Pre-lecture brain teaser

Find the regular expression for the language containing all binary strings that do not contain the subsequence 111000

ECE-374-B: Lecture 3 - NFAs

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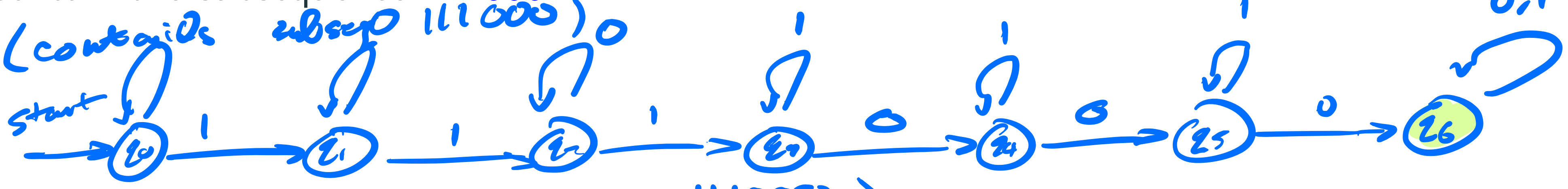
University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

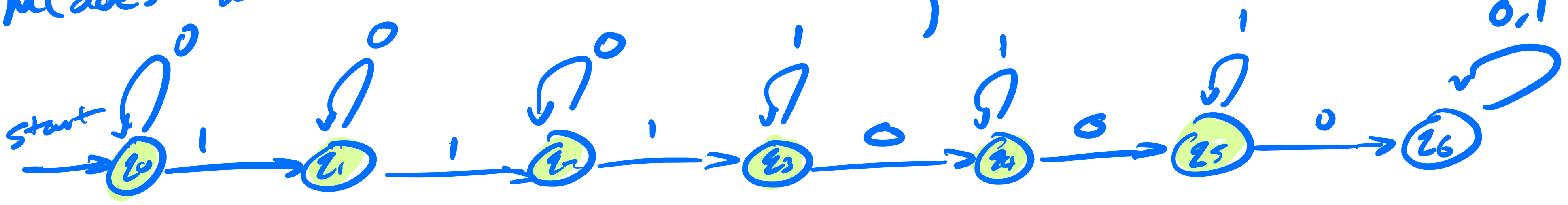
1011000

Find the regular expression for the language containing all binary strings that **do not** contain the subsequence 111000

M(contains subseq 111000)



M(does not contain subseq 111000)



R(does not contain subseq 111000)

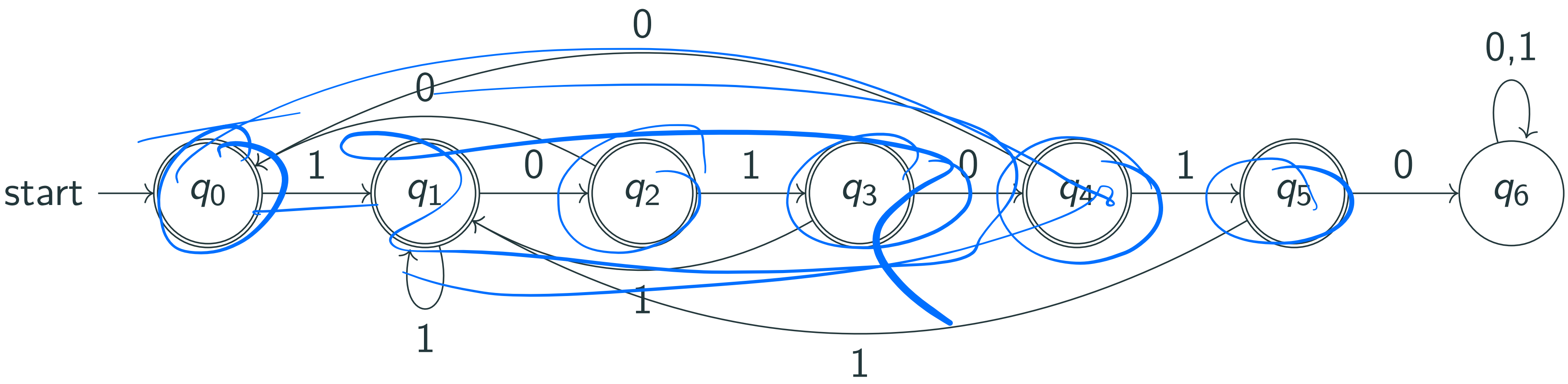
$r = 0^* + 0^*10^* + 0^*10^*10^* + 0^*10^*10^*11^* + 0^*10^*10^*11^*01^* + \ldots$ [illegible]

Pre-lecture brain teaser II

Find the regular expression for the language containing all binary strings that **do not** contain the substring 101010

Pre-lecture brain teaser II

Find the regular expression for the language containing all binary strings that **do not** contain the substring 101010



$r = \varepsilon + 1$

Pre-lecture brain teaser III

Find the regular expression for the language contains all binary strings whose $\#_0(w)\%7 = 0$(number of 0's divisible by 7).

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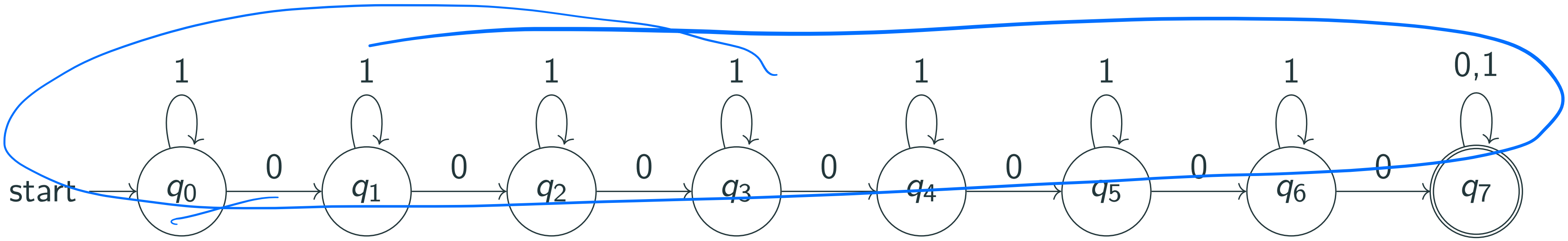

Pre-lecture brain teaser III

Show that the following string(w) is a member of the language that:

- does not contain the subsequence 111000 **or**
- does not contain the substring 101010 **or**
- or has a number of 0's divisible by 7

Pre-lecture brain teaser III

Show that the following string(w) is a member of the language that:

- does not contain the subsequence 111000 **or**
- does not contain the substring 101010 **or**
- or has a number of 0's divisible by 7

$$
\begin{aligned}
w = &1001110110111001\\
&1000010111110010\\
&0101010011001111\\
&1001001011111100
\end{aligned}
$$

You have 30 seconds.

Pre-lecture brain teaser III

Show that the following string(w) is a member of the language that:

- does not contain the subsequence 111000 **or**
- does not contain the substring 101010 **or**
- or has a number of 0's divisible by 7

$$
\begin{aligned}
w =& 1001110110111001 \\
& 1000010111110010 \\
& 0101010011001111 \\
& 1001001011111100
\end{aligned}
$$

You have 30 seconds. Pray, choose a strategy and hope you get **lucky**.

Tangential Thought

Does luck allow us to solve unsolvable problems?

Floyd Warshall 1968

Tangential Thought

Does luck allow us to solve unsolvable problems? New example: Consider two machines: M_1 and M_2

- M_1 is a classic deterministic machine.
- M_2 is a “lucky” machine that will always make the right choice.

Lucky machine programs

Problem: Find shortest path from a to b

$O(n \log n)$

Program on M_1 (Dijkstra's algorithm):

Initialize for each node v, $\mathrm{Dist}(s, v) = d'(s, v) = \infty$
Initialize $X = \emptyset$, $d'(s, s) = 0$
for $i = 1$ to $|V|$ **do**
  Let v be node realizing $d'(s, v) = \min_{u \in V - X} d'(s, u)$
  $\mathrm{Dist}(s, v) = d'(s, v)$
  $X = X \cup \{v\}$
  Update $d'(s, u)$ for each u in $V - X$ as follows:
    $d'(s, u) = min\Big(d'(s, u),\ \mathrm{Dist}(s, v) + \ell(v, u)\Big)$

Lucky machine programs

Problem: Find shortest path from a to b

Program on M_2 (Blind luck):

```
Initialize path = []
path += a
While(notatb)
    take an outgoing edge (u, v) from current node u to v
    current = v
    path += v
return path
```

Tangential Thought

Does luck allow us to solve unsolvable problems?

Consider two machines: M_1 and M_2

- M_1 is a classic deterministic machine.
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Question:

Tangential Thought

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Consider two machines: M_1 and M_2

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Question: Are there problems which M_2 can solve that M_1 cannot.

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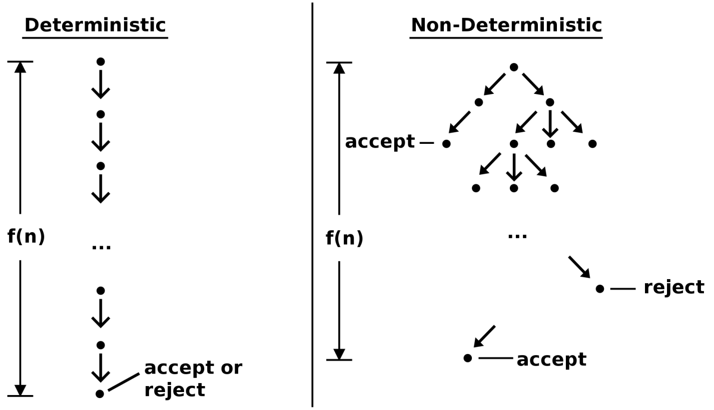
The notion was first posed by **Robert W. Floyd** in 1967.

Non-determinism in computing

In computer science, a nondeterministic machine is a theoretical device that can have more than one output for the same input.

A machine that is capable of taking multiple states concurrently. Whenever it reaches a choice, it takes both paths.

If there is a path for the string to be accepted by the machine, then the string is part of the language.



Non-determinism in media

Placeholder slide for youtube.

Why non-determinism?

- Non-determinism adds power to the model; richer programming language and hence (much) easier to "design" programs
- Fundamental in **theory** to prove many theorems
- Very important in **practice** directly and indirectly
- Many deep connections to various fields in Computer Science and Mathematics

Many interpretations of non-determinism. Hard to understand at the outset. Get used to it and then you will appreciate it slowly.

Non-deterministic finite automata (NFA) Introduction

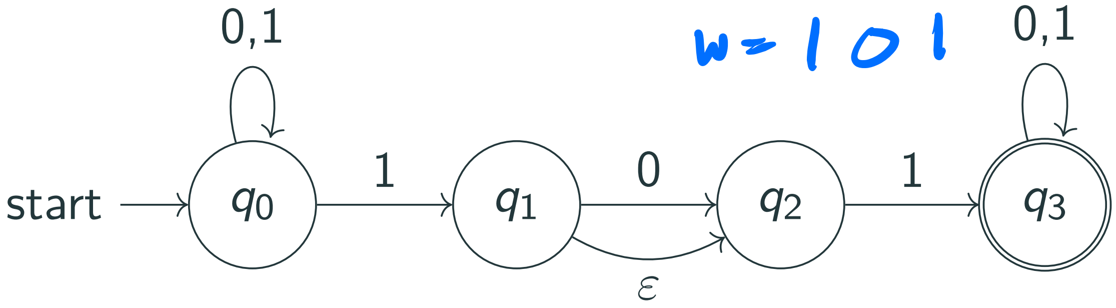
Non-deterministic Finite State Automata by example

When you come to a fork in the road, take it.

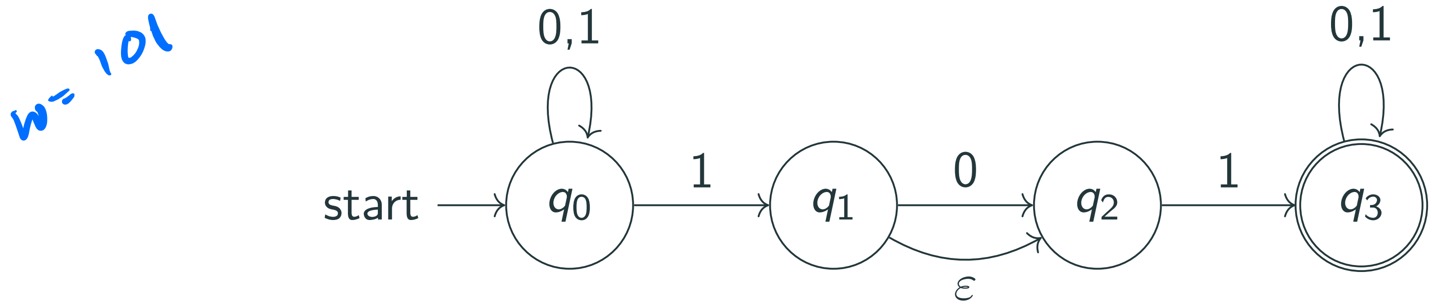
Non-deterministic Finite State Automata by example

When you come to a fork in the road, take it.

Today we'll talk about automata whose logic **is not** deterministic.



NFA acceptance: Informal



Informal definition: An NFA N accepts a string w iff some accepting state is reached by N from the start state on input w.

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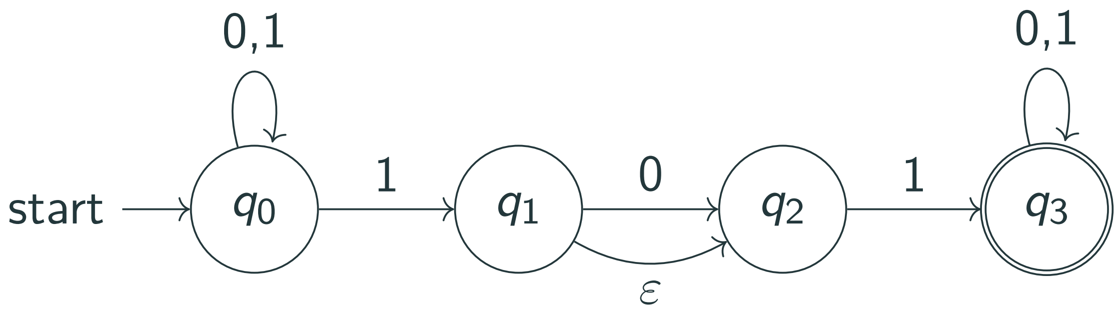
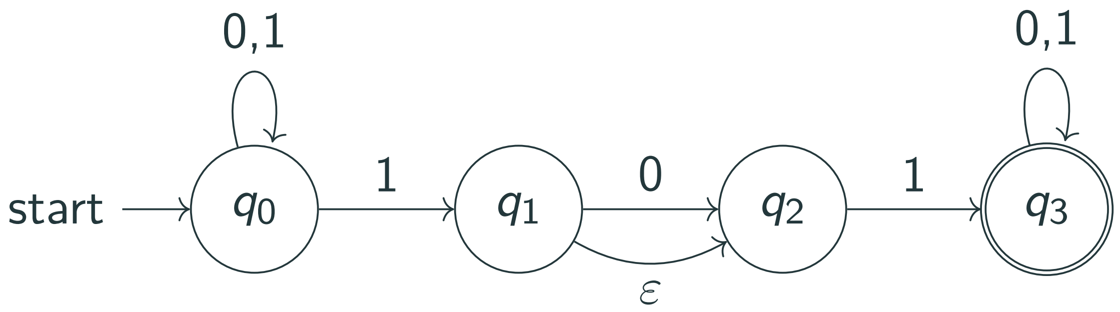


Informal definition: An NFA N accepts a string w iff some accepting state is reached by N from the start state on input w.

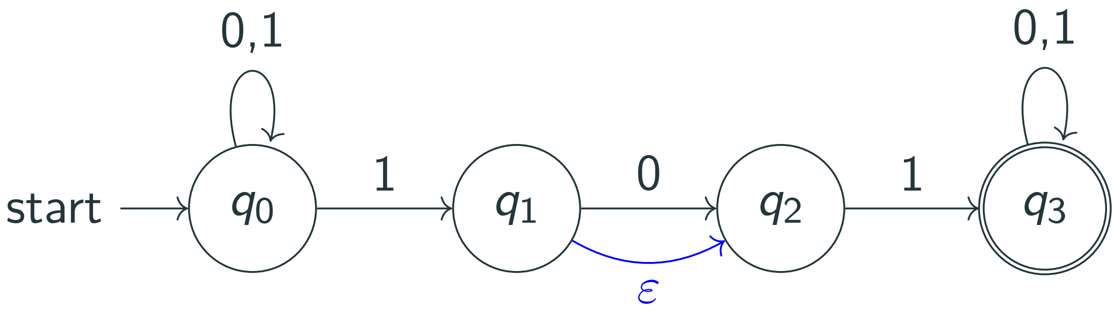
The language accepted (or recognized) by a NFA N is denote by $L(N)$ and defined as: $L(N) = \{w \mid N \text{ accepts } w\}$.

NFA acceptance: Example

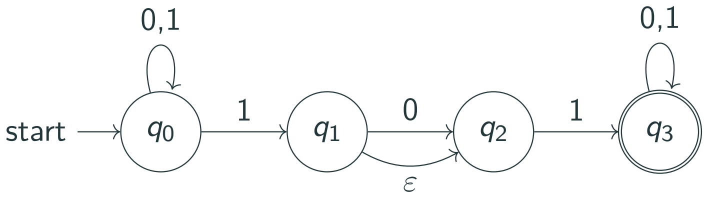


- Is 010110 accepted?

NFA acceptance: Wait! what about the ϵ?!



NFA acceptance: Example



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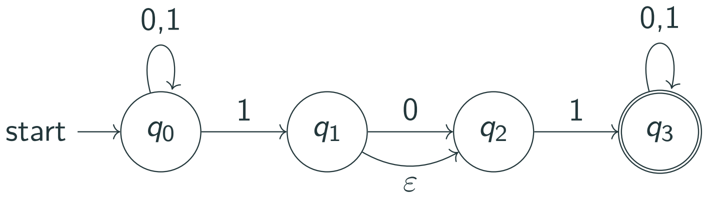

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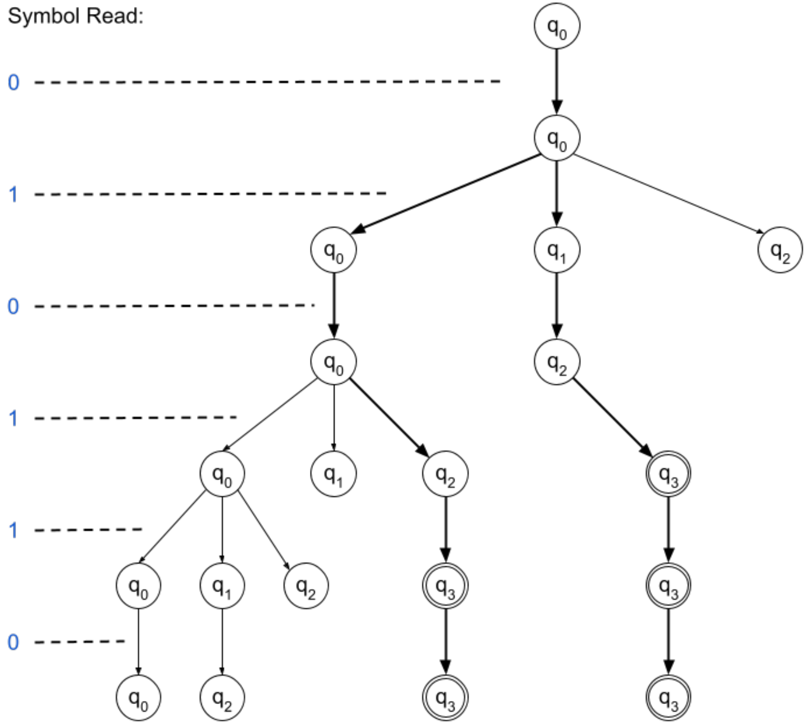
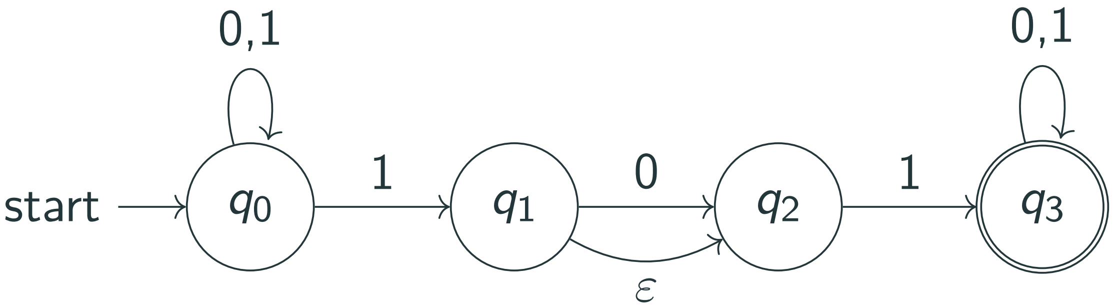
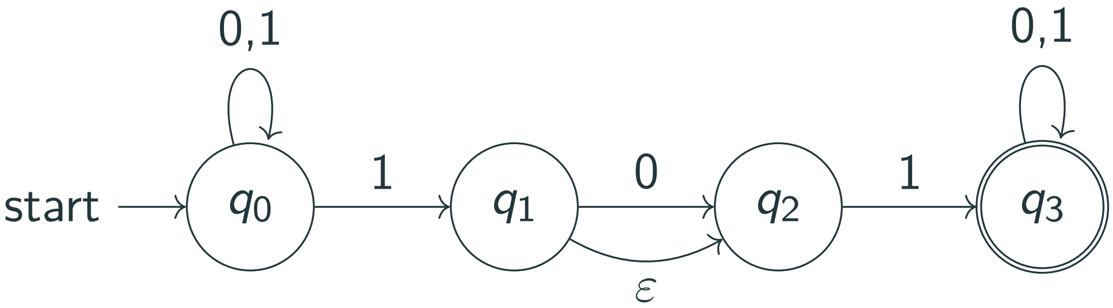


NFA acceptance: Example



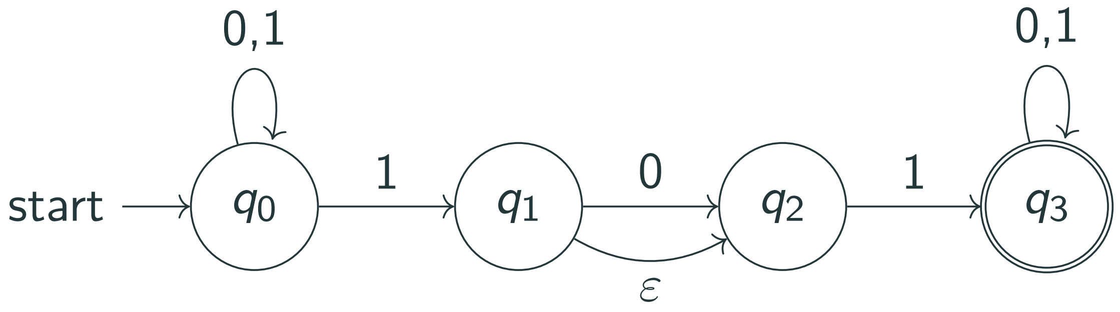
- Is 010110 accepted? Yes

NFA acceptance: Example



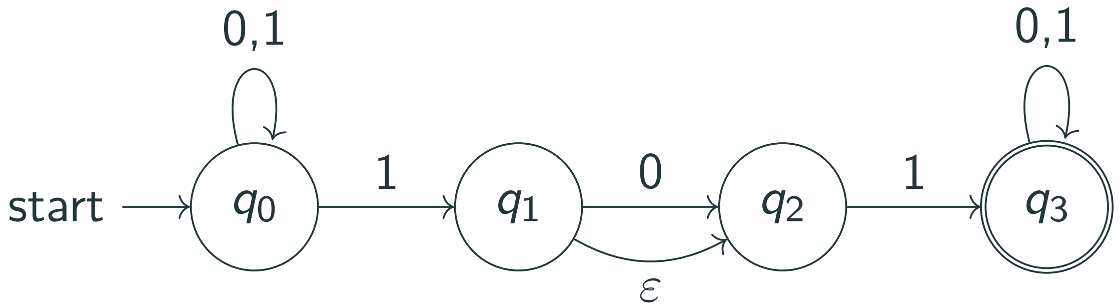
- Is 010110 accepted?
- Is 010 accepted? No

NFA acceptance: Example



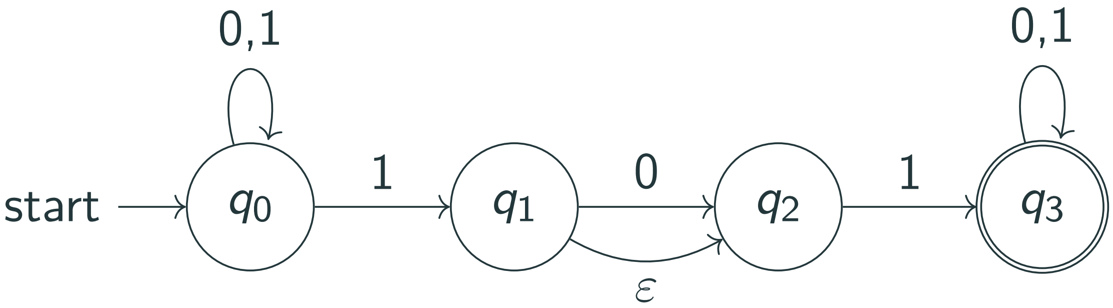
- Is 010110 accepted?
- Is 010 accepted?
- Is 101 accepted? Yes

NFA acceptance: Example



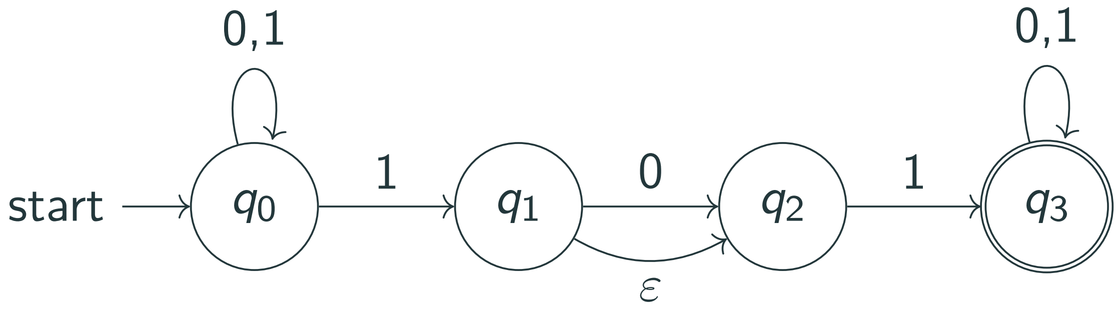
- Is 010110 accepted?
- Is 010 accepted?
- Is 101 accepted?
- Is 10011 accepted? Yes

NFA acceptance: Example



- Is 010110 accepted?
- Is 010 accepted?
- Is 101 accepted?
- Is 10011 accepted?
- What is the language accepted by N? Accepts all w that contains the substring 11 or 101

NFA acceptance: Example



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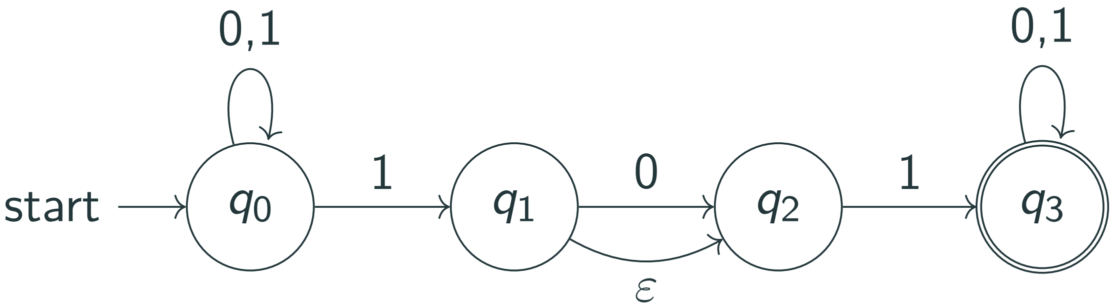

- Is 010110 accepted?
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- Is 101 accepted?
- Is 10011 accepted?
- What is the language accepted by N?

Comment: Unlike DFAs, it is easier in NFAs to show that a string is accepted than to show that a string is **not** accepted.

Formal definition of NFA

Formal Tuple Notation

Definition
A non-deterministic finite automata (NFA) $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

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$\{q_0, q_1, q_2, \ldots\}$

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$\mathcal{P}(Q)$?

Reminder: Power set

Q: a set. Power set of Q is: $\mathcal{P}(Q) = 2^Q = \{X \mid X \subseteq Q\}$ is set of all subsets of Q.

Example

$Q = \{1, 2, 3, 4\}$

$$\mathcal{P}(Q) = \left\{ \begin{array}{c} \{1,2,3,4\}\,, \\ \{2,3,4\}\,,\{1,3,4\}\,,\{1,2,4\}\,,\{1,2,3\}\,, \\ \{1,2\}\,,\{1,3\}\,,\{1,4\}\,,\{2,3\}\,,\{2,4\}\,,\{3,4\}\,, \\ \{1\}\,,\{2\}\,,\{3\}\,,\{4\}\,, \\ \{\} \end{array} \right\}$$

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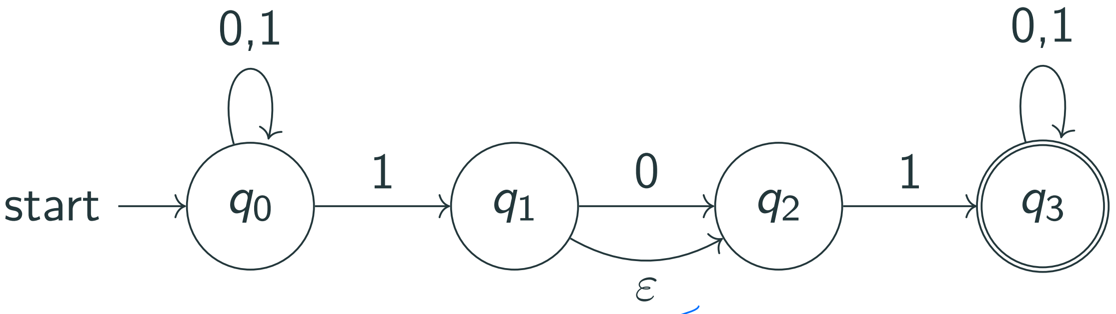
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\varepsilon\}$ is a subset of Q — a set of states.

Example



- $Q = \{q_0, q_1, q_2, q_3\}$
- $\Sigma = \{0, 1\}$
- $\delta =$

	ε	0	1
q_0	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1, q_2\}$	$\{q_2\}$	$\{\}$
q_2	$\{q_2\}$	$\{\}$	$\{q_3\}$
q_3	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$

- $s = q_0$

$A = \{q_3\}$

Extending the transition function to strings

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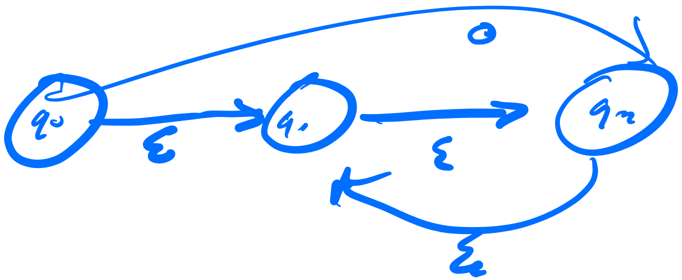
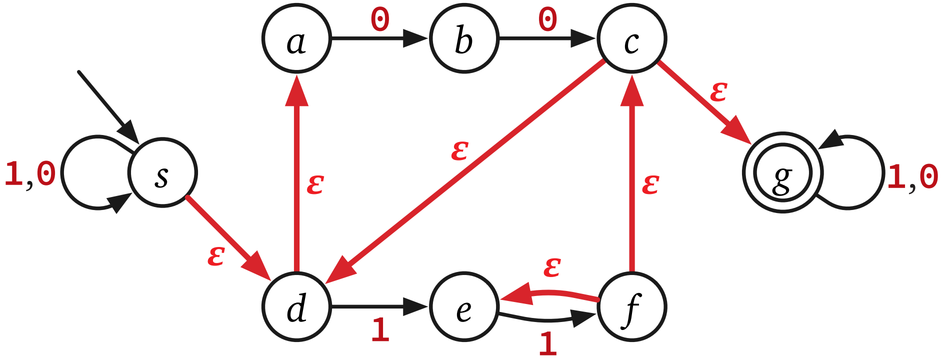
- Want transition function $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$
- $\delta^*(q, w)$: set of states reachable on input w starting in state q.



Extending the transition function to strings

Definition
For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the ϵreach(q) is the set of all states that q can reach using only ε-transitions.



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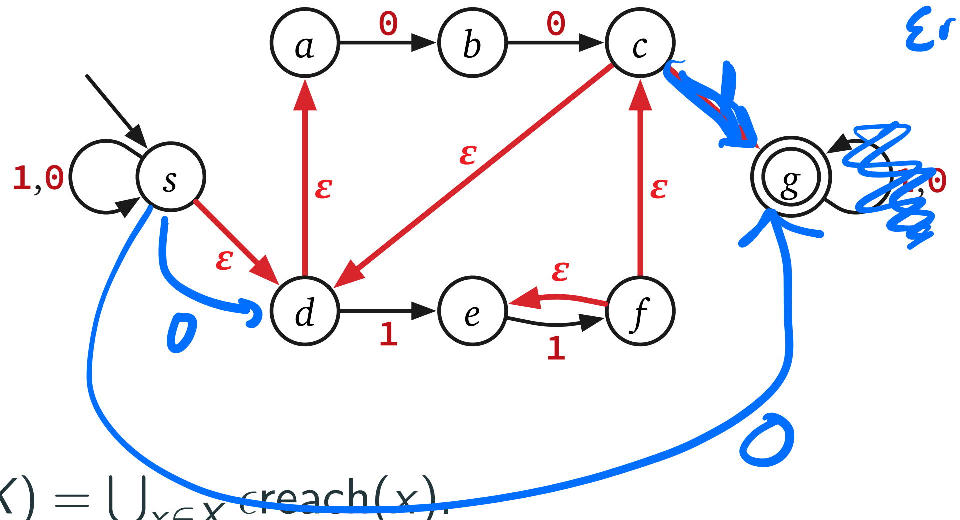

Definition
For $X \subseteq Q$: $\epsilon\text{reach}(X) = \bigcup_{x \in X} \epsilon\text{reach}(x)$.

Extending the transition function to strings

$\epsilon\text{reach}(q)$: set of all states that q can reach using only ε-transitions.

Definition
Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \varepsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$

Extending the transition function to strings

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- if $w = a$ where $a \in \Sigma$: $\delta^*(q, a) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a) \right)$

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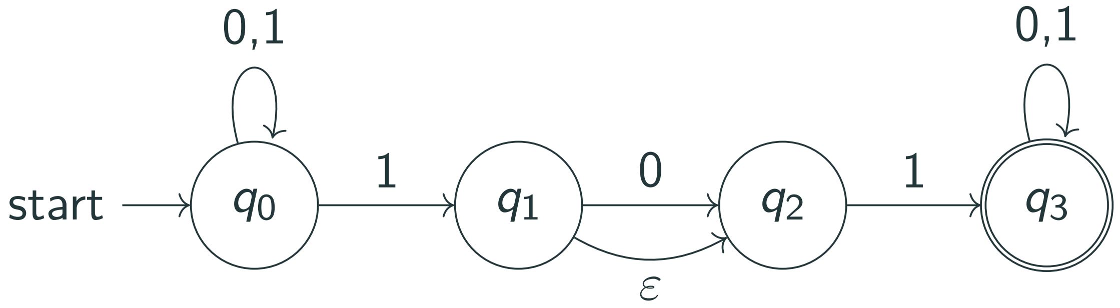
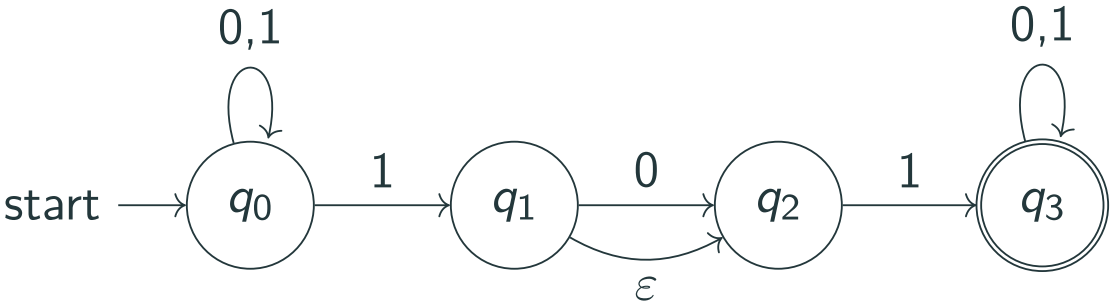
- if $w = ax$: $\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \left(\bigcup_{r \in \delta^*(p,a)} \delta^*(r, x) \right) \right)$

Example of extended transition function



Find $\delta^* (q_0, 11)$:

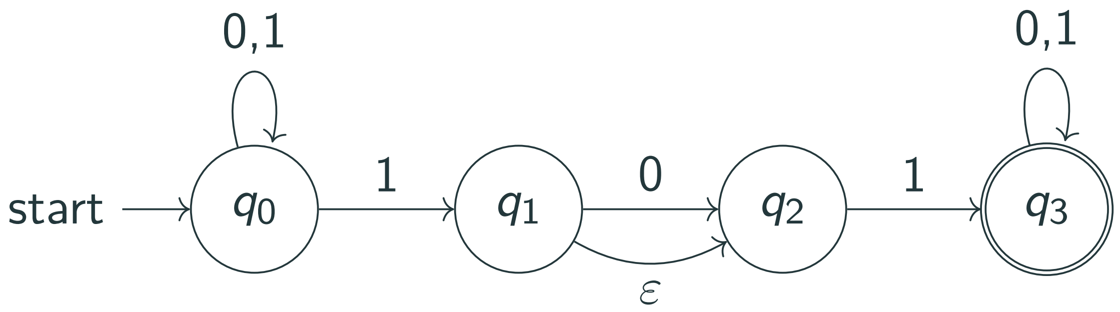
Example of extended transition function



Find $\delta^*(q_0, 11)$:

$$\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \left(\bigcup_{r \in \delta^*(p,a)} \delta^*(r, x)\right)\right)$$

Example of extended transition function



We know $w = 11 = ax$ so $a = 1$ and $x = 1$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q_0)} \left(\bigcup_{r \in \delta^*(p,1)} \delta^*(r, 1) \right) \right)$$

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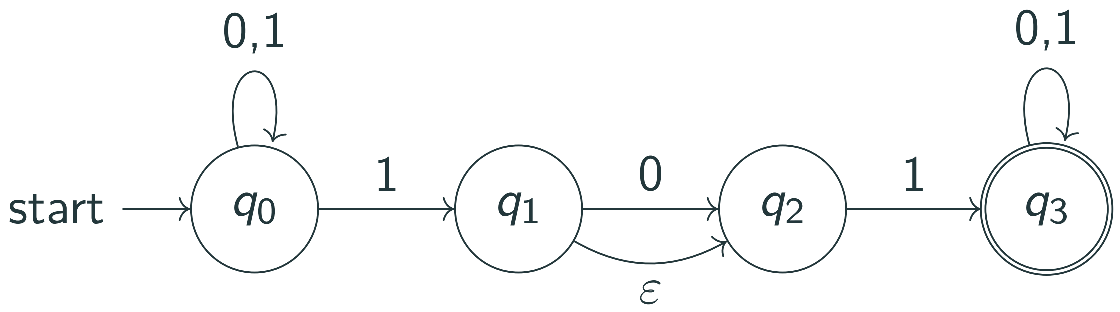

$$\epsilon\text{reach}(q_0) = \{q_0\}$$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{p\in\{q_0\}}\left(\bigcup_{r\in\delta^*(p,1)}\delta^*(r,1)\right)\right)$$

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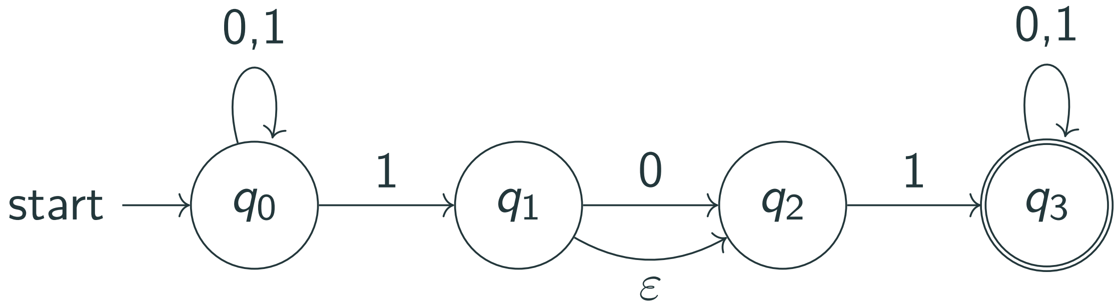

Simplify:

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{r \in \delta^*(\{q_0\}, 1)} \delta^*(r, 1) \right)$$

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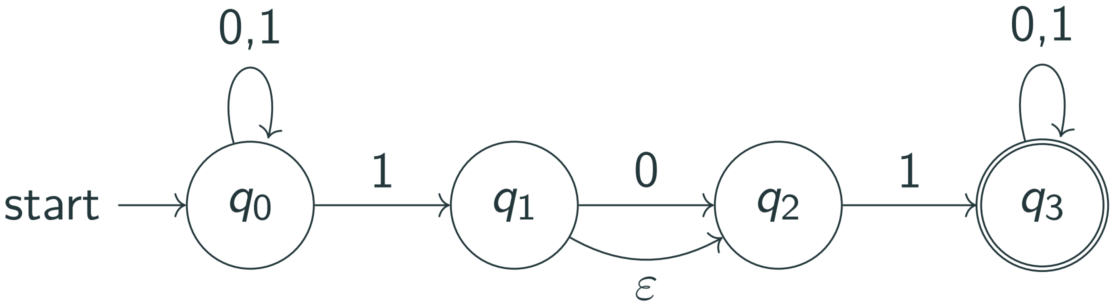

Need $\delta^*(q_0, 1) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a)\right) = \epsilon\text{reach}(\delta\,(q_0, 1))$:

$= \epsilon\text{reach}(\{q_0, q_1\}) = \{q_0, q_1, q_2\}$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{r \in \delta^*(\{q_0\}, 1)} \delta^*(r, 1)\right)$$

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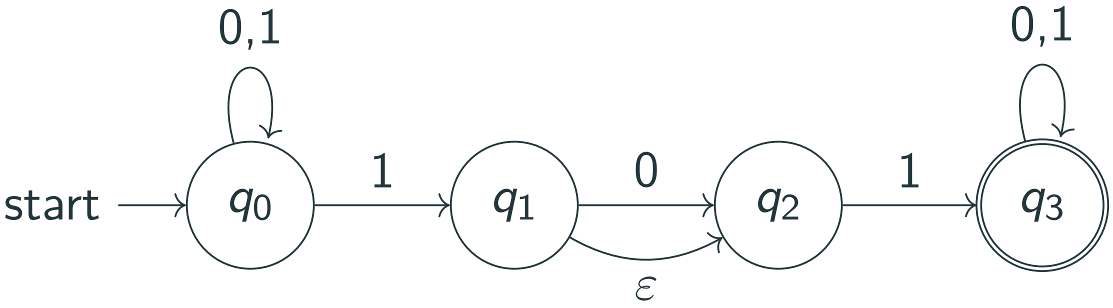
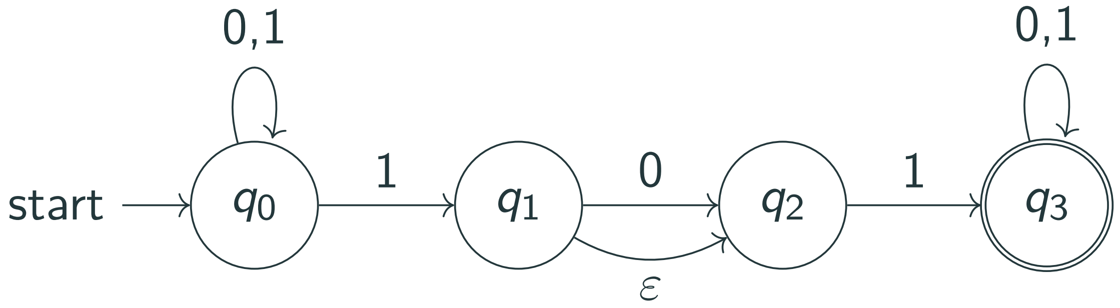


Need $\delta^*(q_0, 1) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a)\right) = \epsilon\text{reach}(\delta\,(q_0, 1))$:

$= \epsilon\text{reach}(\{q_0, q_1\}) = \{q_0, q_1, q_2\}$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{r \in \{q_0, q_1, q_2\}} \delta^*(r, 1)\right)$$

Example of extended transition function



Simplify

$\delta^*(q_0, 11) = \epsilon\text{reach}(\delta^*(q_0, 1) \cup \delta^*(q_1, 1) \cup \delta^*(q_2, 1))$ $= \{q_0, q_1, q_3\}$

Transition for strings: $w = ax$

$$\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \left(\bigcup_{r \in \delta^*(p,a)} \delta^*(r, x) \right) \right)$$

- $R = \epsilon\text{reach}(q) \implies \delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{p \in R} \bigcup_{r \in \delta^*(p,a)} \delta^*(r, x) \right)$
- $N = \bigcup_{p \in R} \delta^*(p, a)$: All the states reachable from q with the letter a.
- $\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{r \in N} \delta^*(r, x) \right)$

Formal definition of language accepted by N

Definition
A string w is accepted by NFA N if $\delta^*_N(s, w) \cap A \neq \emptyset$.

Definition
The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

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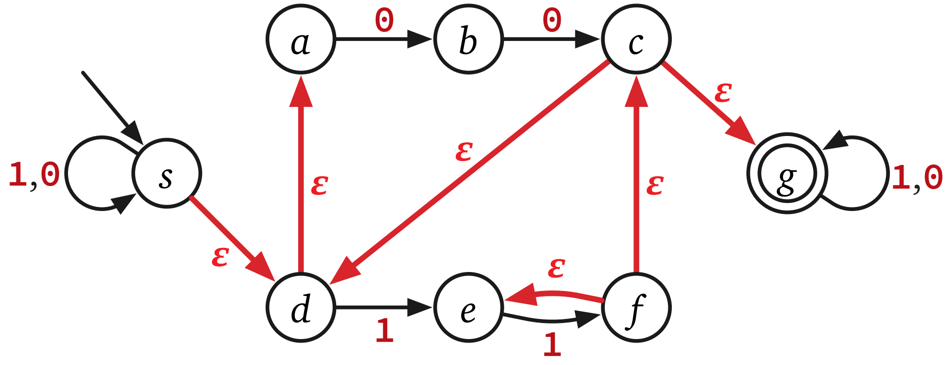
Important: Formal definition of the language of NFA above uses δ^* and not δ. As such, one does not need to include ε-transitions closure when specifying δ, since δ^* takes care of that.

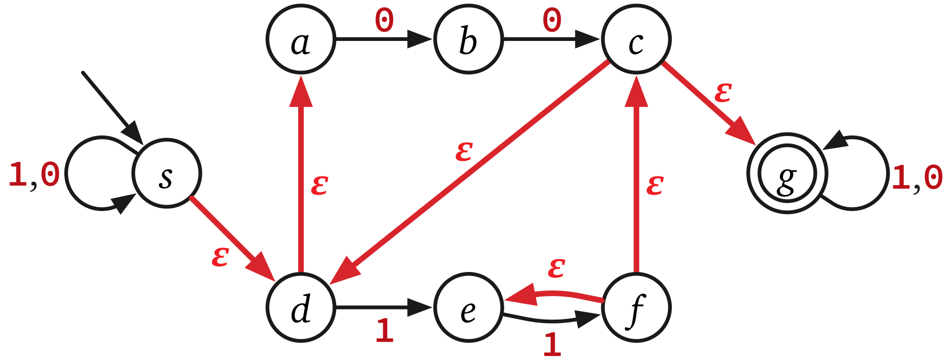
Example



What is:

- $\delta^*(s, \epsilon) =$

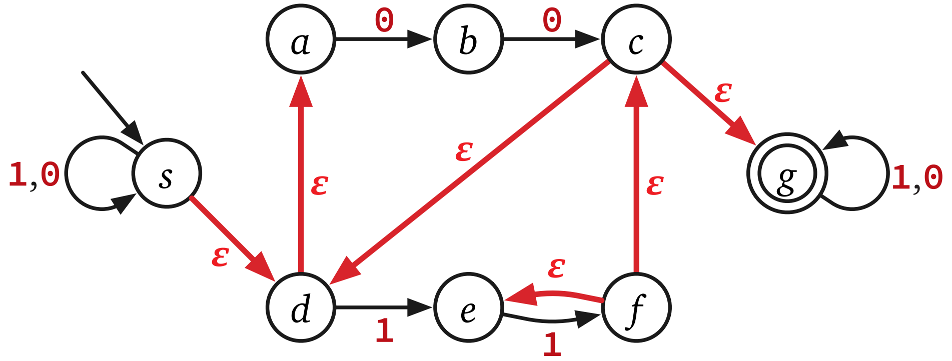
Example



What is:

- $\delta^*(s, \epsilon) =$
- $\delta^*(s, 0) =$

Example



What is:

- $\delta^*(s, \epsilon) =$
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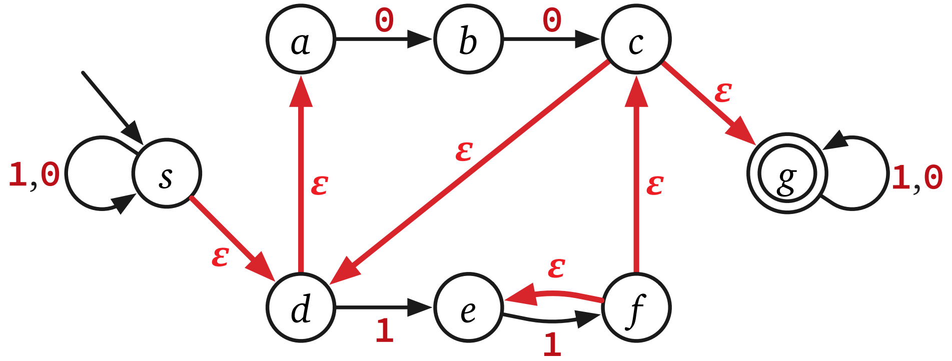

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- $\delta^*(s, \epsilon) =$
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- $\delta^*(b, 0) =$
- $\delta^*(b, 00) =$

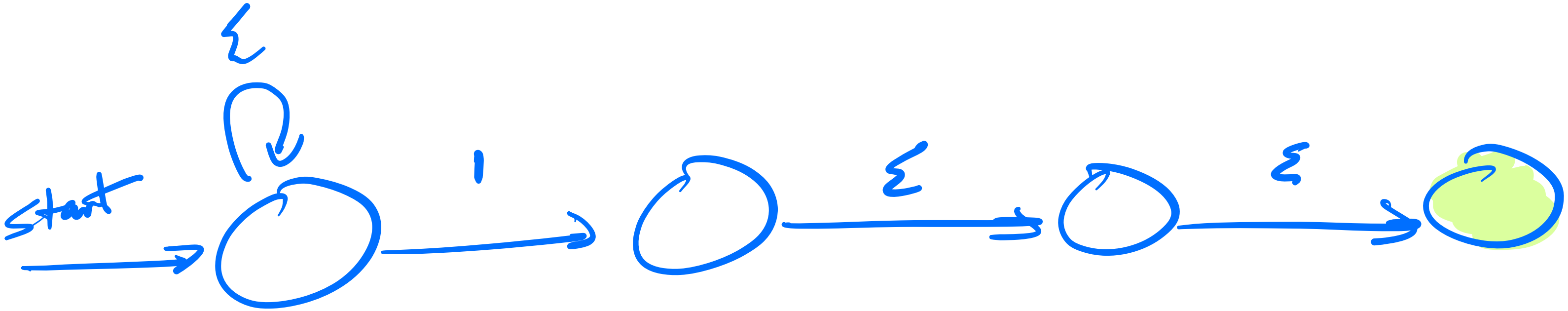
Constructing generalized NFAs

DFAs and NFAs

- Every DFA is a NFA so NFAs are at least as powerful as DFAs.
- NFAs prove ability to "guess and verify" which simplifies design and reduces number of states
- Easy proofs of some closure properties

Example

$L = \{\text{bitstrings that have a } 1 \text{ three positions from the end}\}$



A simple transformation

Theorem

For every NFA N there is another NFA N' such that $L(N) = L(N')$ and such that N' has the following two properties:

- *N' has single final state f that has no outgoing transitions*
- *The start state s of N is different from f*

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Why couldn't we say this for DFA's?

A simple transformation

Hint: Consider the $\mathsf{L} = 0^* + 1^*$.

Closure Properties of NFAs

Closure properties of NFAs

Are the class of languages accepted by NFAs closed under the following operations?

- union
- intersection
- concatenation
- Kleene star
- complement

Closure under union

Theorem
For any two NFAs N_1 and N_2 there is a NFA N such that $L(N) = L(N_1) \cup L(N_2)$.

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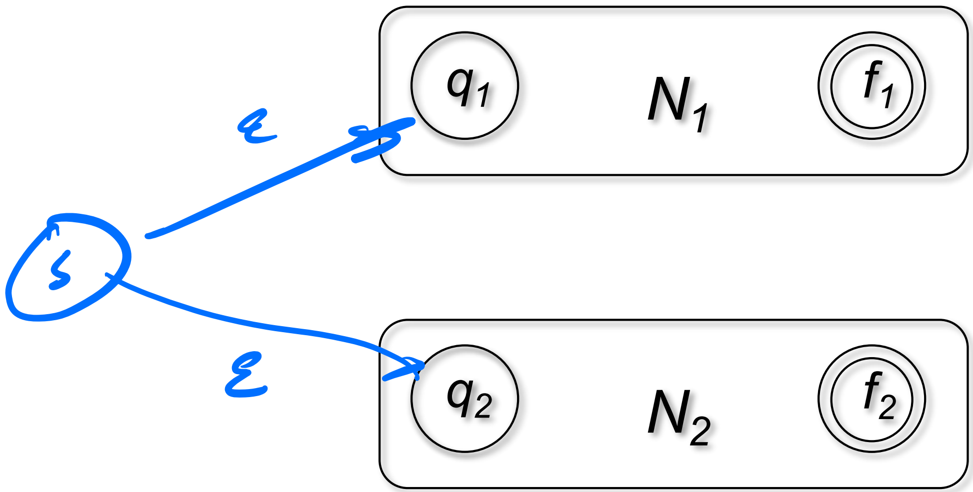

Closure under concatenation

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Closure under concatenation

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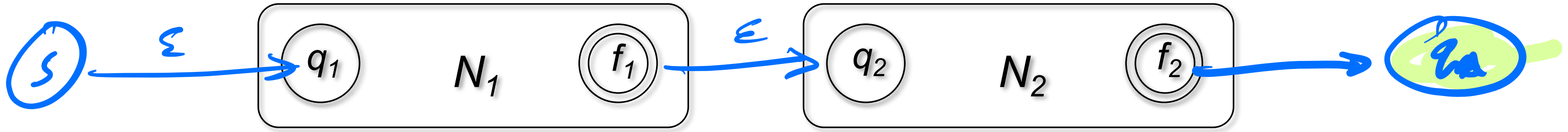
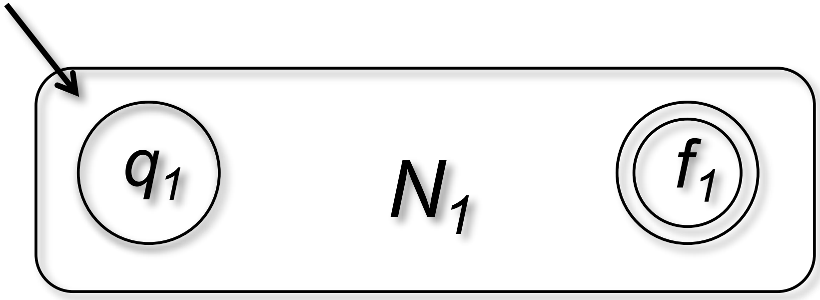


Closure under Kleene star

Theorem
For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.



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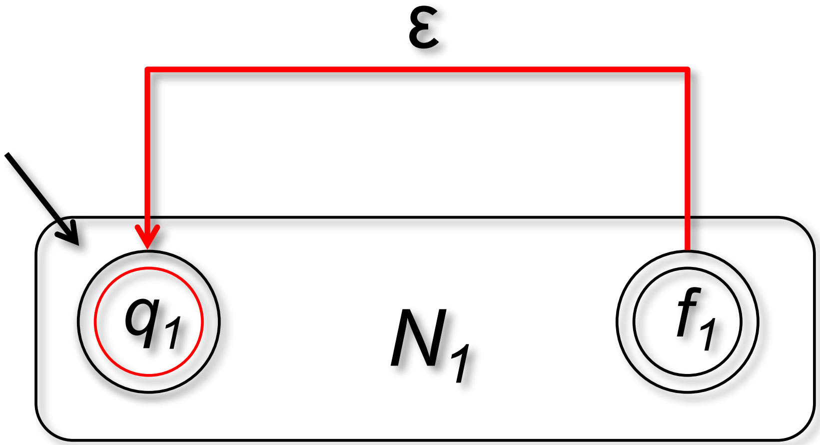

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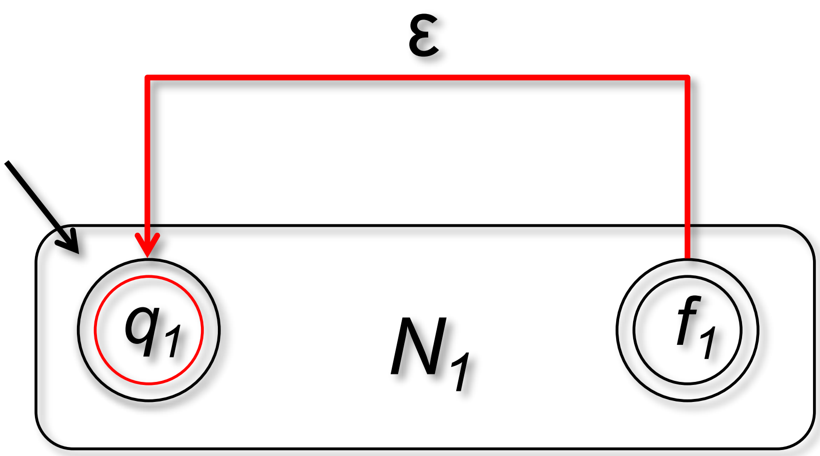

Does not work! Why?

Closure under Kleene star

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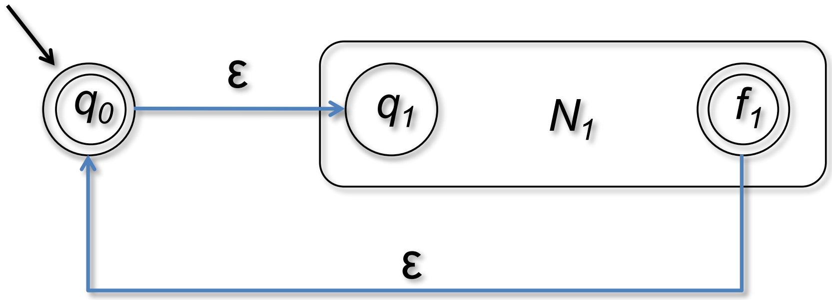


Transformations

All these examples are examples of language *transformations*.

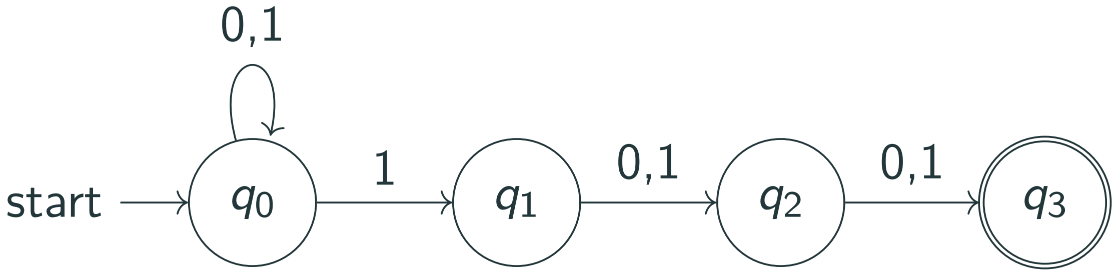
A language transformation is one where you take one class or languages, perform some operation and get a new language **that belongs to that same class (closure)**.

Tomorrow's lab will go over more examples of language transformations.

Last thought

Equivalence

Do all NFAs have a corresponding DFA?



Equivalence

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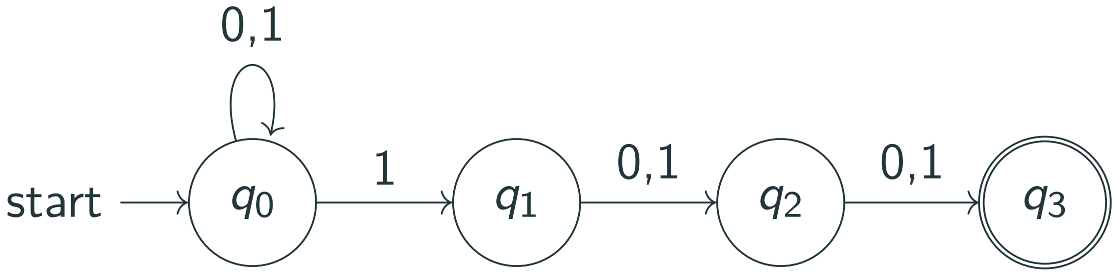

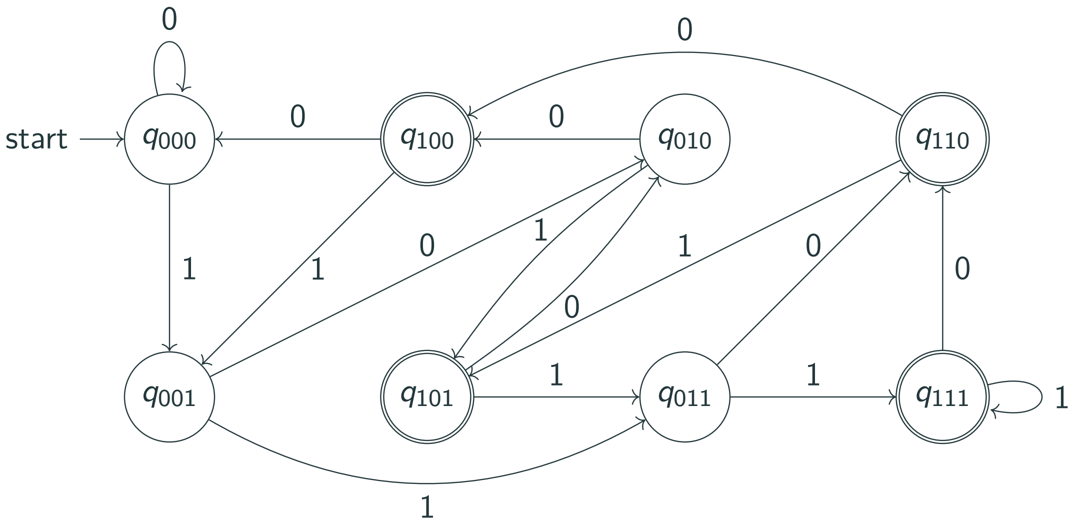

Yes but it likely won't be pretty.