Pushdown automata and context-free languages

Sides based on material by Kani, Erickson, Chekuri, et. al.

All mistakes are my own! - Ivan Abraham (Fall 2024)

Image by ChatGPT (probably collaborated with DALL-E)

Introduction

Will this code execute successfully?

main.c

```c
// Online C compiler to run C program online
#include <stdio.h>

int main() {
    int a = 2;
    int b = 5 ;
}
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main.c

Share Run

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Output Clear

```
/tmp/3w7RzFLskv.o


=== Code Execution Successful ===
```

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ERROR!
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What was the compiler expecting?

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- **Recursive Expression**

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- **Recursive Expression**

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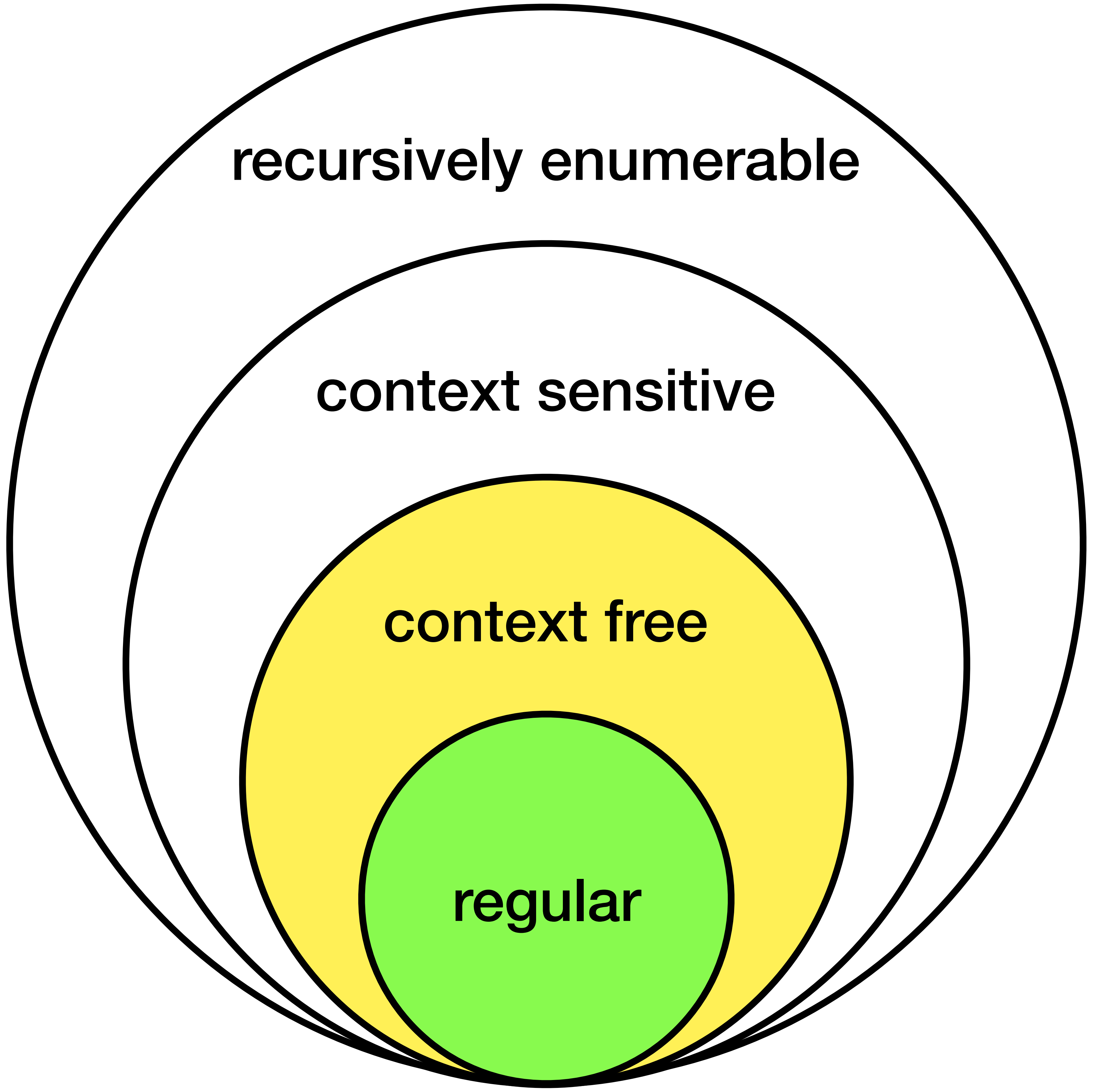
This one says "if you see **Op**, you can replace it with **+** or **-** or **×** or **/**

Grammar - rules for a language

A ***context-free grammar*** (or ***CFG***) is a recursive set of rules that define a language.

Definition:



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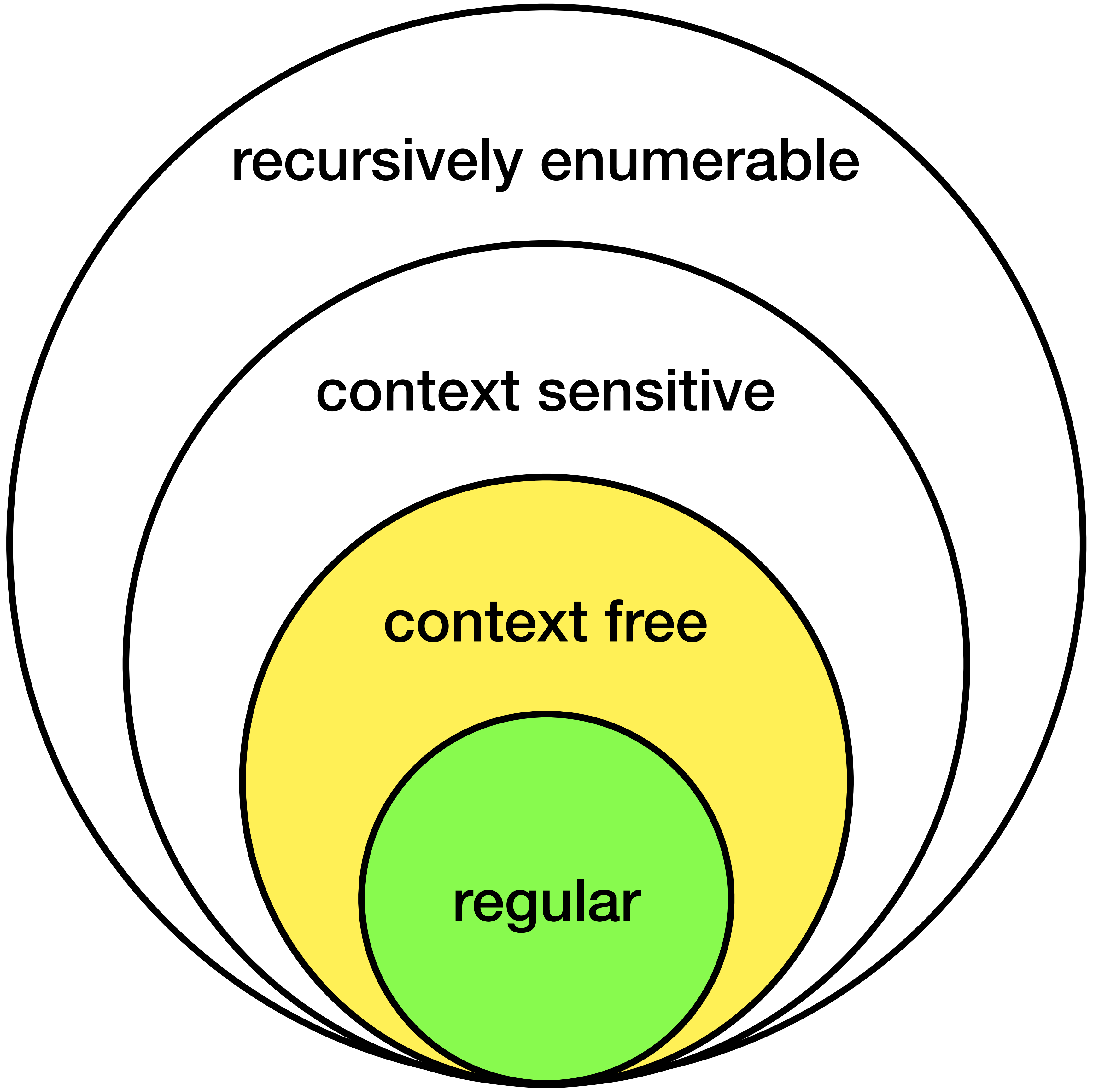

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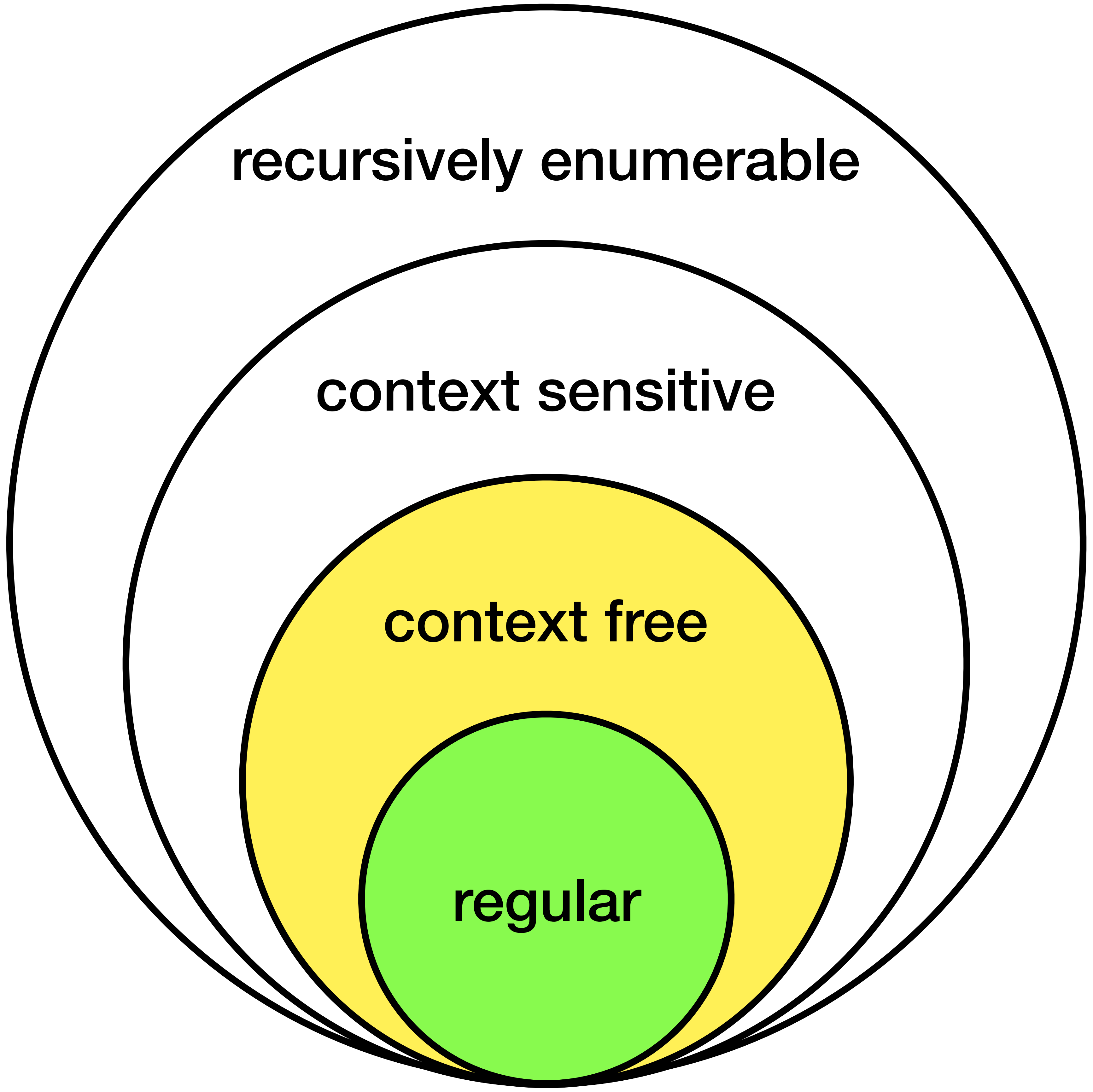
G = (*Variables*, *Terminals*, *Productions*, *Start Var*)



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- $S \in V$ is a ***start symbol*** .

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cannot produce an odd length string

010

$S \to 0S0$

$\not\downarrow$

?

$$S \to 0S0 \to 01S10 \to 011S110 \to 011\epsilon110 \to 011110$$

Derives relation

Formalism for how strings are derived/generated

Definition: Let $G = (V, T, P, S)$ be a CFG. For strings, $\alpha_1, \alpha_2 \in (V \cup T)*$ we say α_2 *derives from* α_1, denoted by $\alpha_1 \rightsquigarrow \alpha_2$, if there exist strings β, γ, δ in $(V \cup T)*$ such that

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- $A \to \gamma \in P$

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- $\alpha_1 \overset{k}{\rightsquigarrow} \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \overset{k-1}{\rightsquigarrow} \alpha_2$

single step first

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- $\alpha_1 \overset{k}{\rightsquigarrow} \alpha_2$ if $\alpha_1 \rightsquigarrow \beta_1$ and $\beta_1 \overset{k-1}{\rightsquigarrow} \alpha_2$
- Alternatively, $\alpha_1 \overset{k}{\rightsquigarrow} \alpha_2$ if $\alpha_1 \overset{k-1}{\rightsquigarrow} \beta_1$ and $\beta_1 \rightsquigarrow \alpha_2$

single step
last.

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Finally, we use the notation $\alpha_1 \overset{*}{\rightsquigarrow} \alpha_2$ to mean that α_2 can be derived from α_1. In other words,

$$\alpha_1 \overset{*}{\rightsquigarrow} \alpha_2 \text{ if } \alpha_1 \overset{k}{\rightsquigarrow} \alpha_2 \text{ for some } k$$

"α_1 'becomes' α_2 in zero or more steps"

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Thus, a language L is context-free (called a context-free language or CFL) if it is generated by a context-free grammar.

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Alternatively, a language L is said to be a CFL, if there exists a CFG G such that $L = L(G)$.

Context Free Languages

Production rule examples

- $L = \{0^n 1^n \mid n \geq 0\}$

- n can be zero

  $S \rightarrow \varepsilon$

- if there is a zero, then there must be a 1

  $S \rightarrow 01$

- Need to repeat "01" as many times as necessary in the where $0 \square 1$

  $S \rightarrow 0S1$

$$S \rightarrow \varepsilon \mid 0S1$$

Context Free Languages

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- $L = \{0^n 1^n \mid n \geq 0\}$
- $L = \{0^n 1^m \mid m \geq n\}$

- n can be zero (and equal to m)

$S \rightarrow \varepsilon$

- n can be same as m (and not zero)

$S \rightarrow 0S1$

- m can be greater than n.

$S \rightarrow S1$

$S \rightarrow \varepsilon \mid 0S1 \mid S1$

what if $m > n$?

$S \rightarrow 1 \mid 0S1 \mid S1$

$S \rightarrow \varepsilon \mid 0S1 \mid SR$

$R \rightarrow 1$

Context Free Languages

Production rule examples

- $L = \{0^n1^n \mid n \geq 0\}$
- $L = \{0^n1^m \mid m \geq n\}$
- $L = \{0^n1^m \mid m, n \geq 0\}$

m, n can be 0

$S \to \varepsilon$.

$S \to 0S$ zero always at beginning

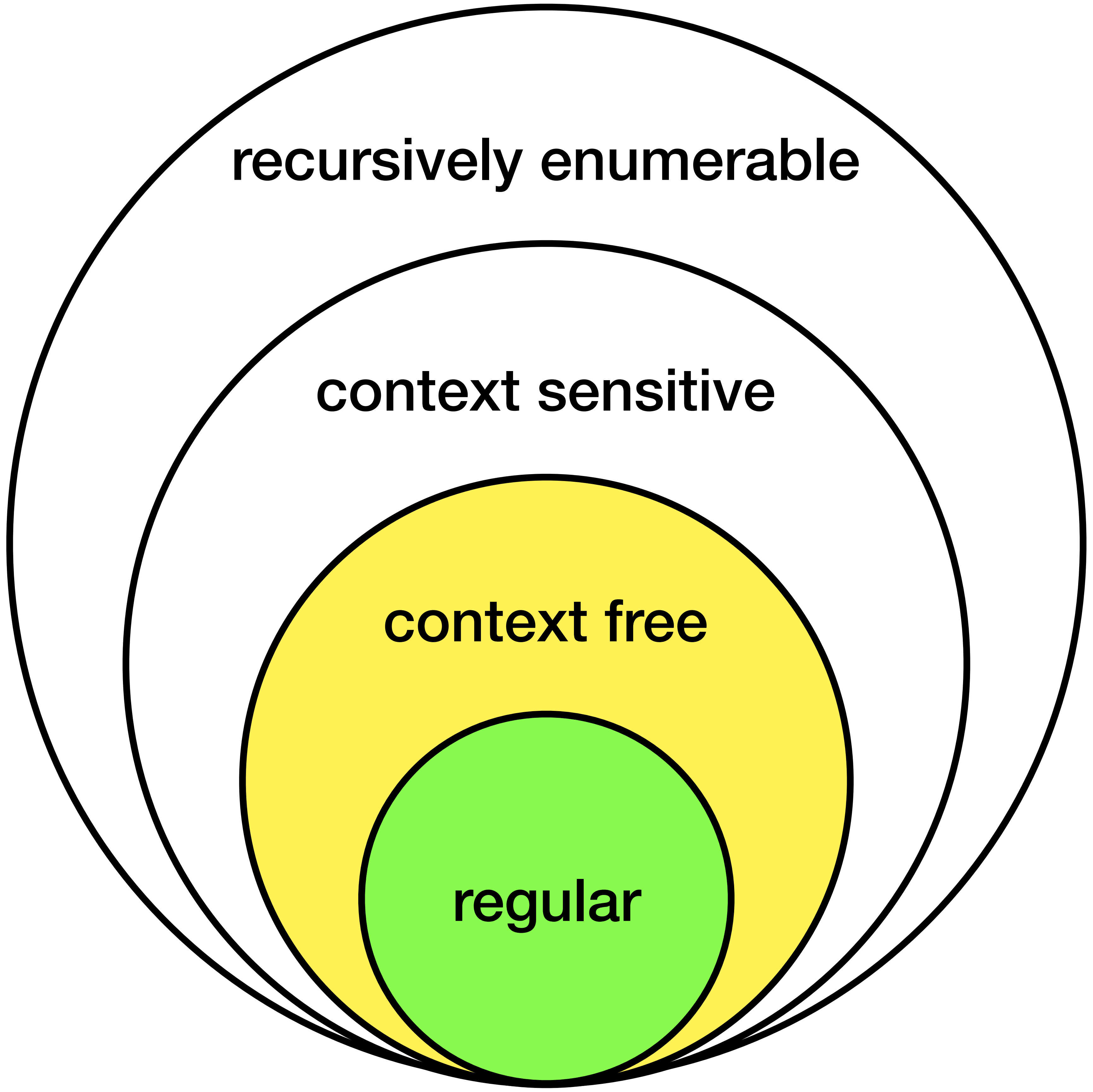
$S \to S1$ one always at end.

$S \to \varepsilon \mid 0S \mid S1$

CFL/CFGs and regular languages

Recall Chomsky Heirarchy

- The picture depicts regular languages as a proper subset of context-free languages.



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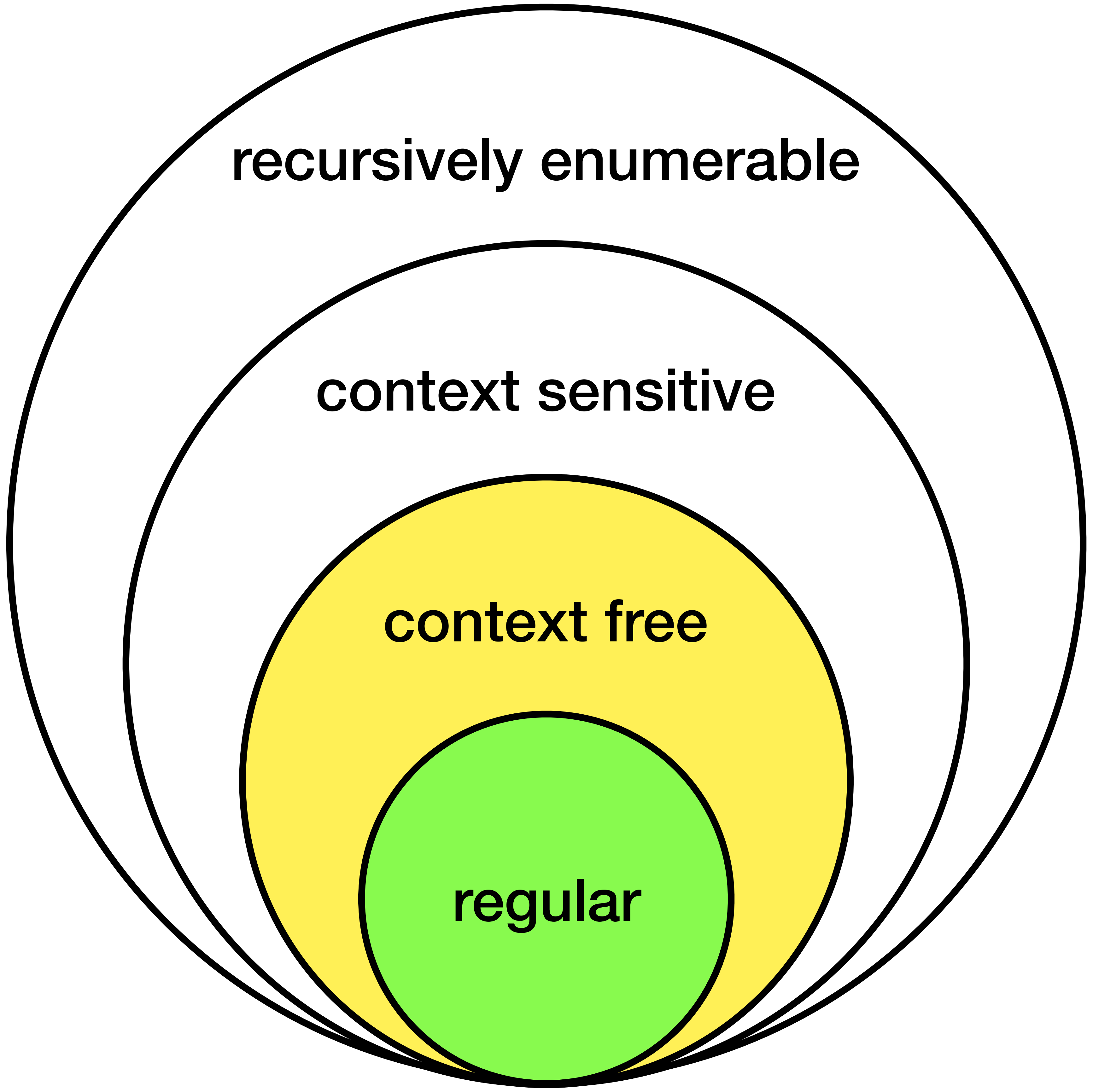

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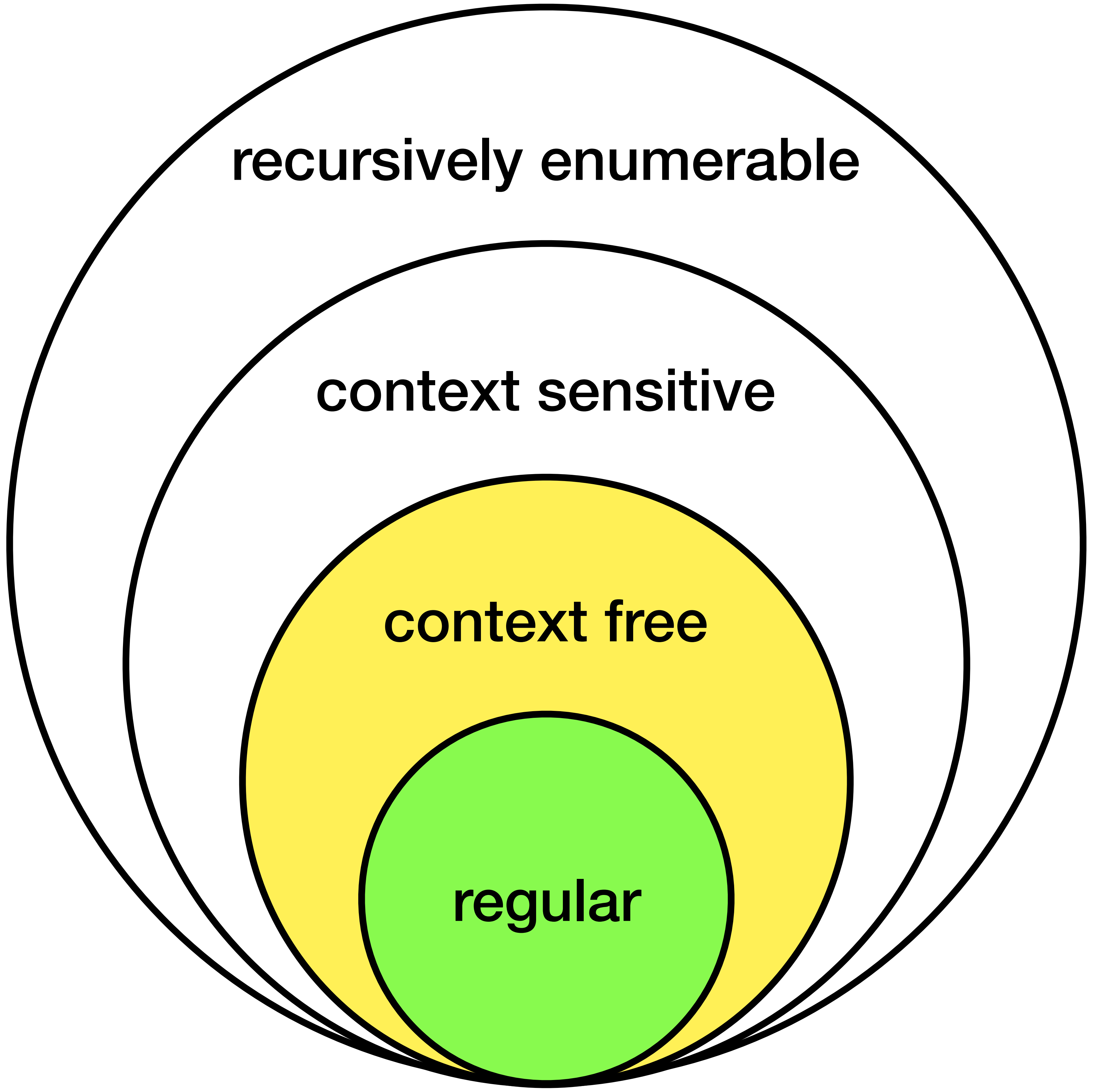

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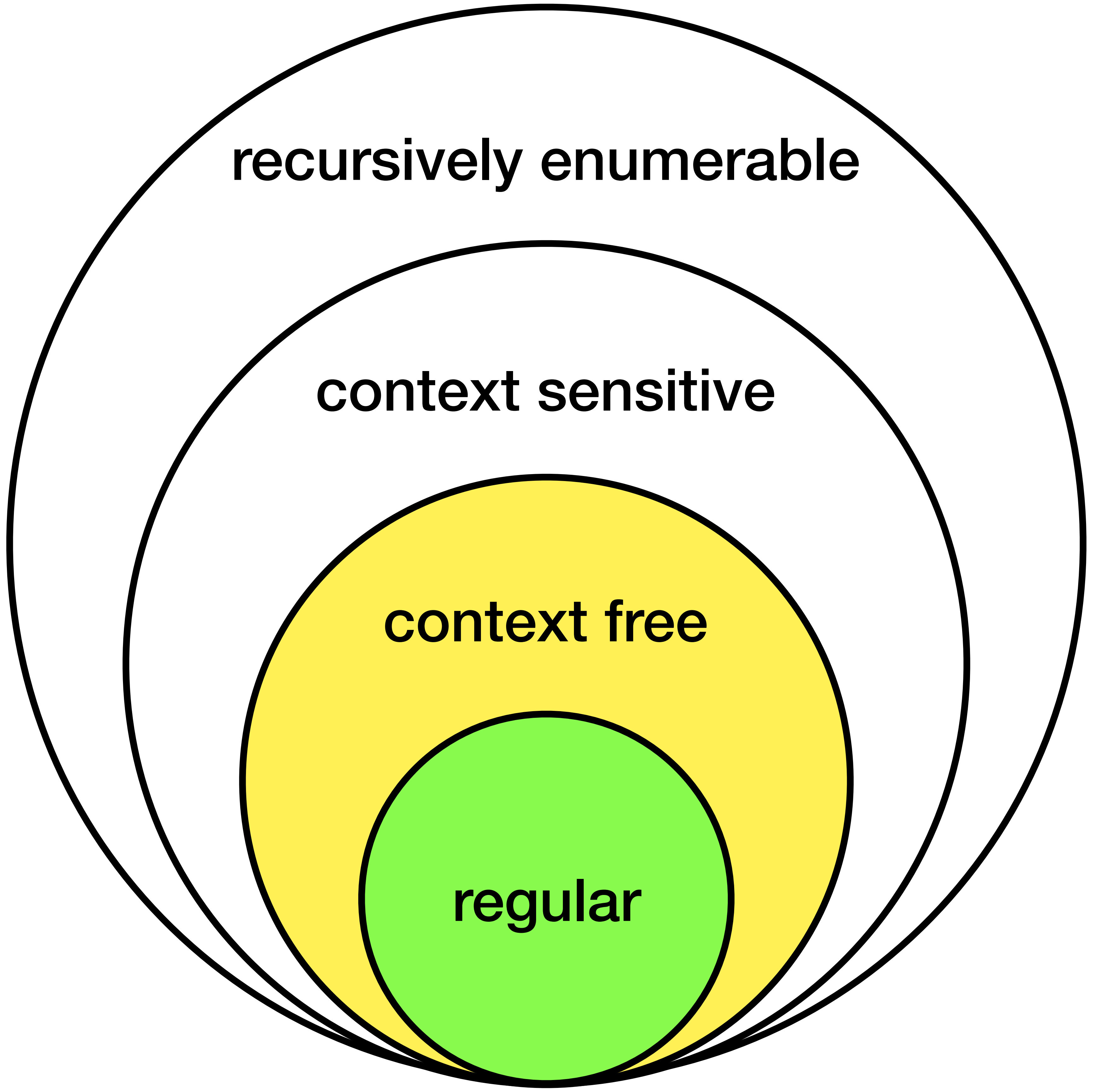

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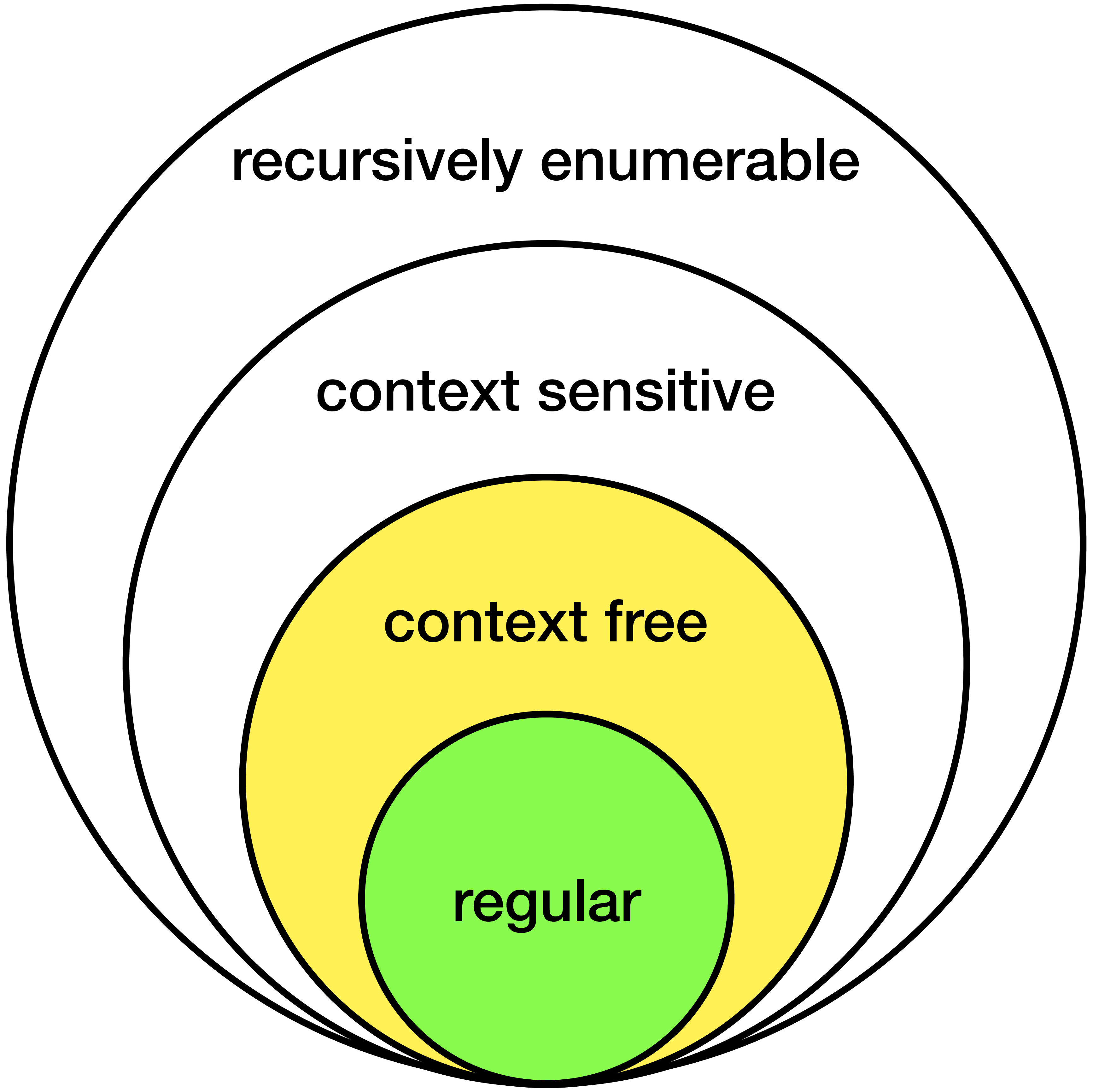
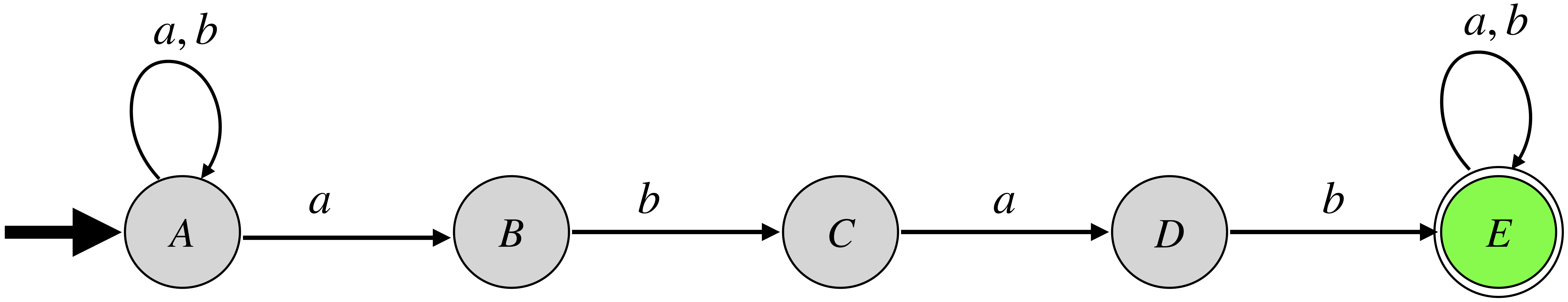
    - What was the grammar that generated a regular language?
        - We can start with the DFA recognizing a regular language.
        - Then, extend the algebraic method.



Converting DFAs into CFL

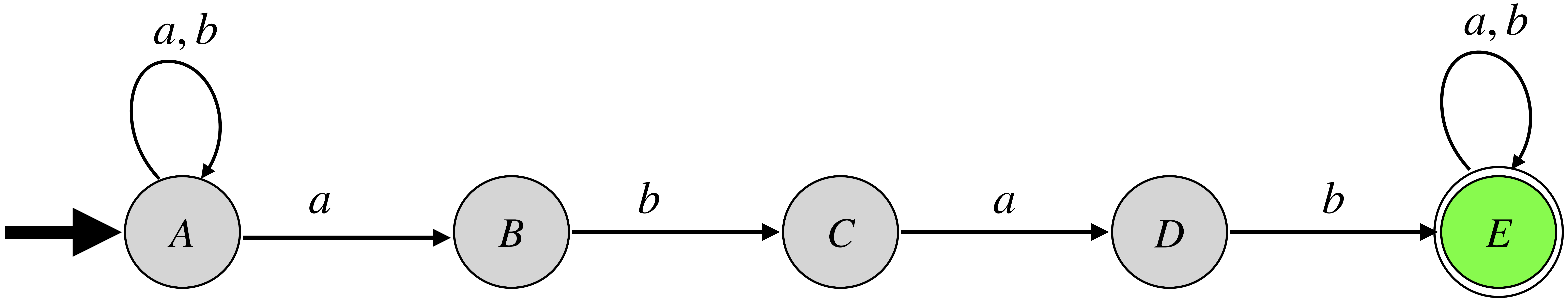


Need for a CFG

$G = (V, T, P, S)$

- V → variables
- T → Terminals
- P → production rules
- S → start variable

Converting regular languages into CFL



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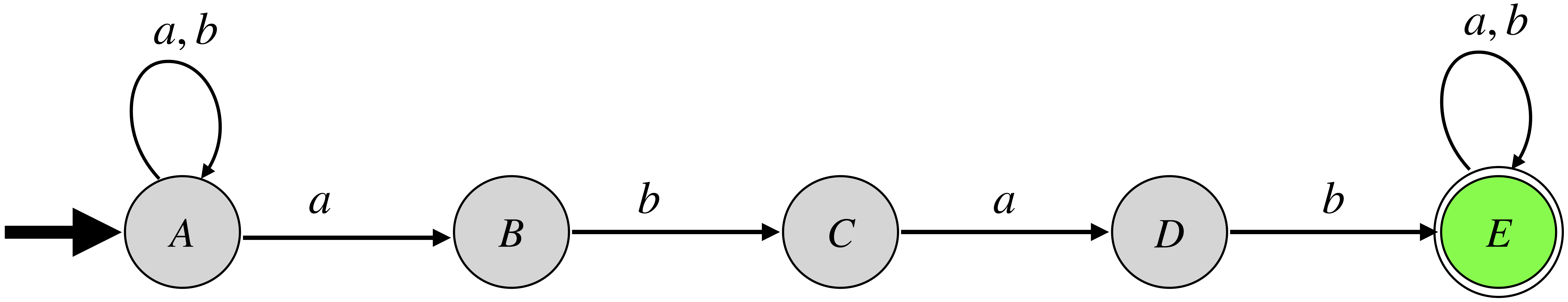

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$$G = \left(\overbrace{Q}^{\text{Variables}}, \overbrace{\Sigma}^{\text{Terminals}}, \overbrace{\{q \to a\underbrace{\delta(q, a)}_{\in Q} \mid q \in Q, a \in \Sigma\} \bigcup_{q \in F} \{q \to \varepsilon\}}^{\text{Productions}}, \overbrace{q_0}^{\text{Start var}} \right)$$

Converting regular languages into CFL

$$G = \left(\{A, B, C, D, E\}, \ \{a, b\} \ \left\{ \begin{array}{r} A \rightarrow aA, A \rightarrow bA, A \rightarrow aB \\ B \rightarrow bC \\ C \rightarrow aD, \\ D \rightarrow bE, \\ E \rightarrow aE, E \rightarrow bE, E \rightarrow \varepsilon \end{array} \right\}, A \right)$$

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In regular languages:

- Terminals can only appear on one side of the production string
- Only one variable allowed in the production result

{0^n 1^n}   S → ε | 0S1 → problem because terminals appear on both sides

Converting regular languages into CFL

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Closure Properties of CFL

Let $G_1 = (V_1, T_1, P_1, S_1)$ and $G_2 = (V_2, T_1, P_2, S_2)$ be CFGs for $L_1 = L(G_1)$ and $L_2 = L(G_2)$

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- CFLs are closed under concatenation: $L_1 \cdot L_2$ is a CFL.

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- CFLs are closed under union: $L_1 \cup L_2$ is a CFL.
- CFLs are closed under concatenation: $L_1 \cdot L_2$ is a CFL.
- CFLs are closed under Kleene star: L_k CFL implies L_k^* is a CFL.

NOT CLOSED

→ intersection

→ complement

Closure Properties of CFL

Closure under concatenation

Let $G_1 = (V_1, T_1, P_1, S_1)$ and $G_2 = (V_2, T_2, P_2, S_2)$ be CFGs for $L_1 = L(G_1)$ and $L_2 = L(G_2)$. Suppose $L = L_1 \cdot L_2$. What is a grammar for L?

Simplifying assumption: $V_1 \cap V_2 = \emptyset$, that is, non-terminals are not shared.

$$G = (V_1 \cup V_2 \cup \{S\}, T_1 \cup T_2, P, S)$$

$$S \to S_1 S_2$$

$$P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$$

Closure Properties of CFL

Closure under Kleene star

Let $G_1 = (V_1, T_1, P_1, S_1)$ be CFG for $L_1 = L(G_1)$. Suppose $L = L_1^*$. What is a grammar for L?

Left as an exercise

Pushdown automata

Pushdown automata

The machine that recognizes CFGs

We established that $\{0^n 1^n \mid n \geq 0\}$ is a CFL but not a regular language.

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(ECE 220) What enables recursion in programming?

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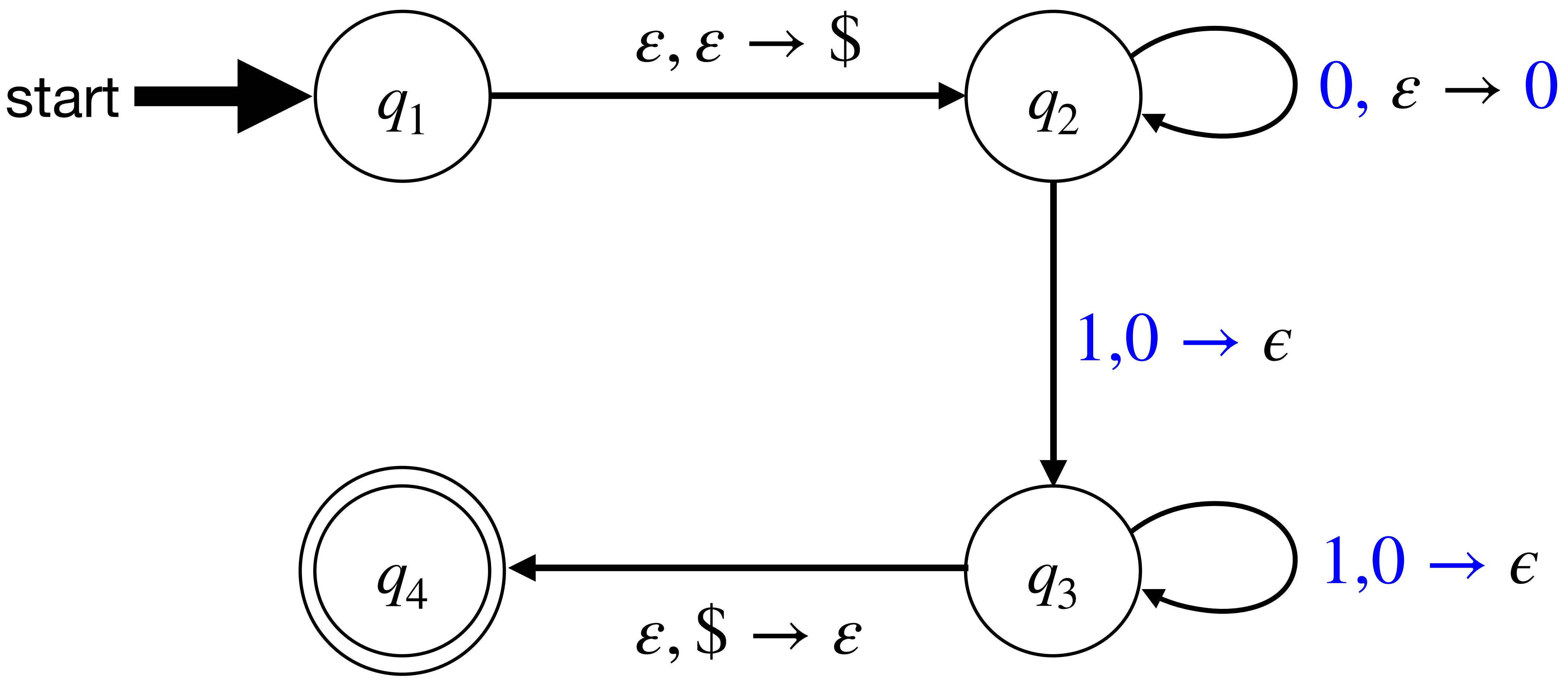
The key idea is that CFGs allow recursive definitions.

(ECE 220) What enables recursion in programming?

We need a stack!

Push-down Automata

The machine that generates CFGs



Each transition is formatted as:

<token read>, <stack pop> → <stack push>

Formal Tuple Notation

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together captures state (Q) and stack configuration

- $\delta : Q \times \Sigma \cup \{\varepsilon\} \times \Gamma \cup \{\varepsilon\} \to \mathscr{P}\left(Q \times (\Gamma \cup \{\varepsilon\})\right)$ is the transition function

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Formal Tuple Notation

Non-deterministic PDAs are more "powerful" than deterministic PDAs. Hence, we'll only be talking about non-deterministic PDAs.

NPDAs can recognize more than DPDAs.

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CFGs and PDAs

Convert a CFG to a PDA

Consider,

$$S \rightarrow 0S \mid 1 \mid \epsilon$$

What is a PDA for this?

CFGs and PDAs

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Key idea: Recreate the string on the stack

CFGs and PDAs

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Consider,

$$S \rightarrow 0S \mid 1 \mid \epsilon$$

What is a PDA for this?

Key idea: Recreate the string on the stack

- Every time we see a non-terminal, we replace it with one of the replacement rules.

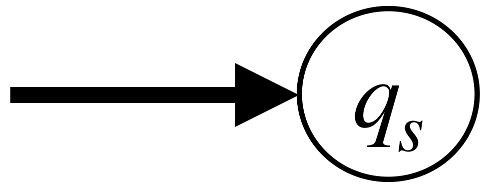
- Every time we see a terminal symbol, we take that symbol from the input.

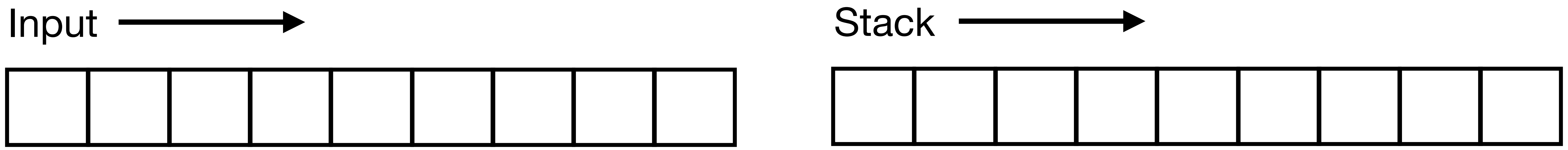
- If we reach a point where the stack and input are empty, then we accept the string.

CFGs and PDAs

Convert a CFG to a PDA

$$S \to 0S \ | \ 1 \ | \epsilon$$





CFGs and PDAs

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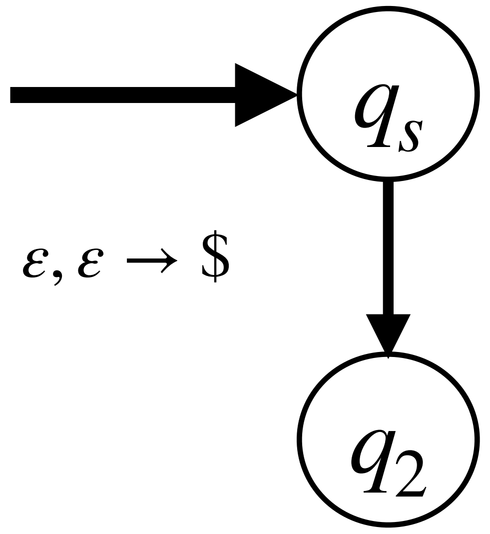
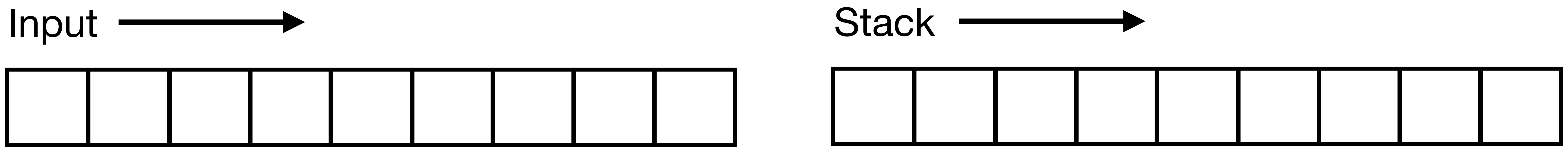


$$S \to 0S \mid 1 \mid \epsilon$$

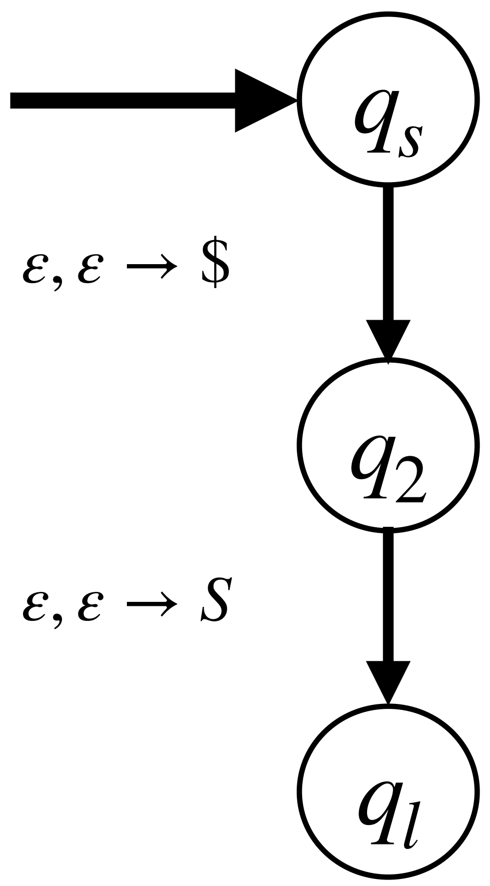
- First let's put in a $\$$ to mark the end of the string



input token, pop item → push item

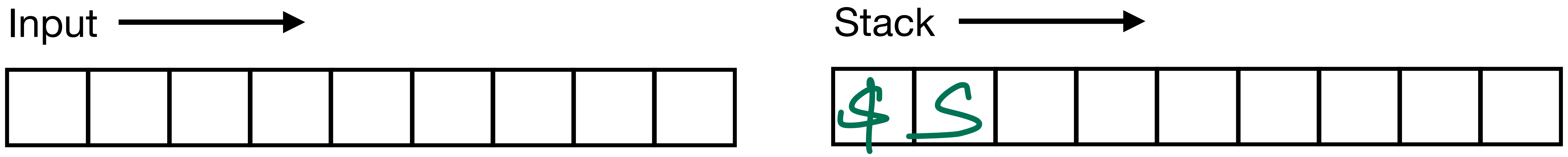
CFGs and PDAs

Convert a CFG to a PDA



$$S \to 0S \mid 1 \mid \epsilon$$

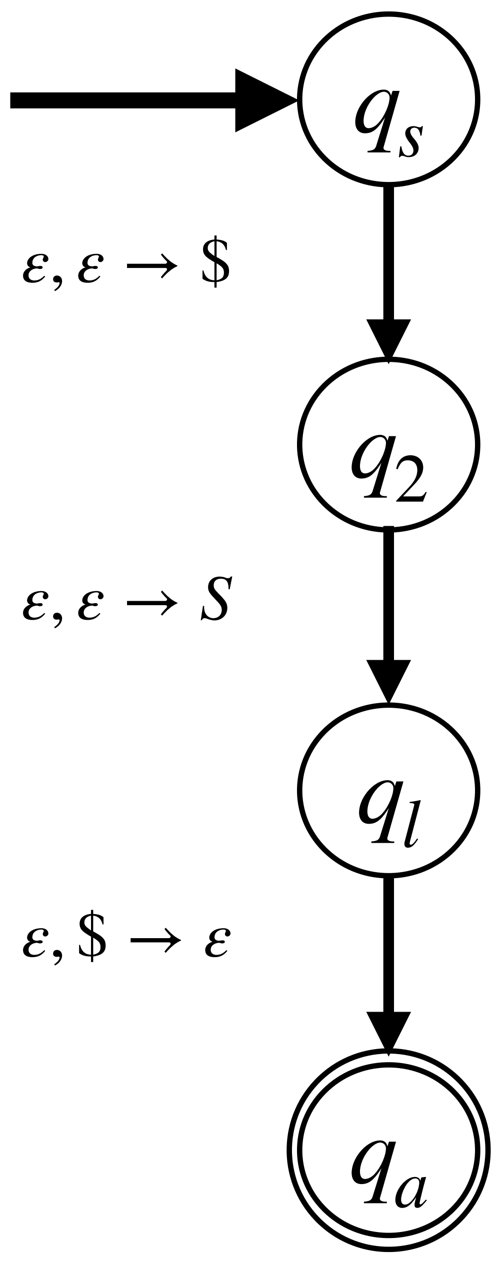
- First let's put in a $\$$ to mark the end of the string
- Also let's put in the start symbol on the stack.



CFGs and PDAs

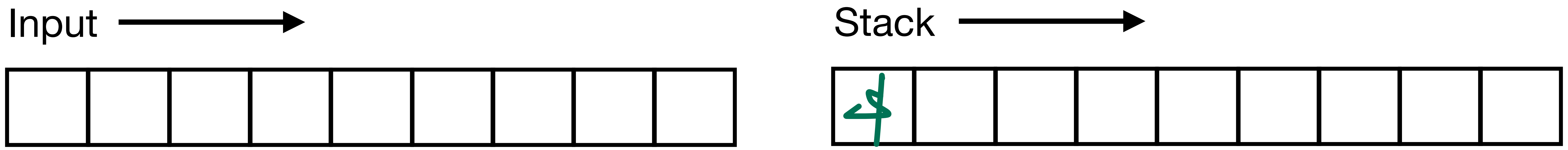
Convert a CFG to a PDA

$S \to 0S \mid 1 \mid \epsilon$



Next we want to add a loop for every non-terminal symbol that replaces that non-terminal with the result.

Consider the rule: $S \to 0S$



CFGs and PDAs

Convert a CFG to a PDA

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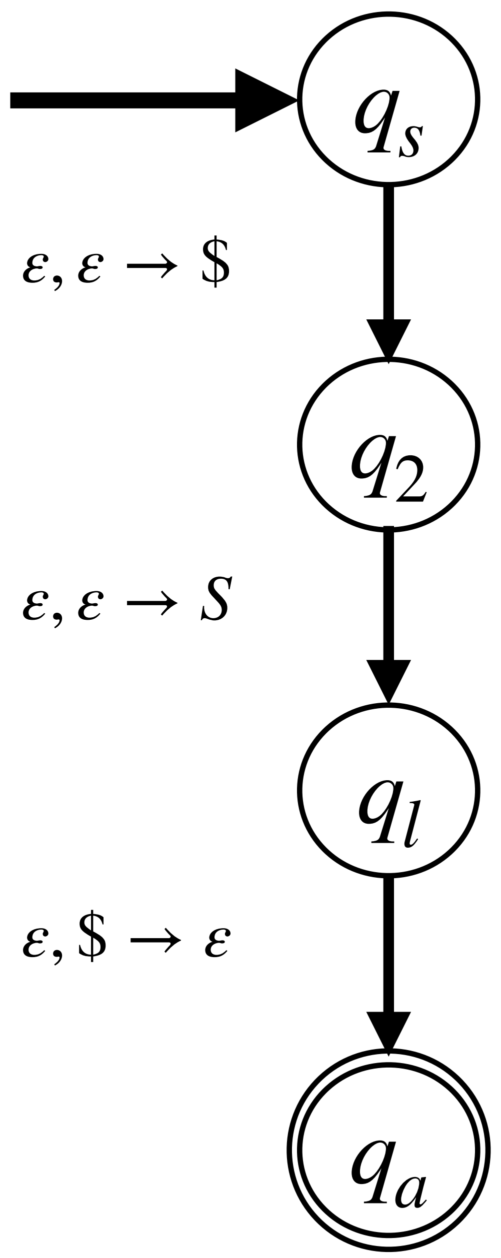

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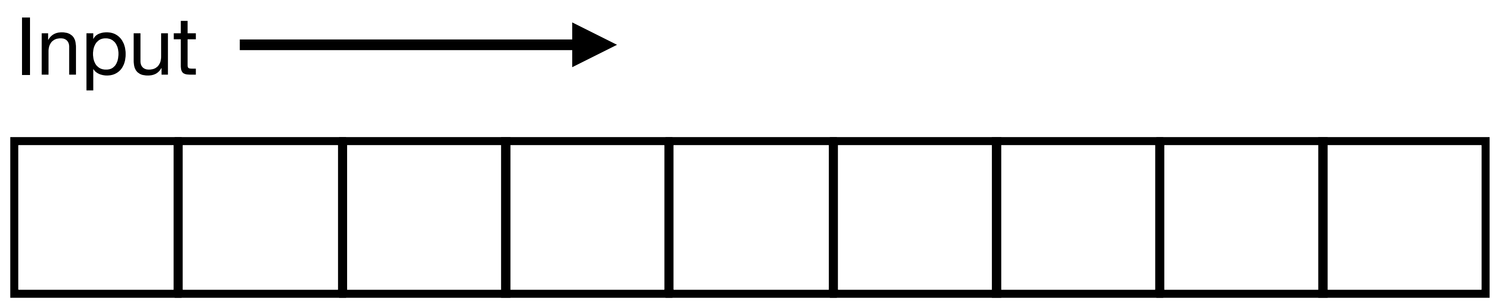

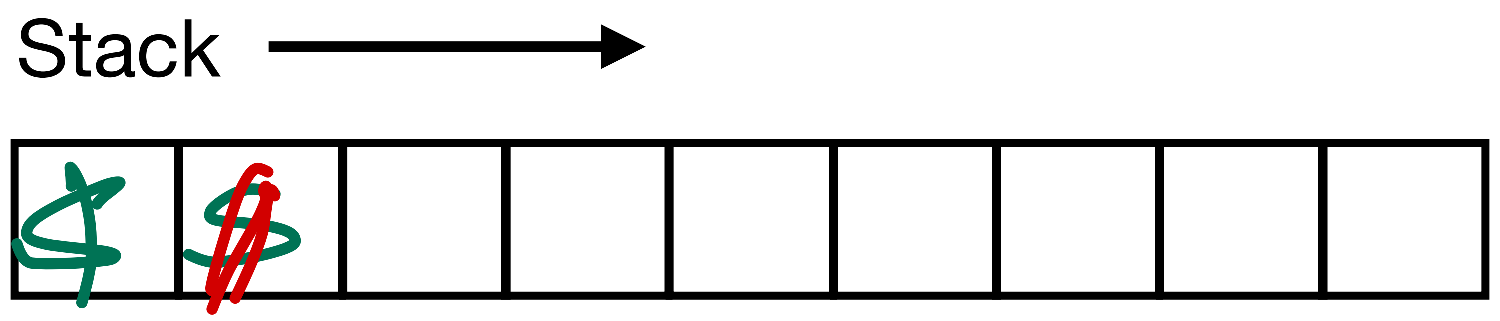

CFGs and PDAs

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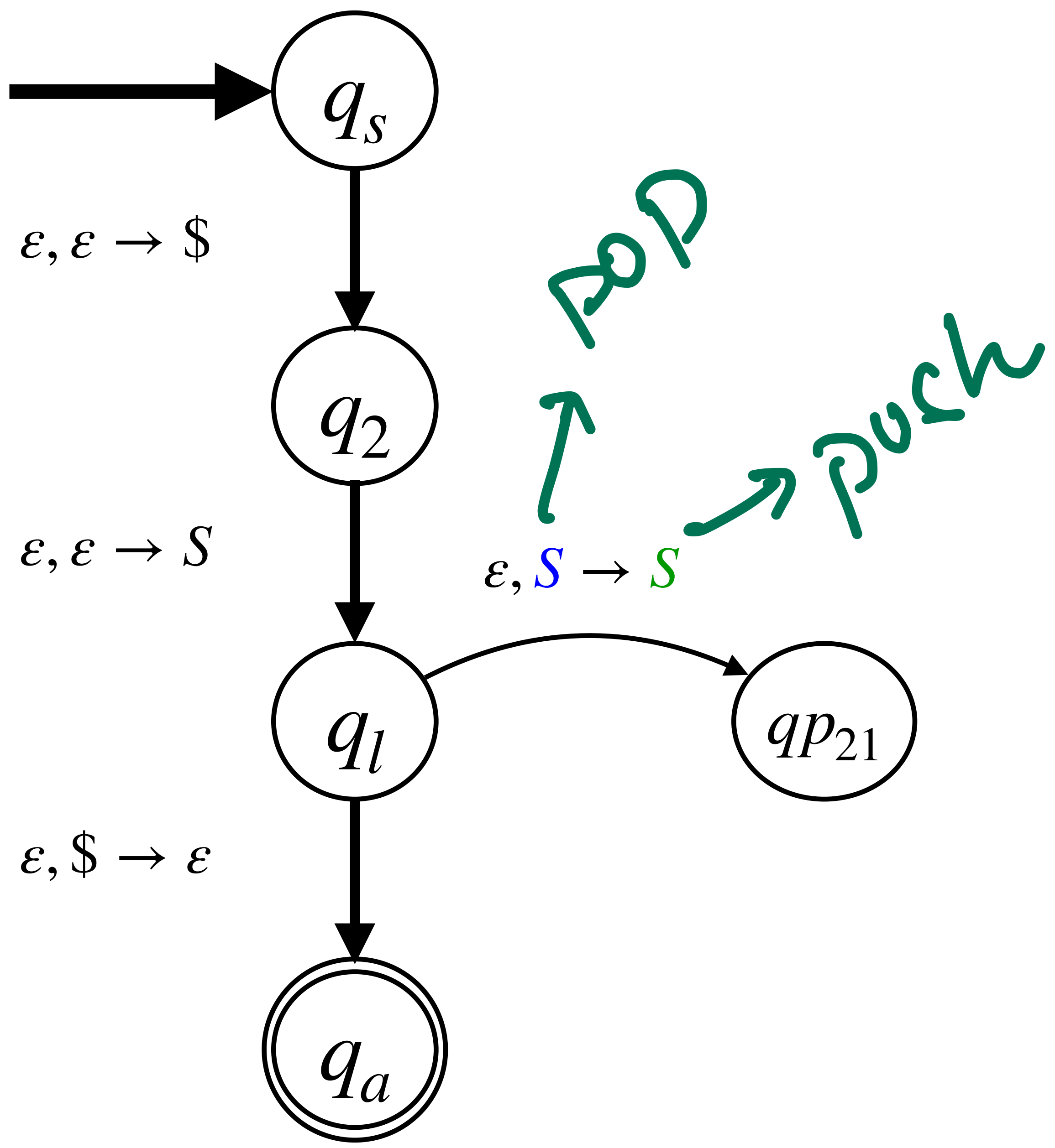

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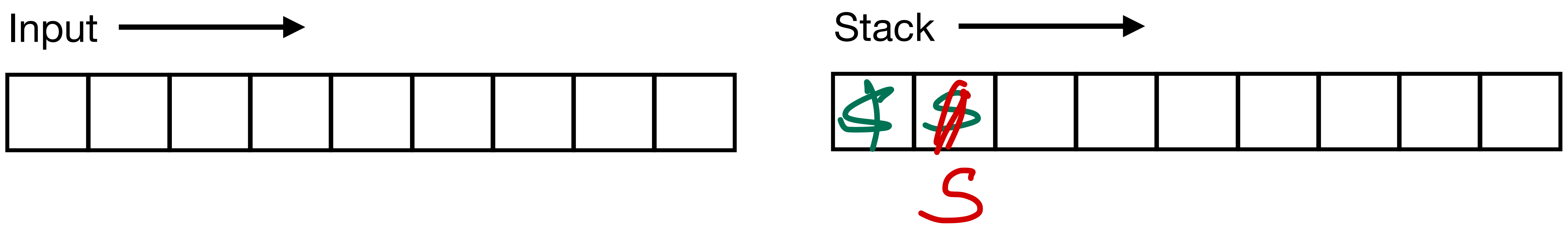

CFGs and PDAs

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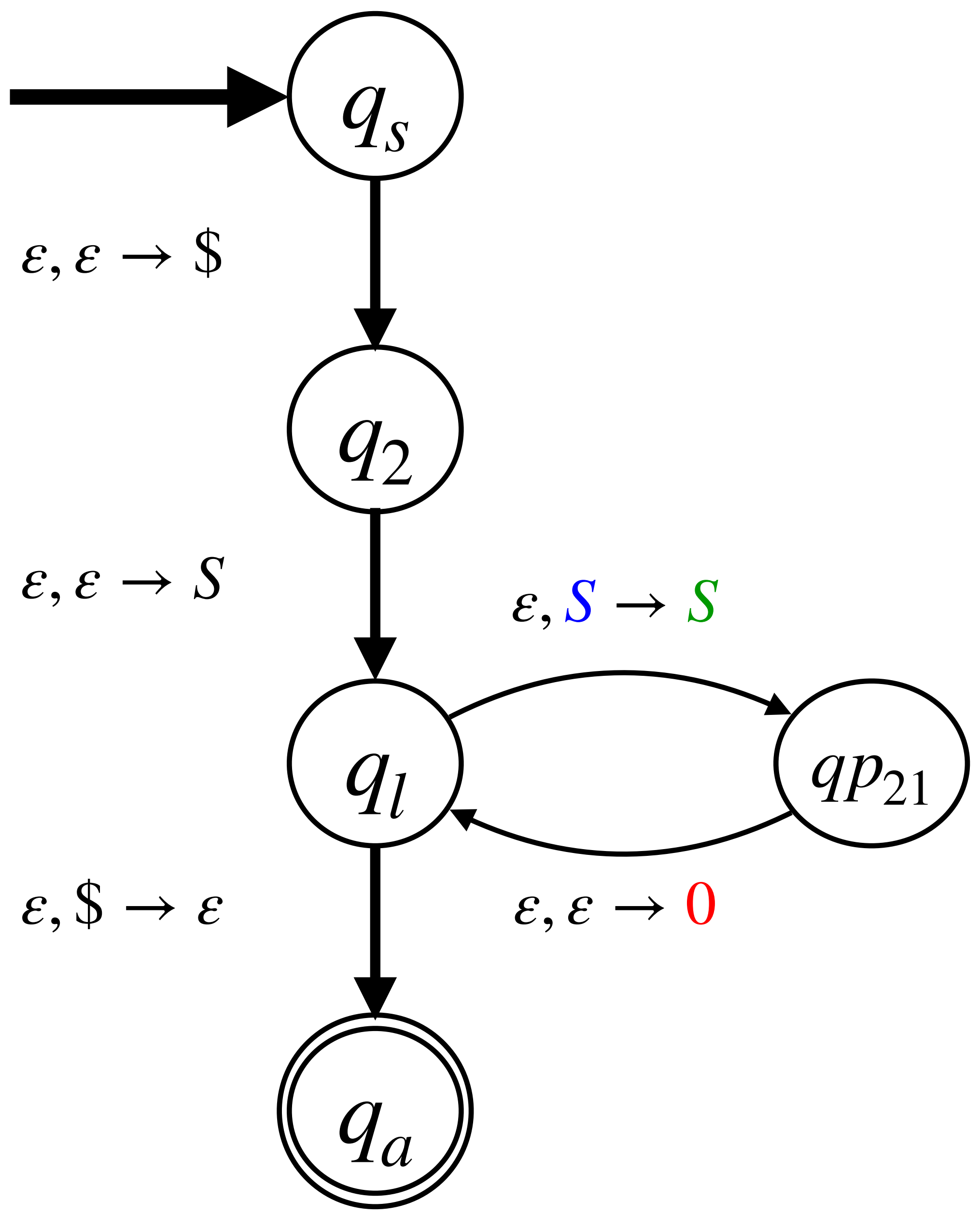

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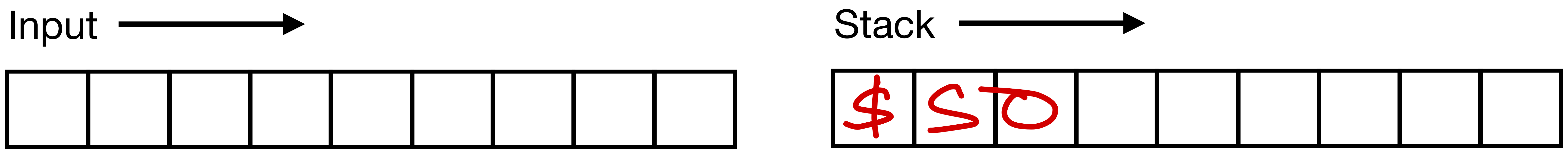
Consider the rule: $S \rightarrow 0S$

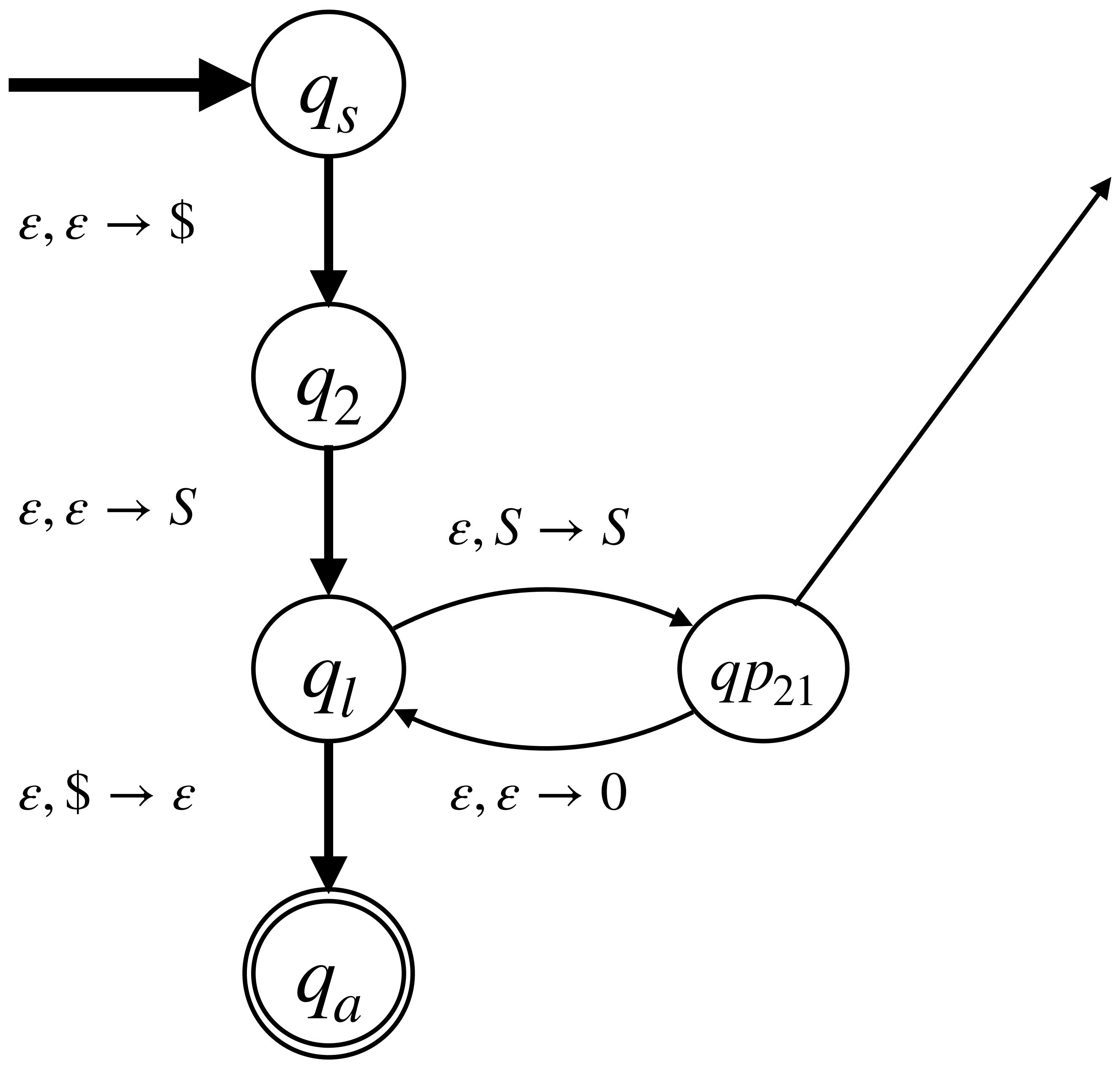
- So we got to pop S the non-terminal and …
- Add a non-terminal S to the stack.
- And add a terminal 0 to the stack.



CFGs and PDAs

Convert a CFG to a PDA

$S \to 0S \mid 1 \mid \epsilon$



Is this state necessary?

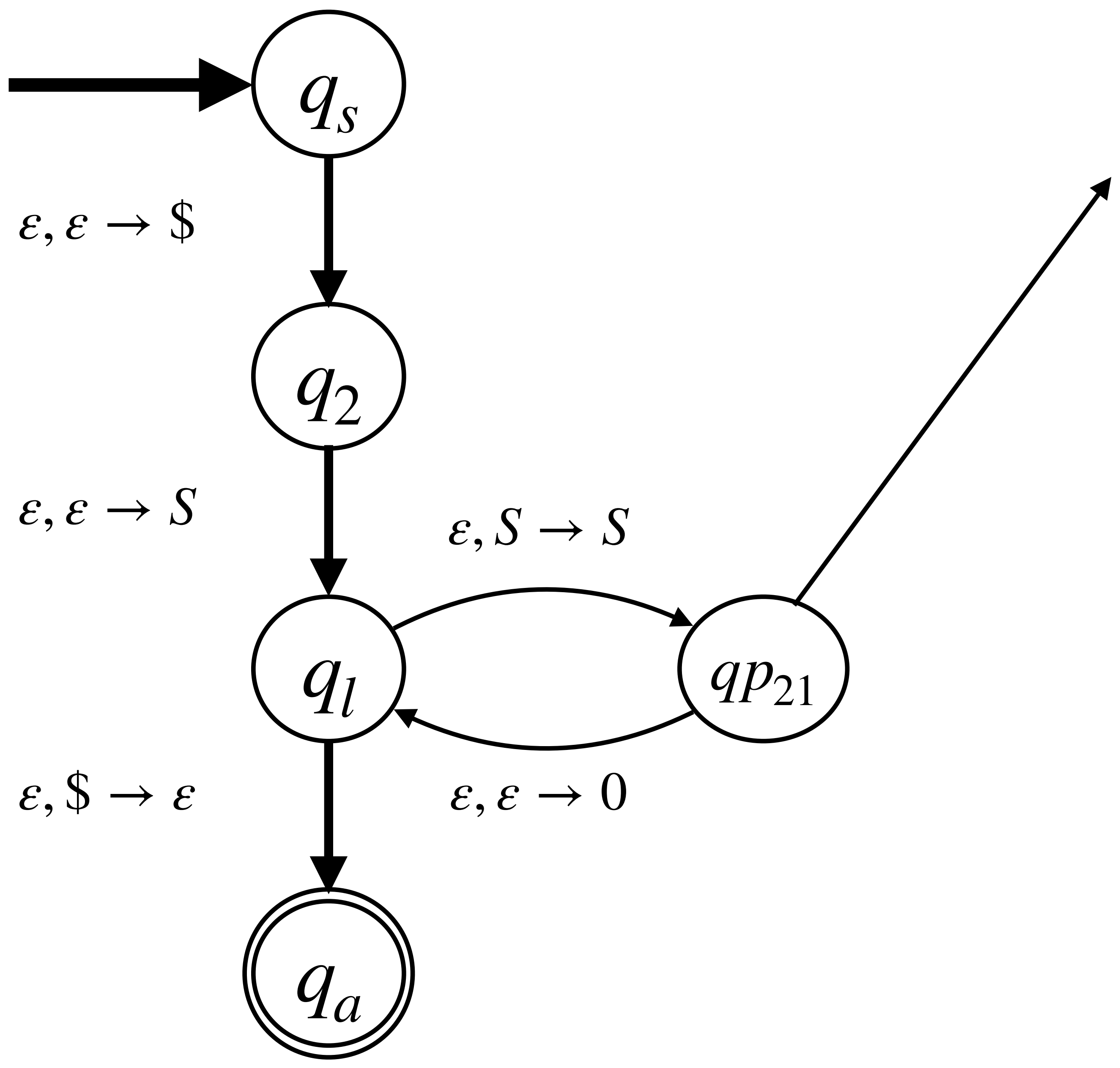
Input

Stack

CFGs and PDAs

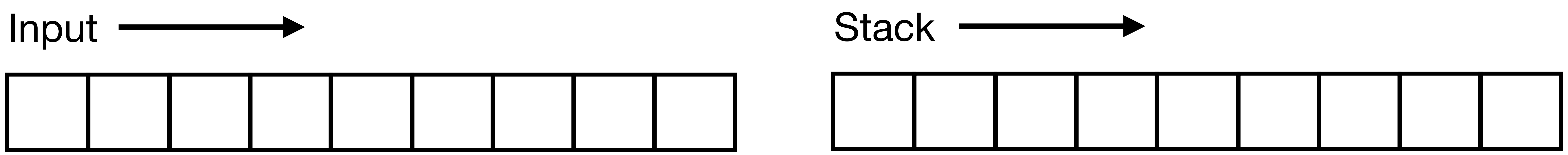
Convert a CFG to a PDA

$S \rightarrow 0S \mid 1 \mid \epsilon$



Is this state necessary?

- Recall generalized NFAs?



CFGs and PDAs

Convert a CFG to a PDA

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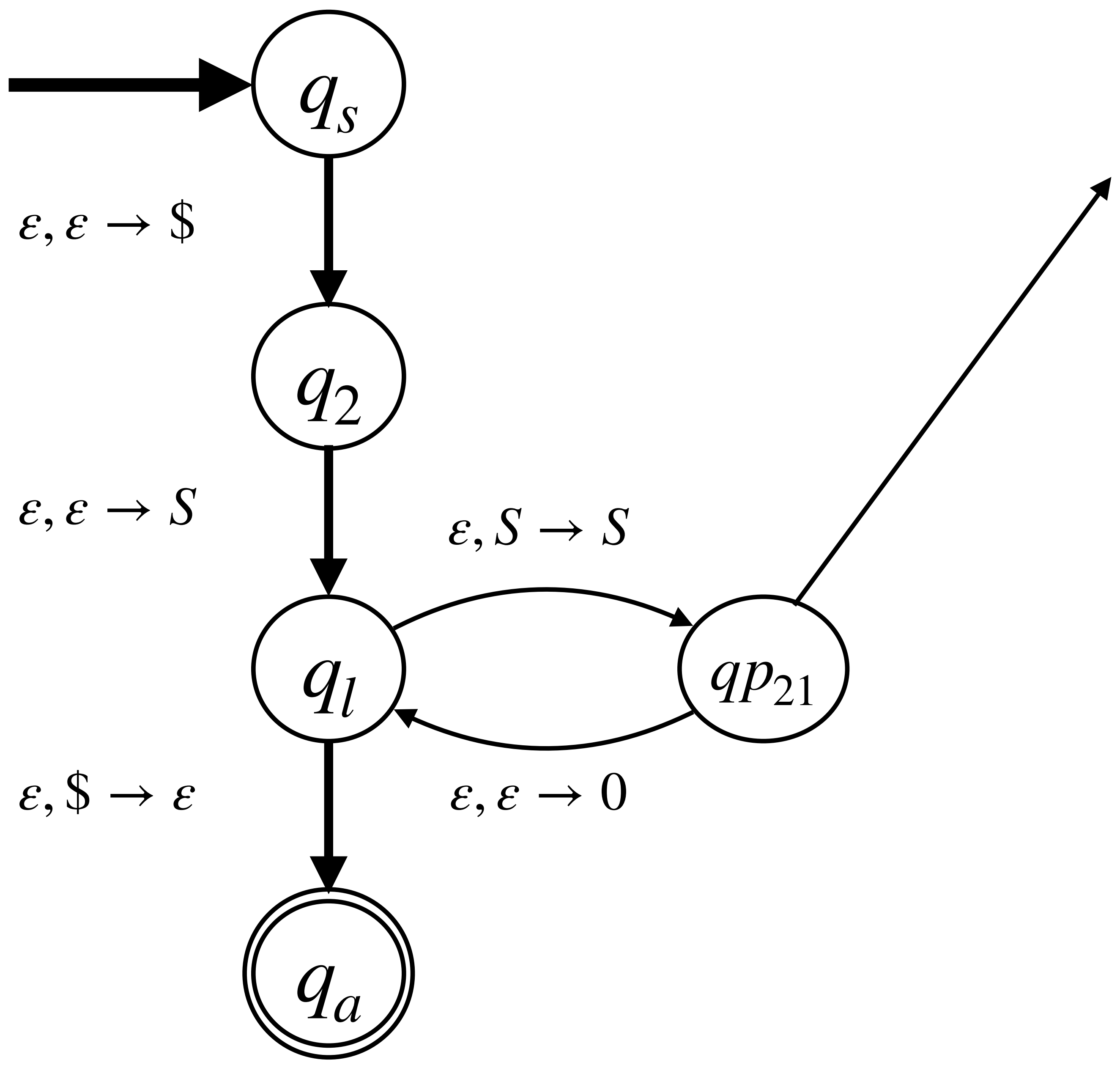
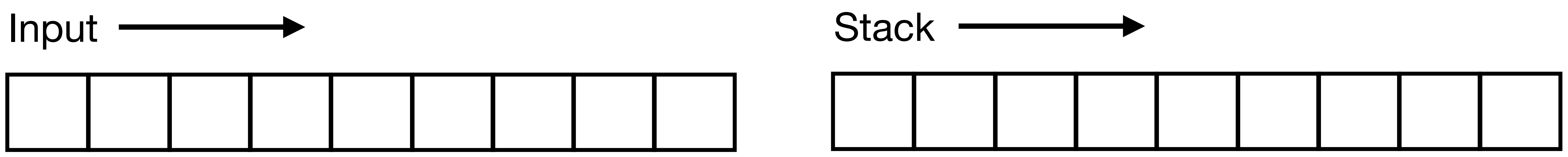


Is this state necessary?

- Recall generalized NFAs?
- Can follow same route to allow entire strings to be pushed onto stack.



CFGs and PDAs

Convert a CFG to a PDA

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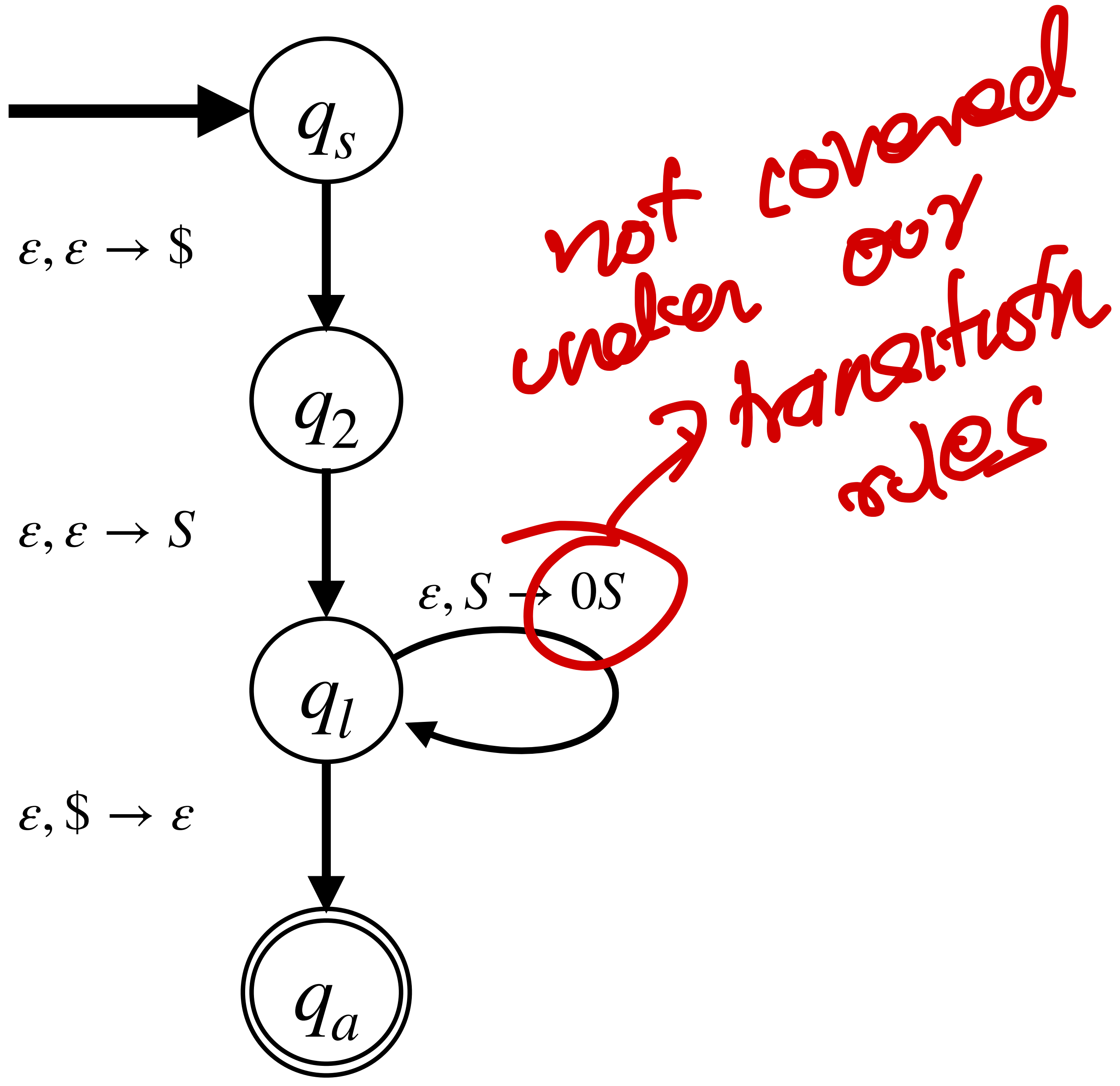

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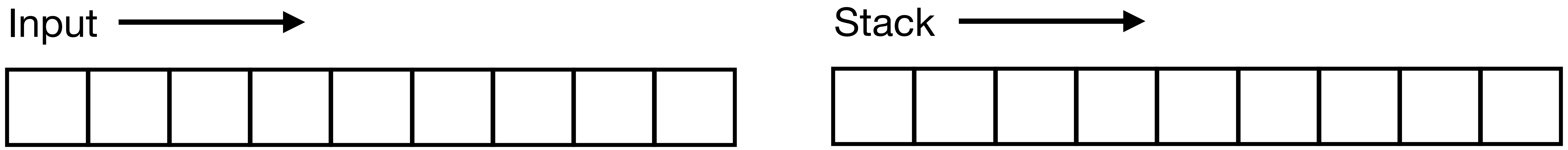

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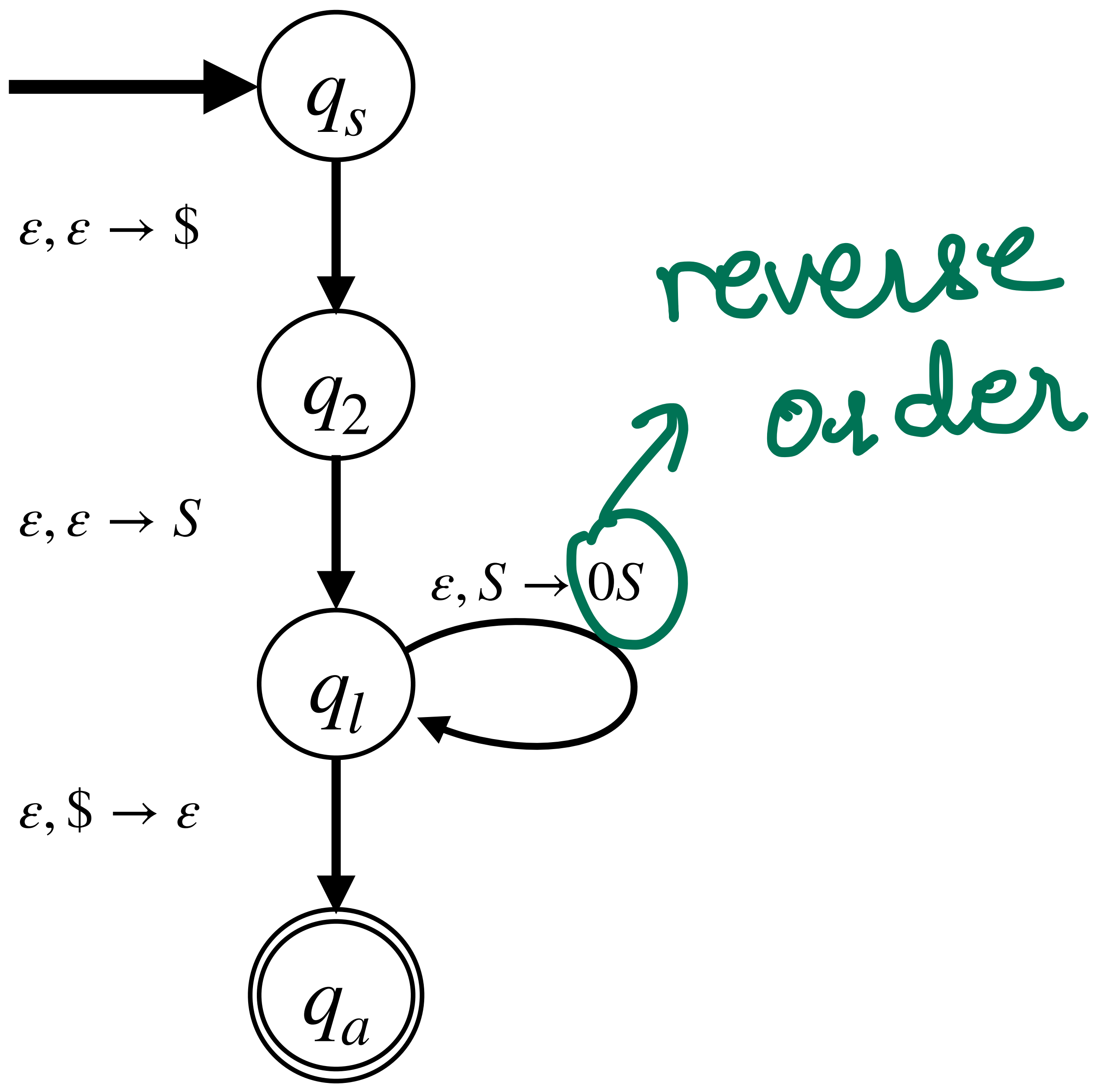
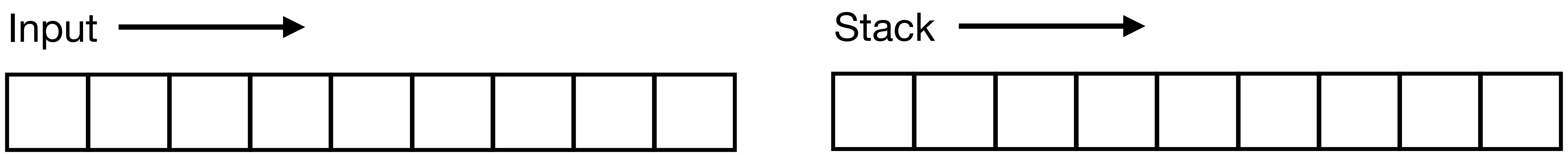


Is this state necessary?

- Recall generalized NFAs?
- Can follow same route to allow entire strings to be pushed onto stack.
- But we are going to stick with PDAs.

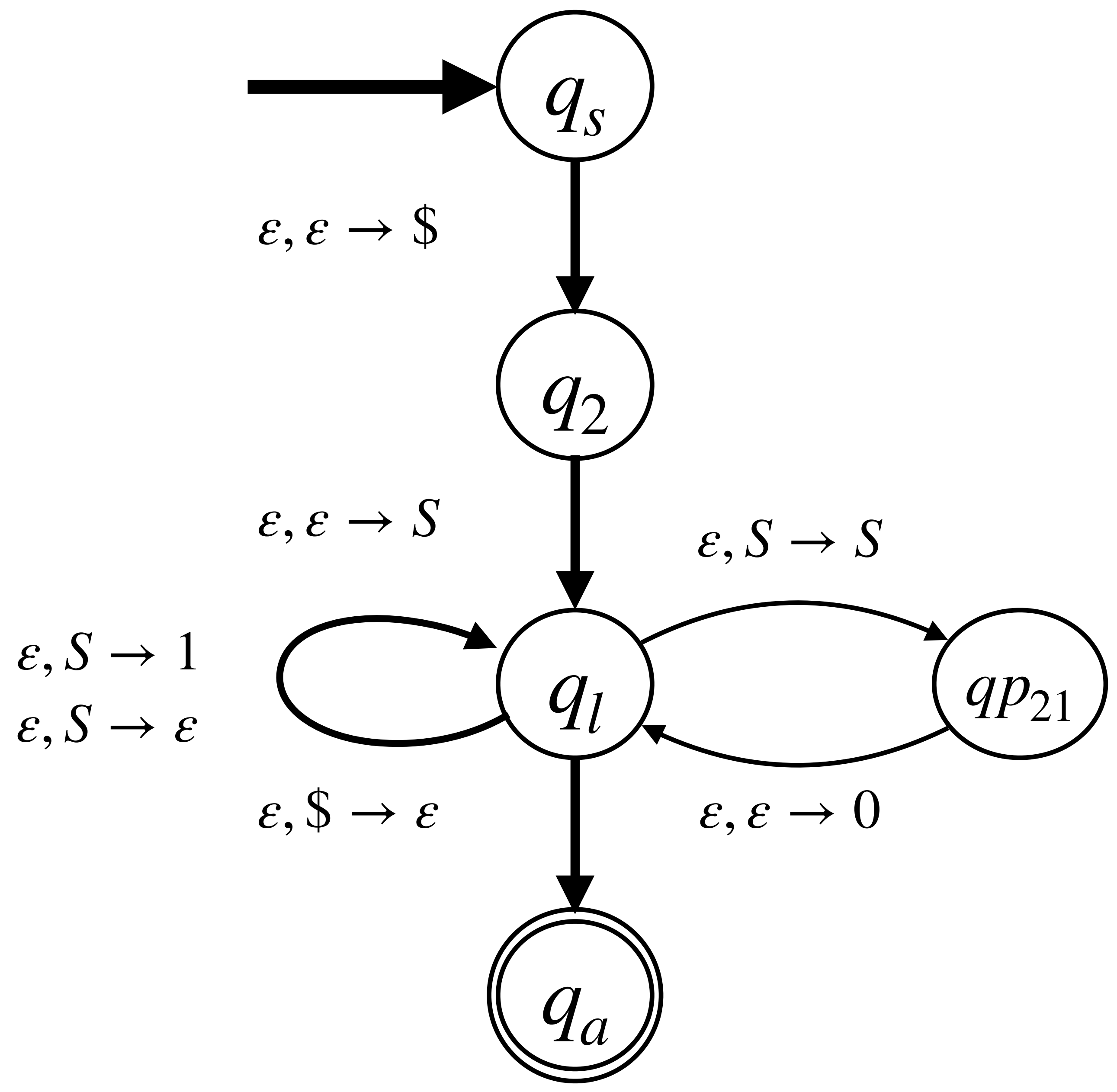


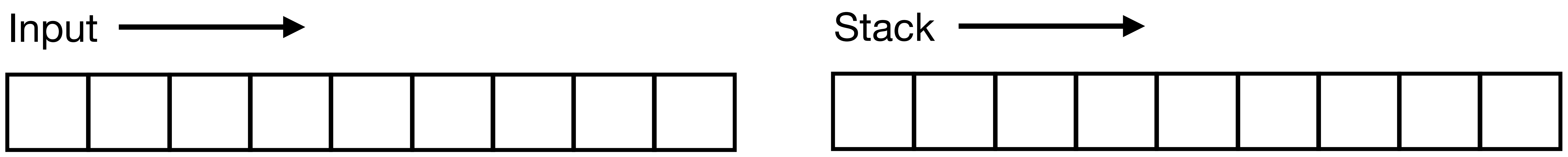
CFGs and PDAs

Convert a CFG to a PDA

$$S \to 0S \;|\; 1 \;|\; \varepsilon$$

- If we see a ~~non~~-terminal symbol on the stack, then we can cross that symbol from the input.





CFGs and PDAs

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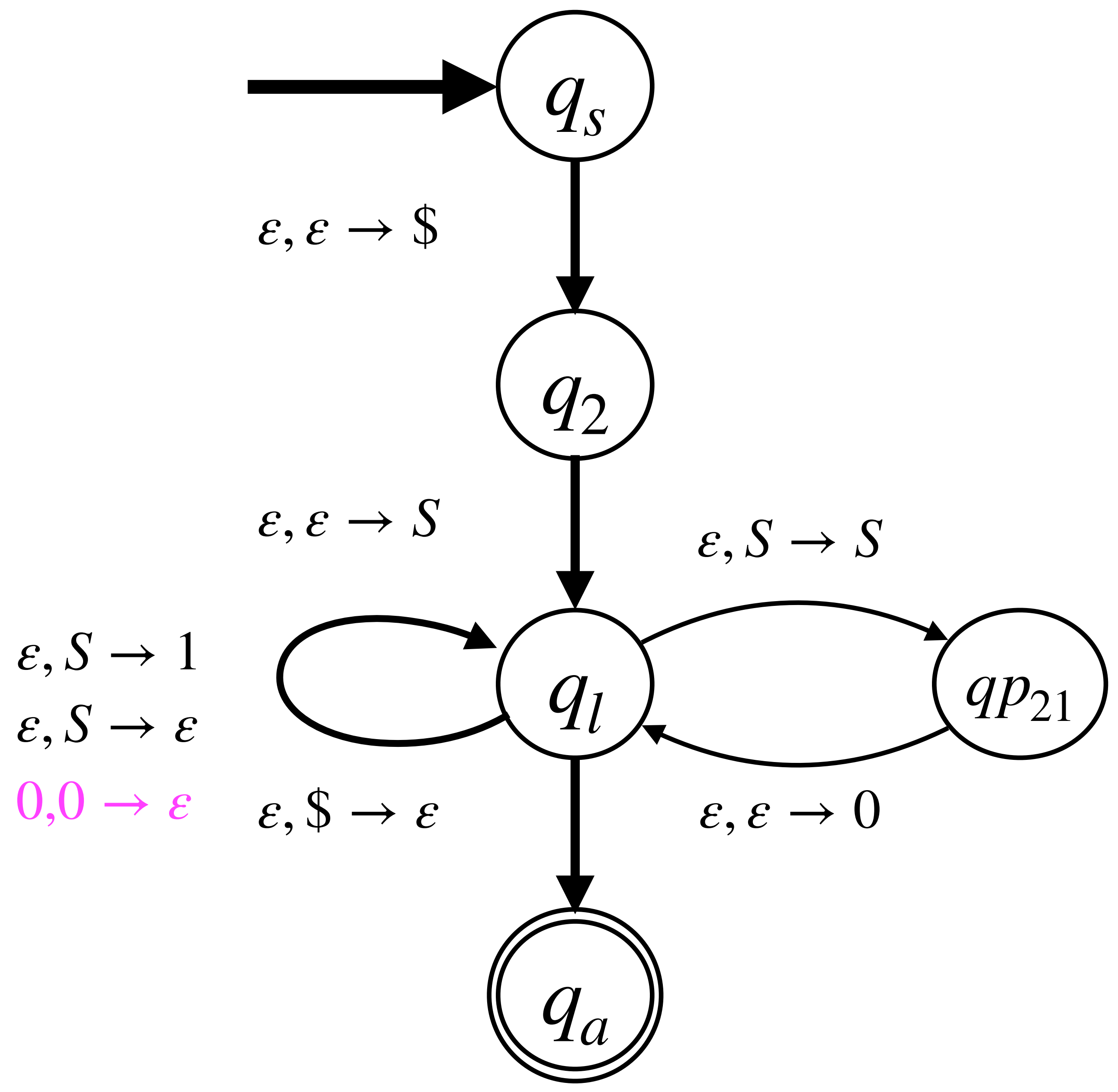
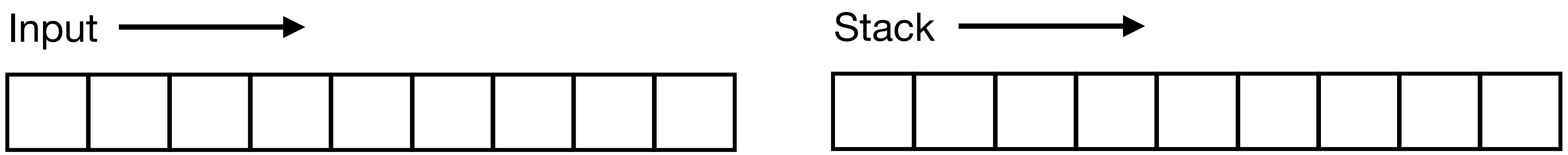


$$S \to 0S \mid 1 \mid \varepsilon$$

typo

- If we see a ~~non~~-terminal symbol on the stack, then we can cross that symbol from the input.



CFGs and PDAs

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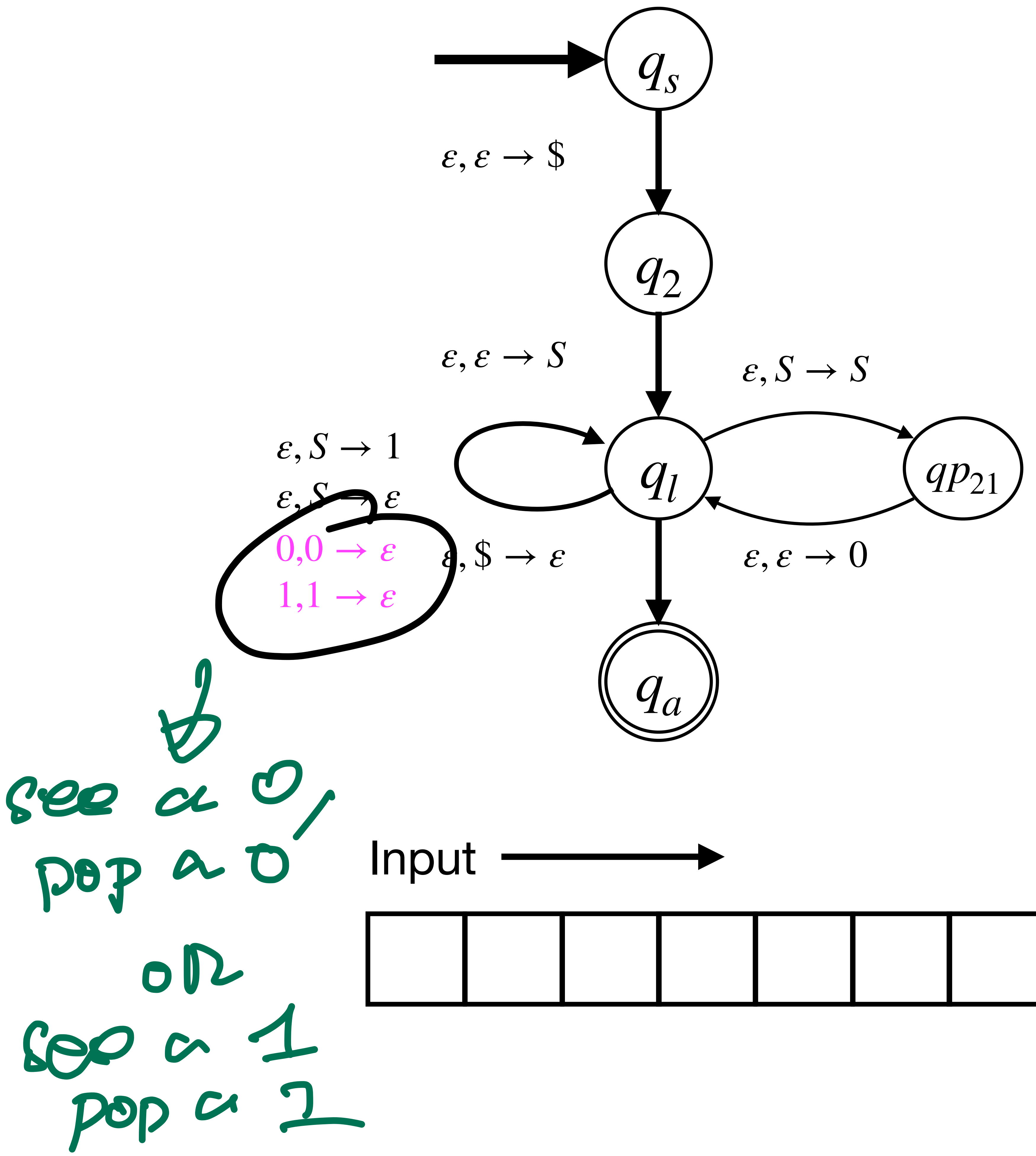

$$S \to 0S \mid 1 \mid \varepsilon$$

- If we see a ~~non~~-terminal symbol on the stack, then we can cross that symbol from the input.

Input →

Stack →

Example from Kani Archive

Start here next lecture.

Convert a CFG to a PDA

Another example