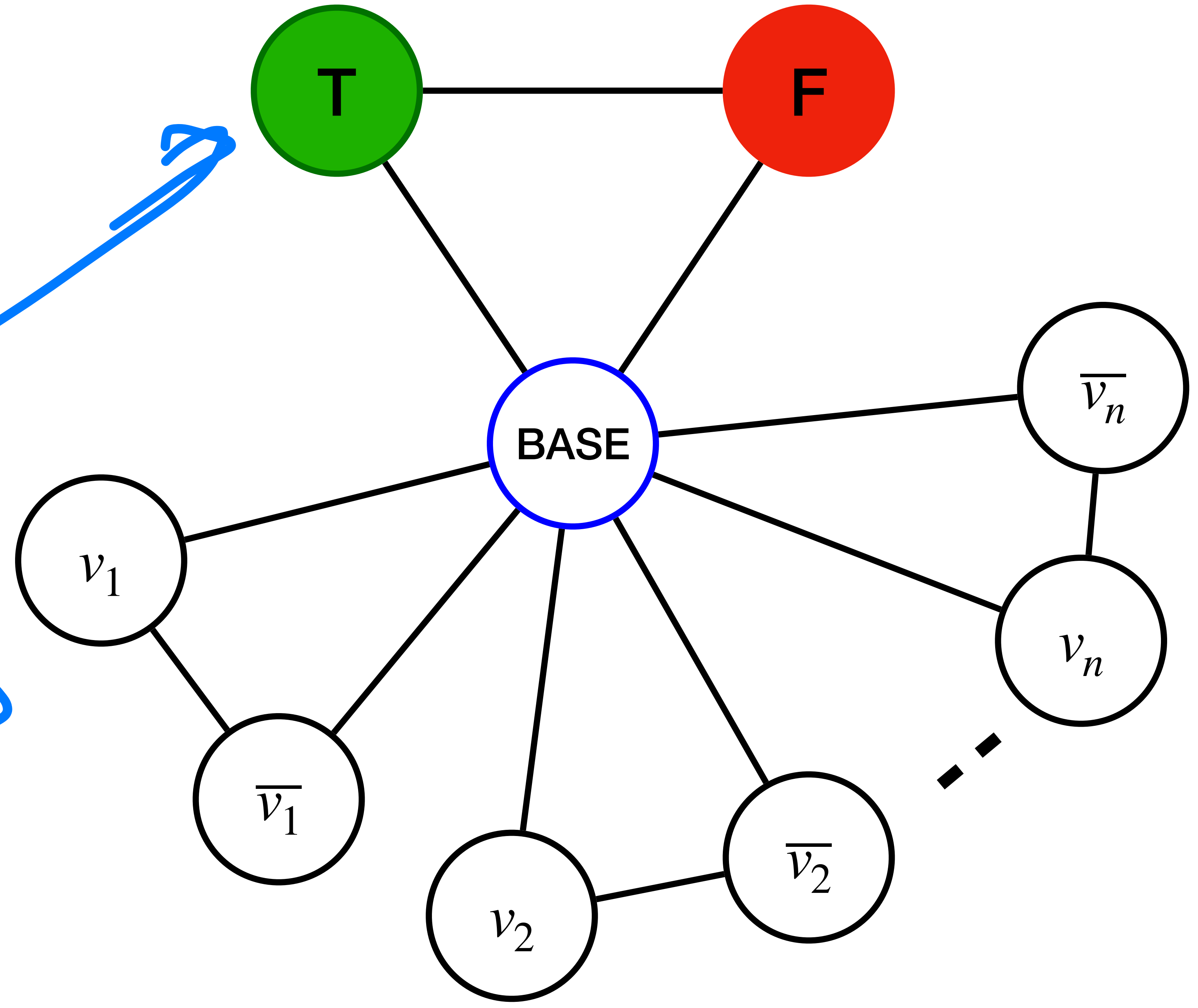
- create triangle with nodes: True, False, Base
- for each variable X_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base



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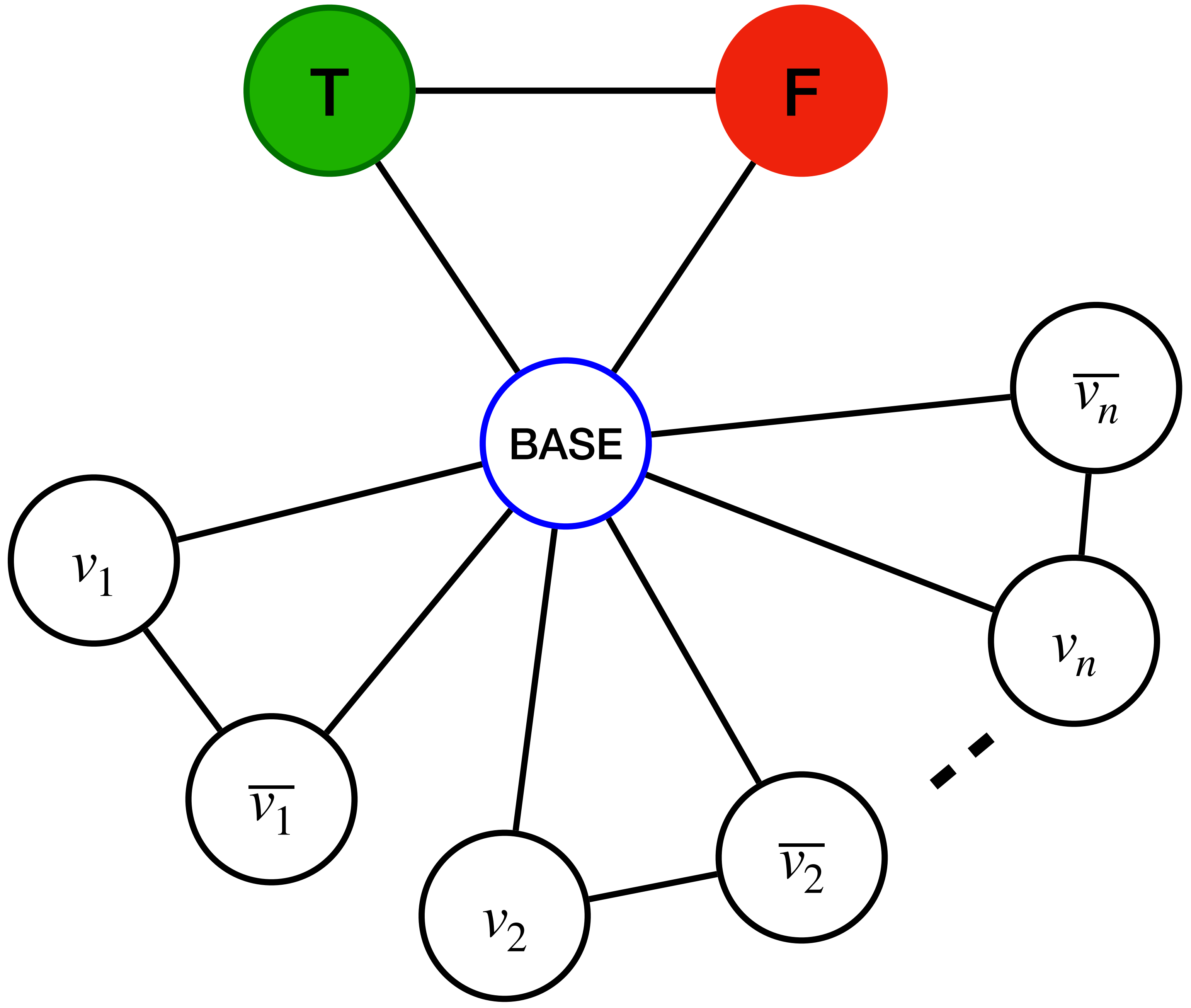
- for each variable X_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i



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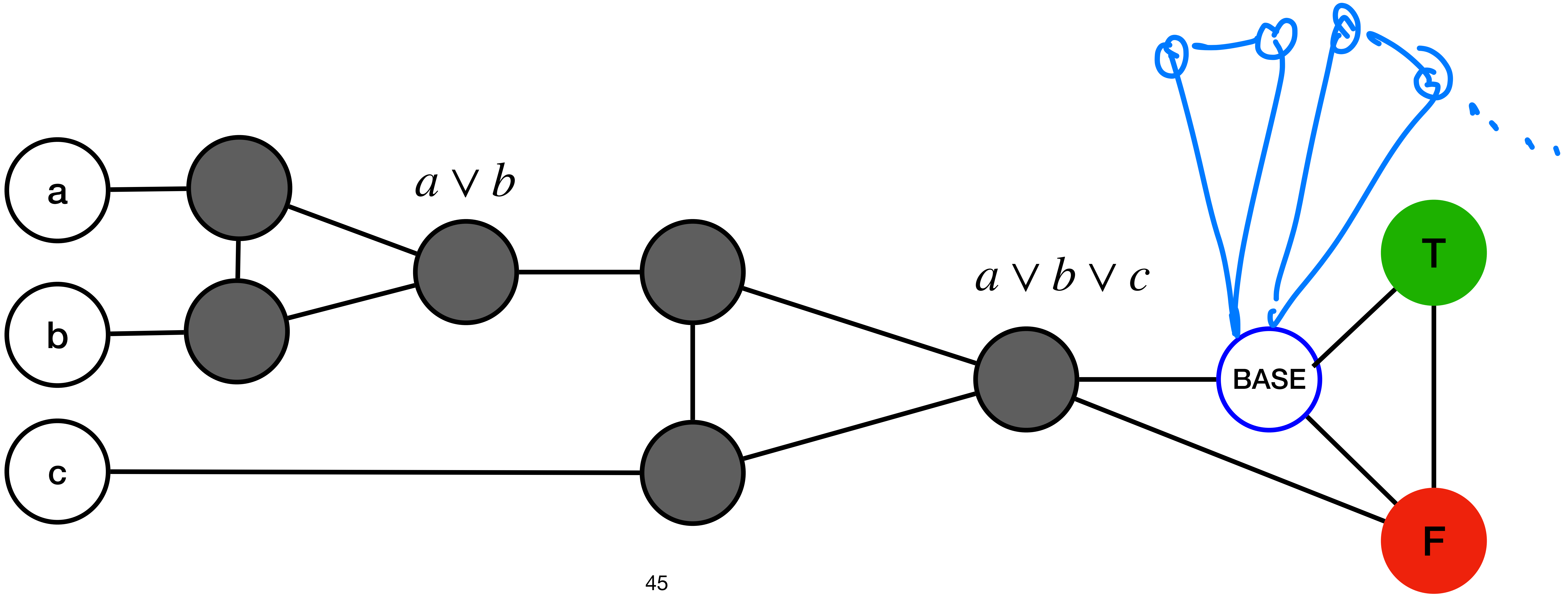
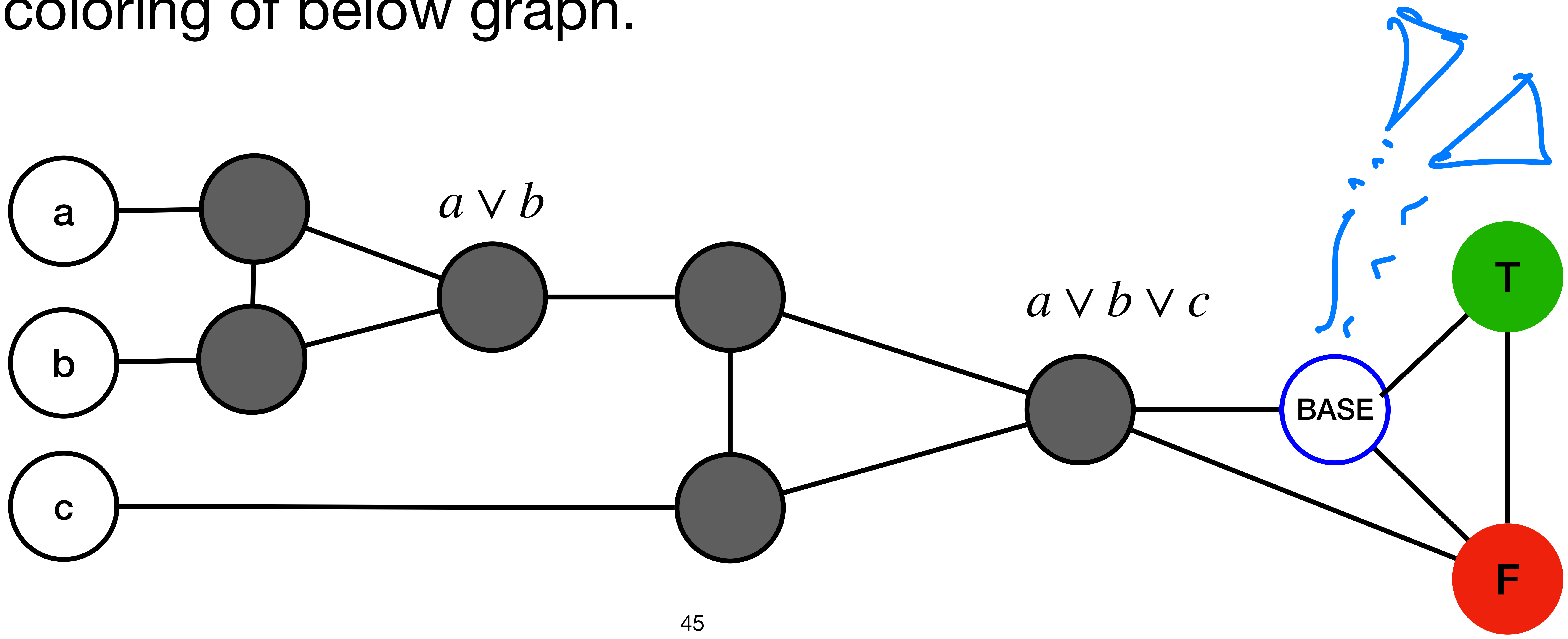
- For each clause $C_j = (a \vee b \vee c)$, add OR-gadget graph with input nodes a, b, c and connect output node of gadget to both False and Base.



Reduction

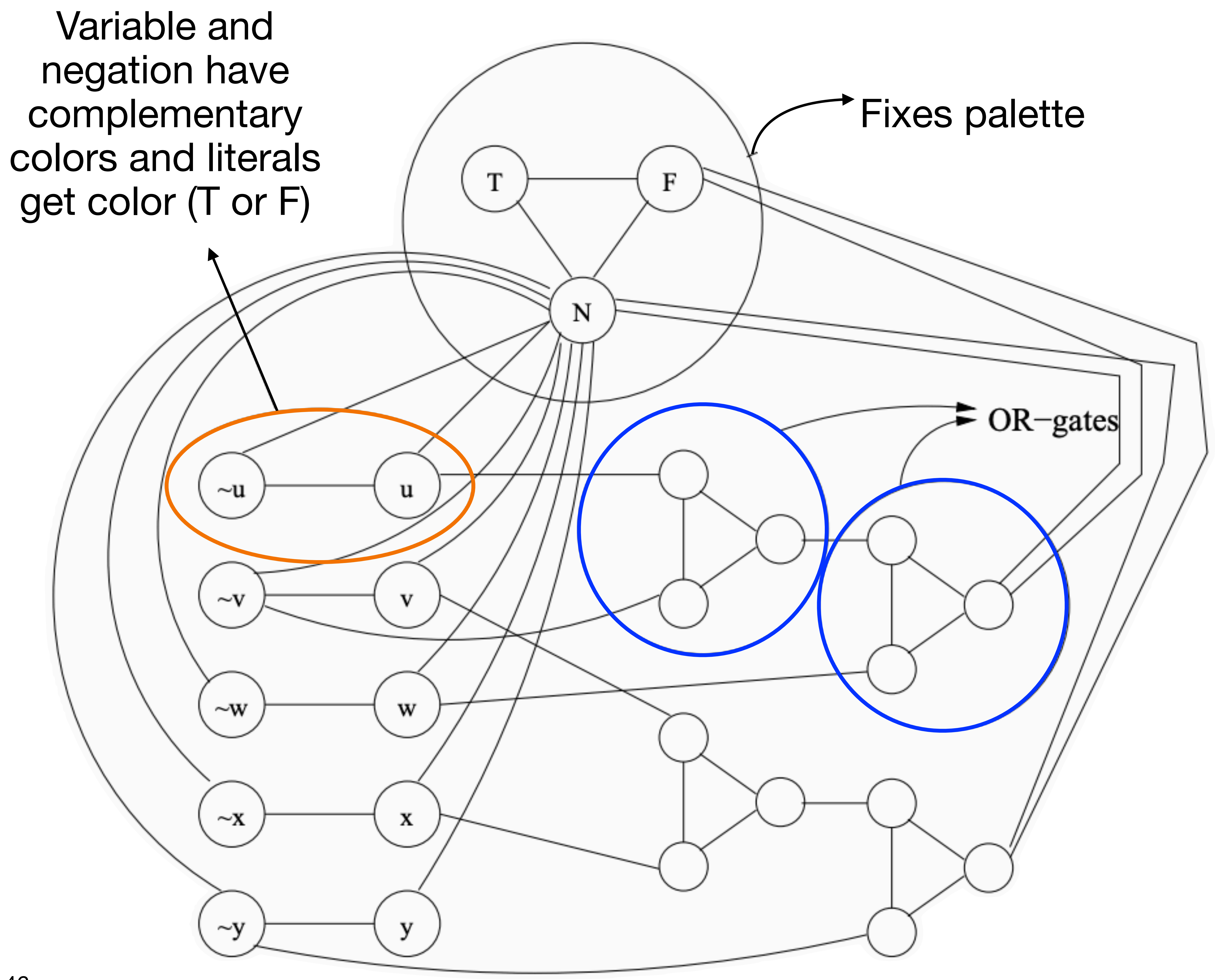
3SAT $\le_p$ 3-COLOR

- For each clause $C_j = (a \vee b \vee c)$, add OR-gadget graph with input nodes a, b, c and connect output node of gadget to both False and Base.
- **Claim:** No legal 3-coloring of below graph (with coloring of nodes T, F, B fixed) in which a, b, c are colored False. If any of a, b, c are colored True then there is a legal 3-coloring of below graph.



Reduction Outline

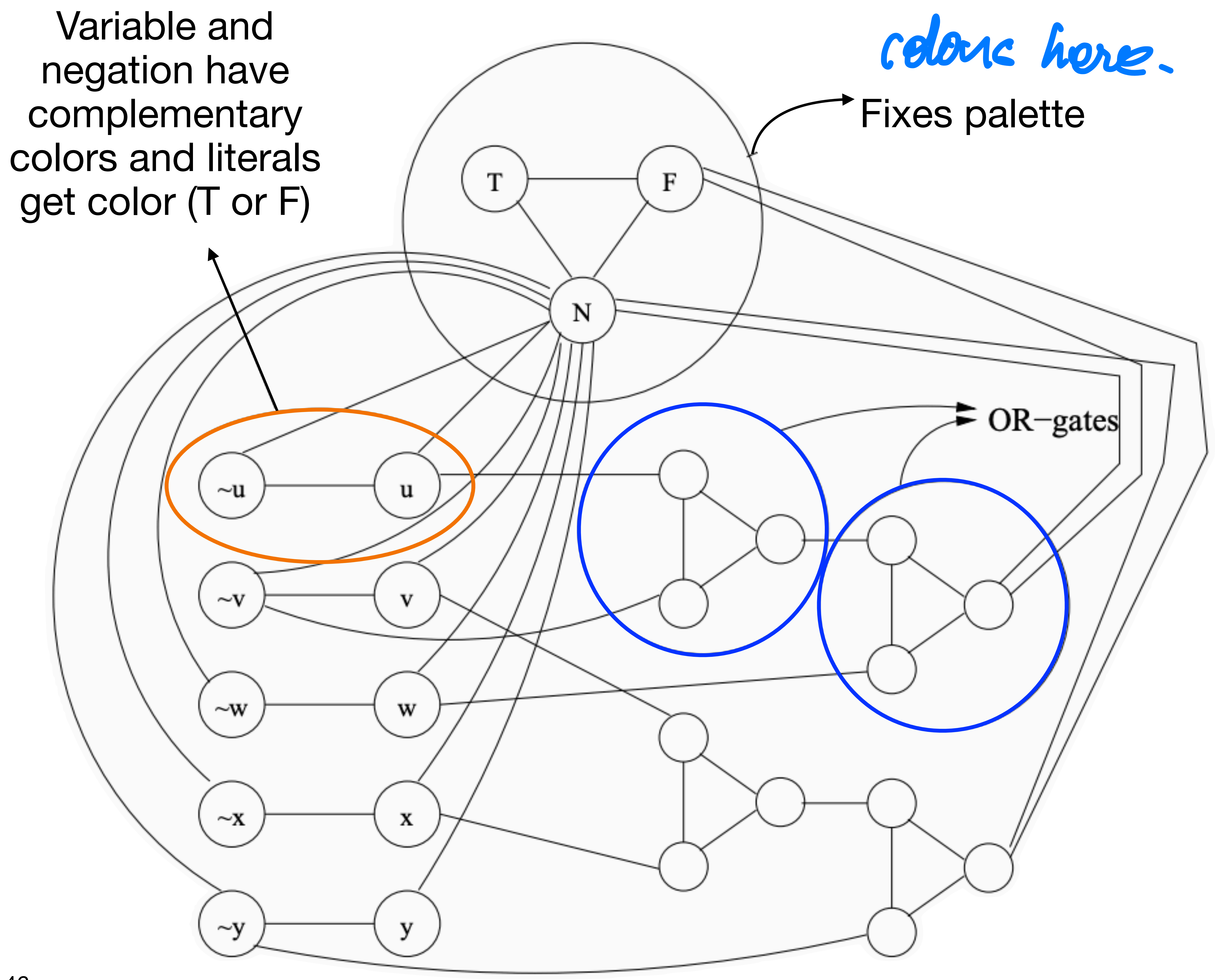
Example:



Reduction Outline

Example:

$$\varphi = \color{blue}{(u \vee \neg v \vee w)} \color{black}{\wedge (v \vee x \vee \neg y)}$$



Correctness of reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False

Circuit-SAT Problem

Circuits

A circuit is a ***directed acyclic graph*** with

Circuit-SAT Problem

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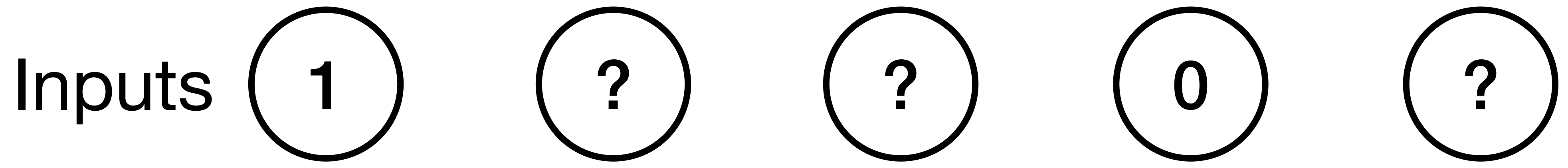

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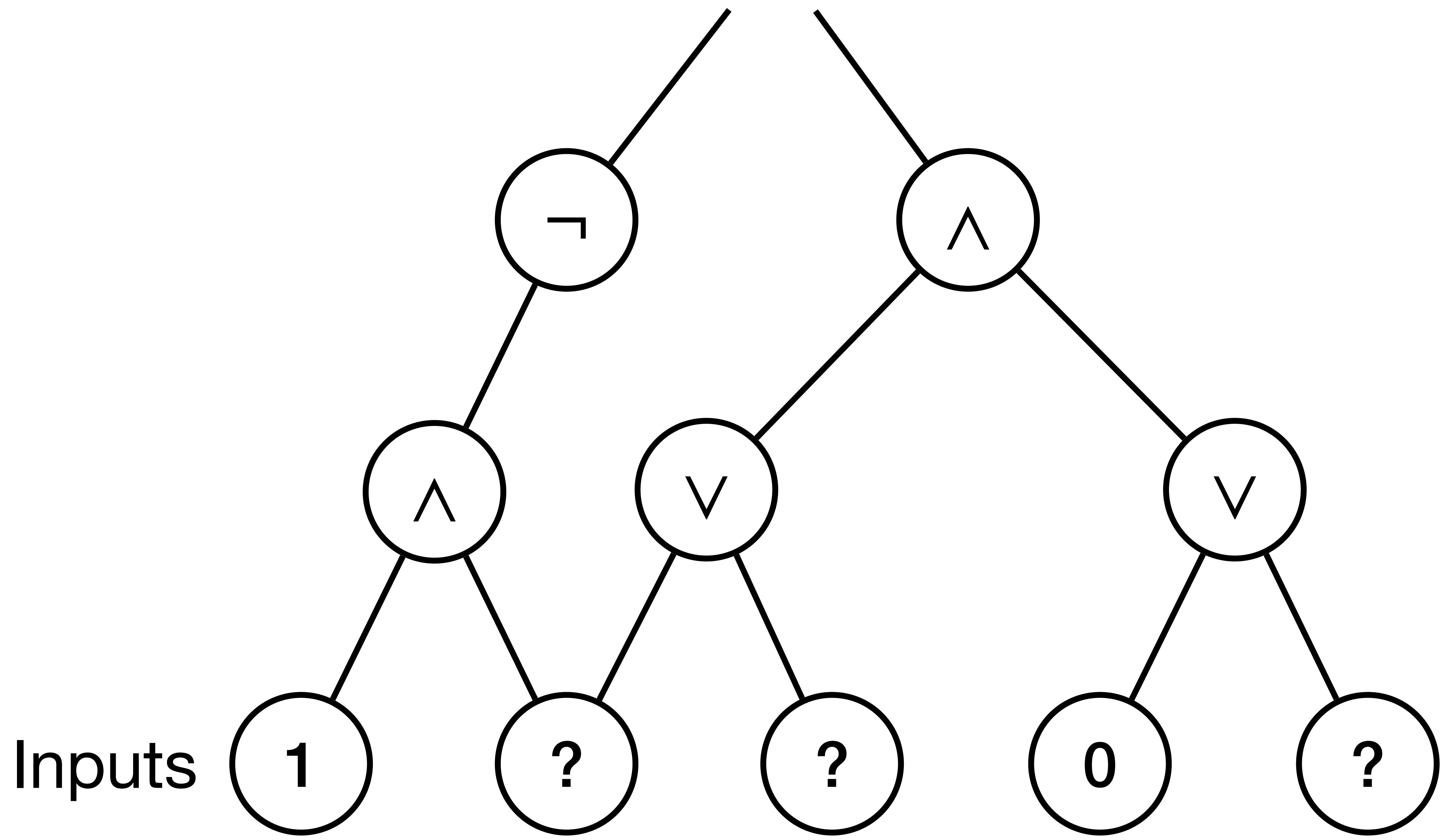

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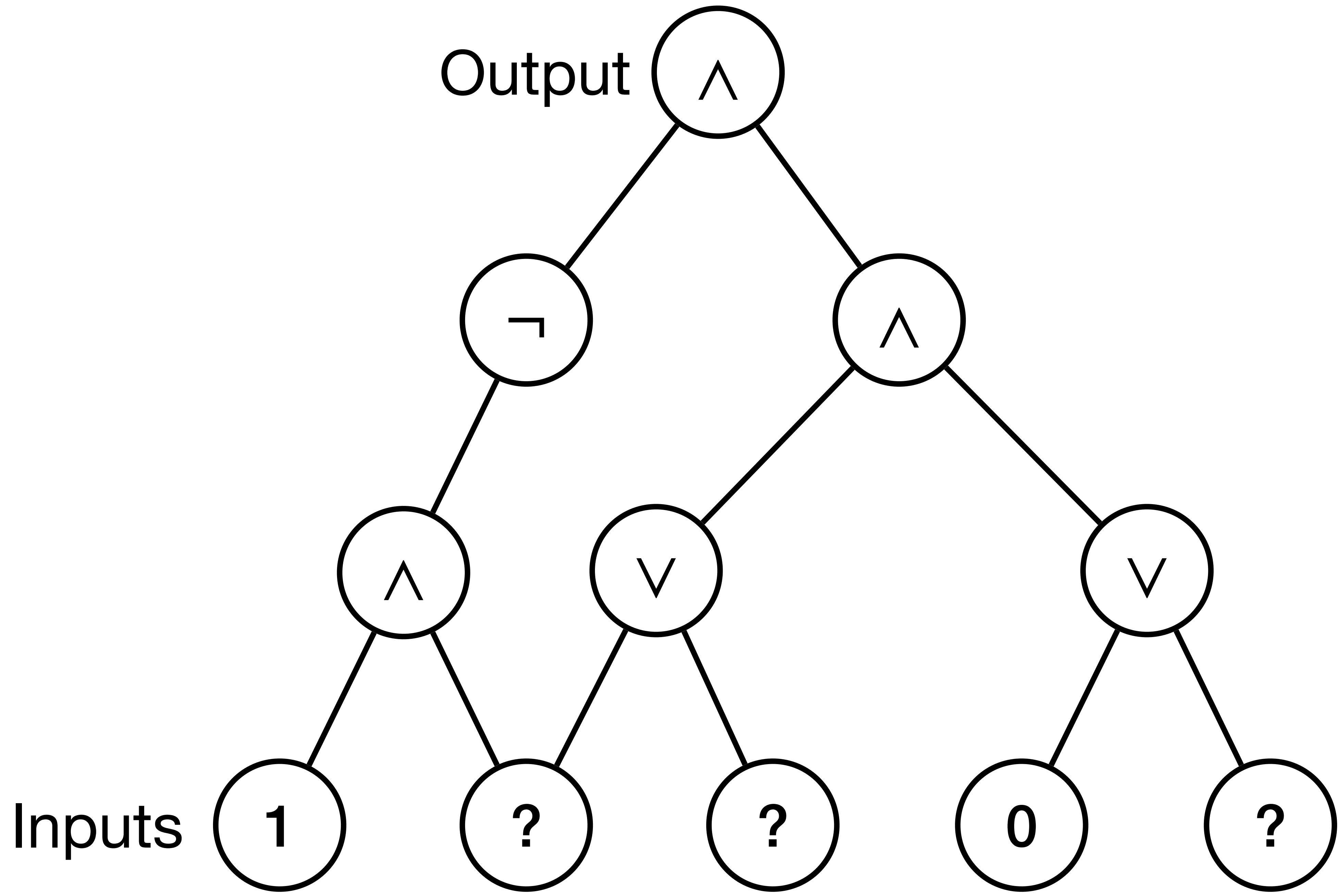

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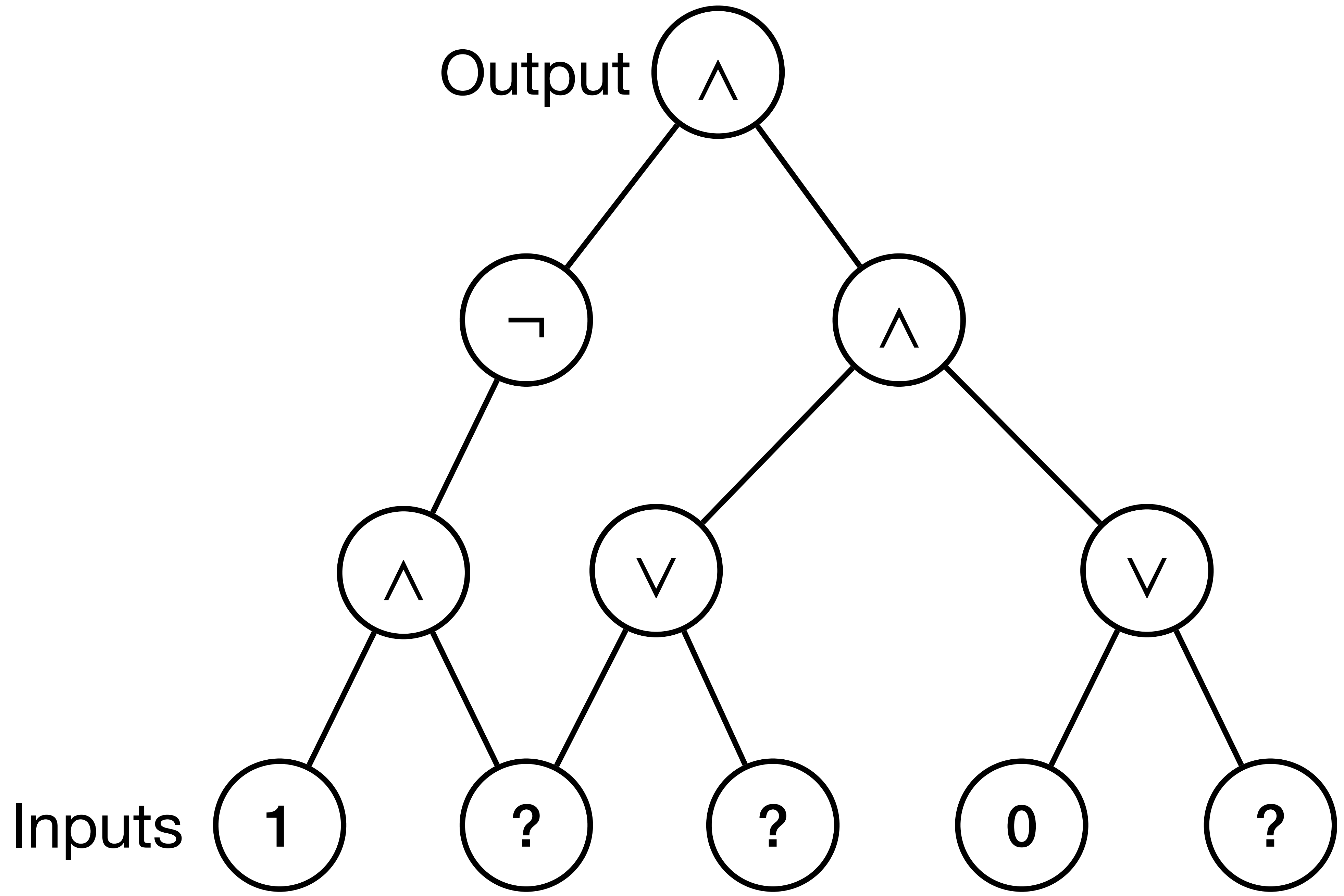

Given a circuit as input, is there an assignment to the input variables that causes the output to get value 1?

Circuit-SAT Problem

Circuits

Problem definition (CSAT): Given a ***circuit*** as is there an assignment to the input variables that causes the output to get value 1?

Lemma: *CSAT* is in **NP**

- Certificate: Assignment to input variables.
- Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.
- Can show: 3SAT $\leq_P$ CSAT

Circuit SAT vs SAT

- CNF formulas are a rather restricted form of Boolean formulas.
- Circuits are a much more powerful (and hence easier) way to express Boolean formulas.
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Theorem

SAT $\leq_P$ 3SAT $\leq_P$ CSAT.

Theorem

CSAT $\leq_P$ SAT $\leq_P$ 3SAT.

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Given 3CNF formula φ with n variables and m clauses, create a Circuit C.

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- For each clause $(l_1 \vee l_2 \vee l_3)$ use two OR gates to mimic formula
- Combine the outputs for the clauses using AND gates to obtain the final output

Example: 3SAT $\leq_P$ CSAT

$$\varphi = (x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4)$$

Example: 3SAT $\leq_P$ CSAT

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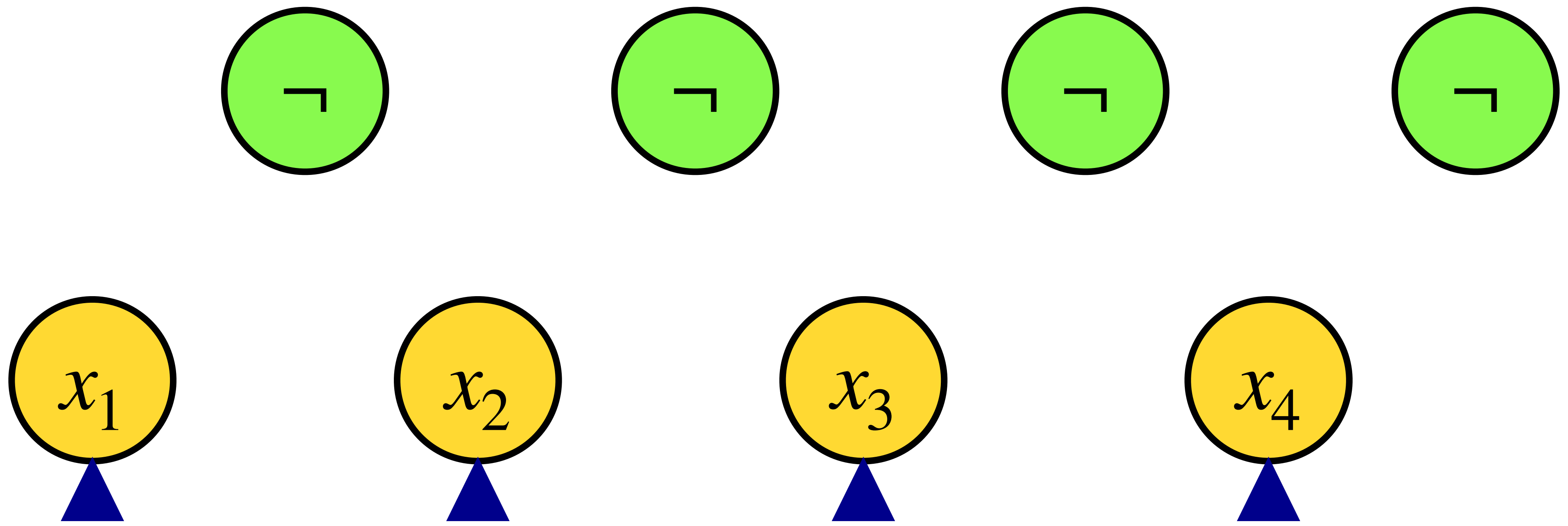

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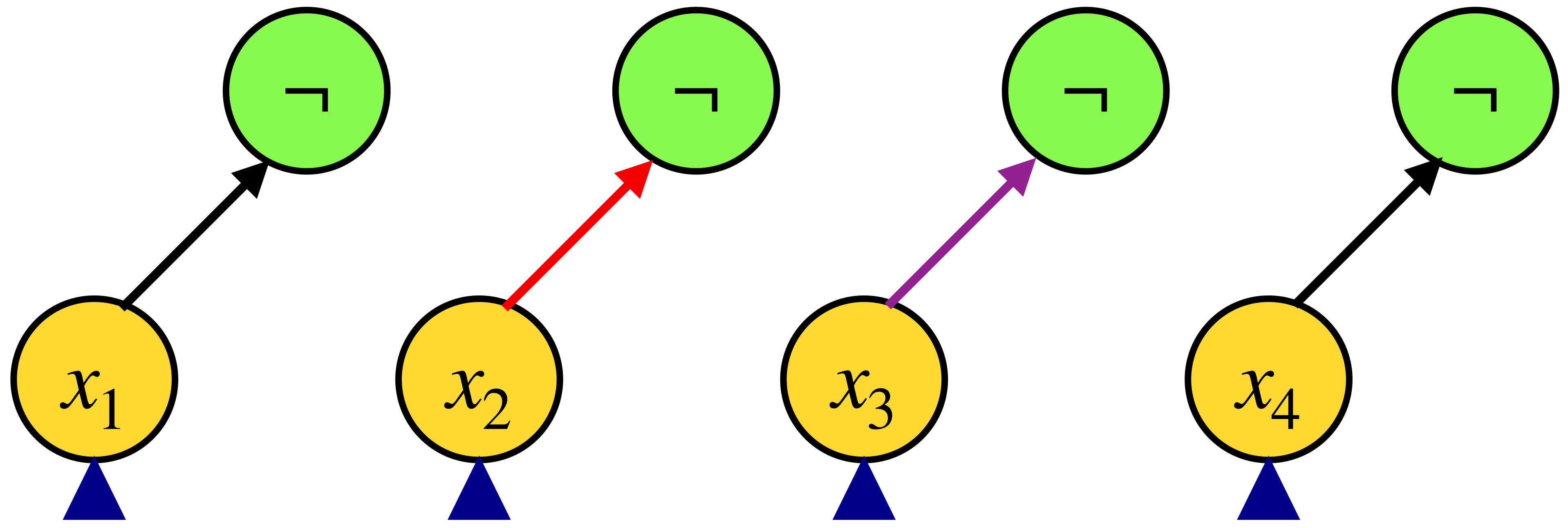

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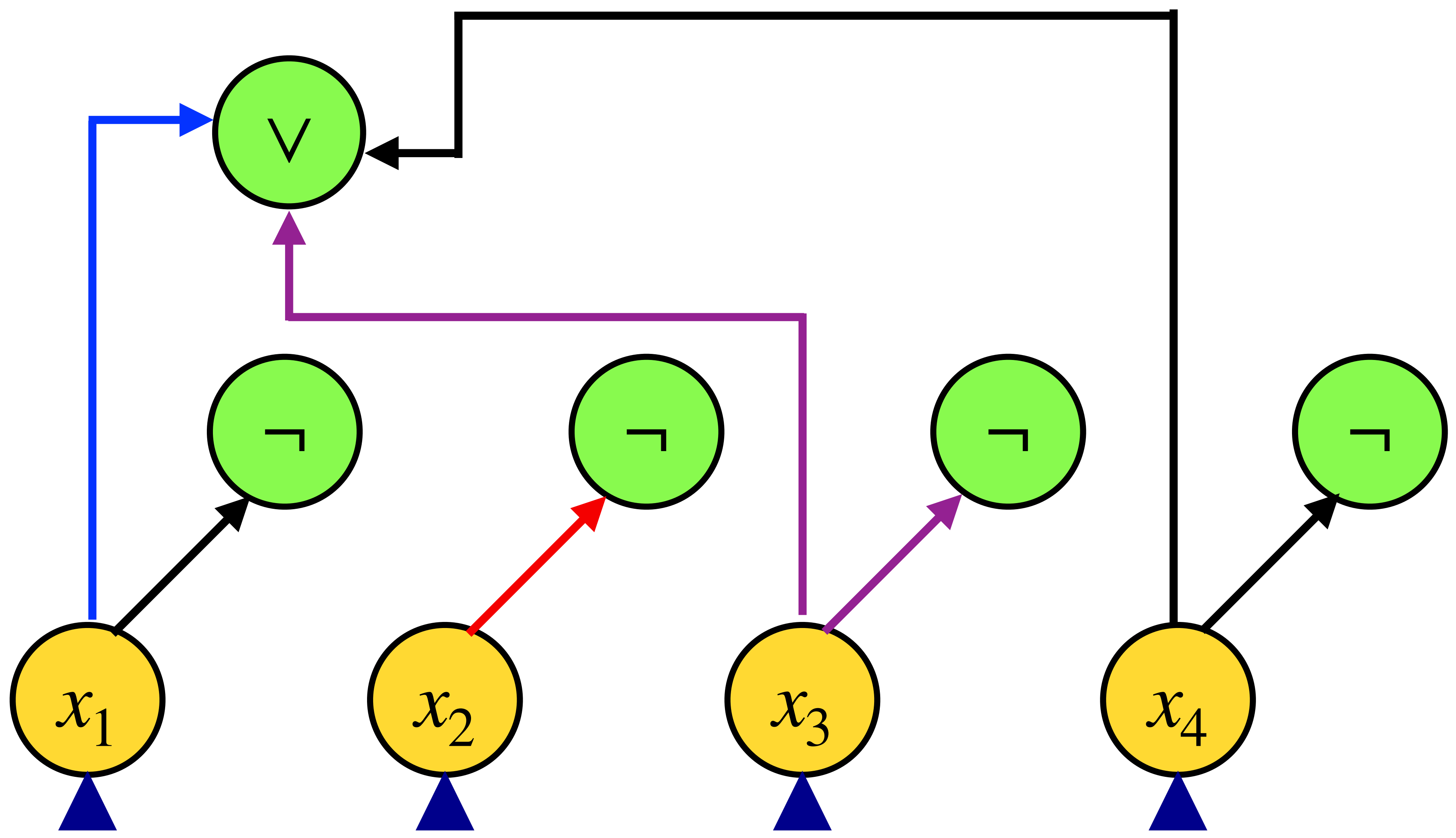

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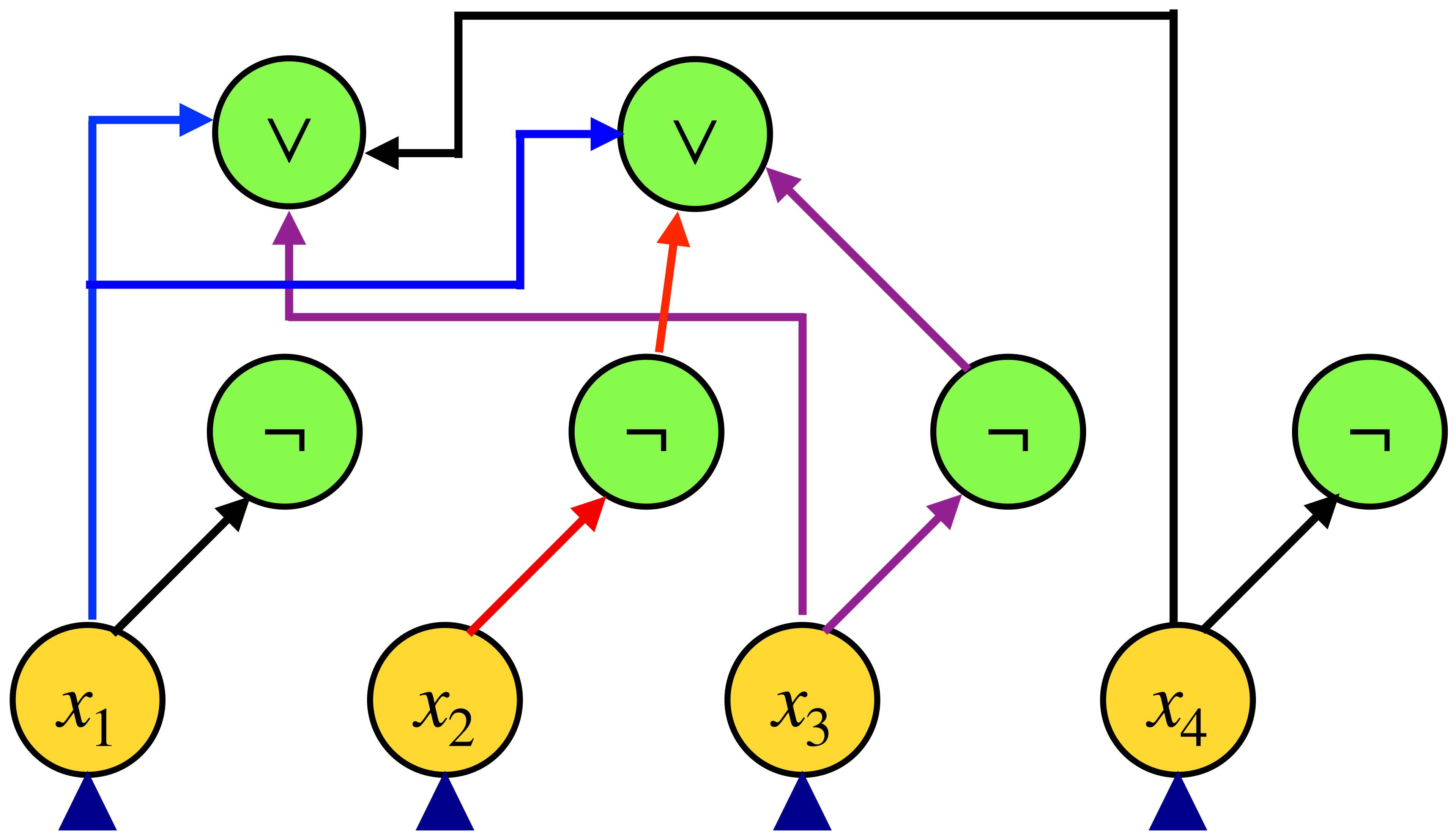

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