Context-sensitive and decidable languages

Sides based on material by Kani, Erickson, Chekuri, et. al.

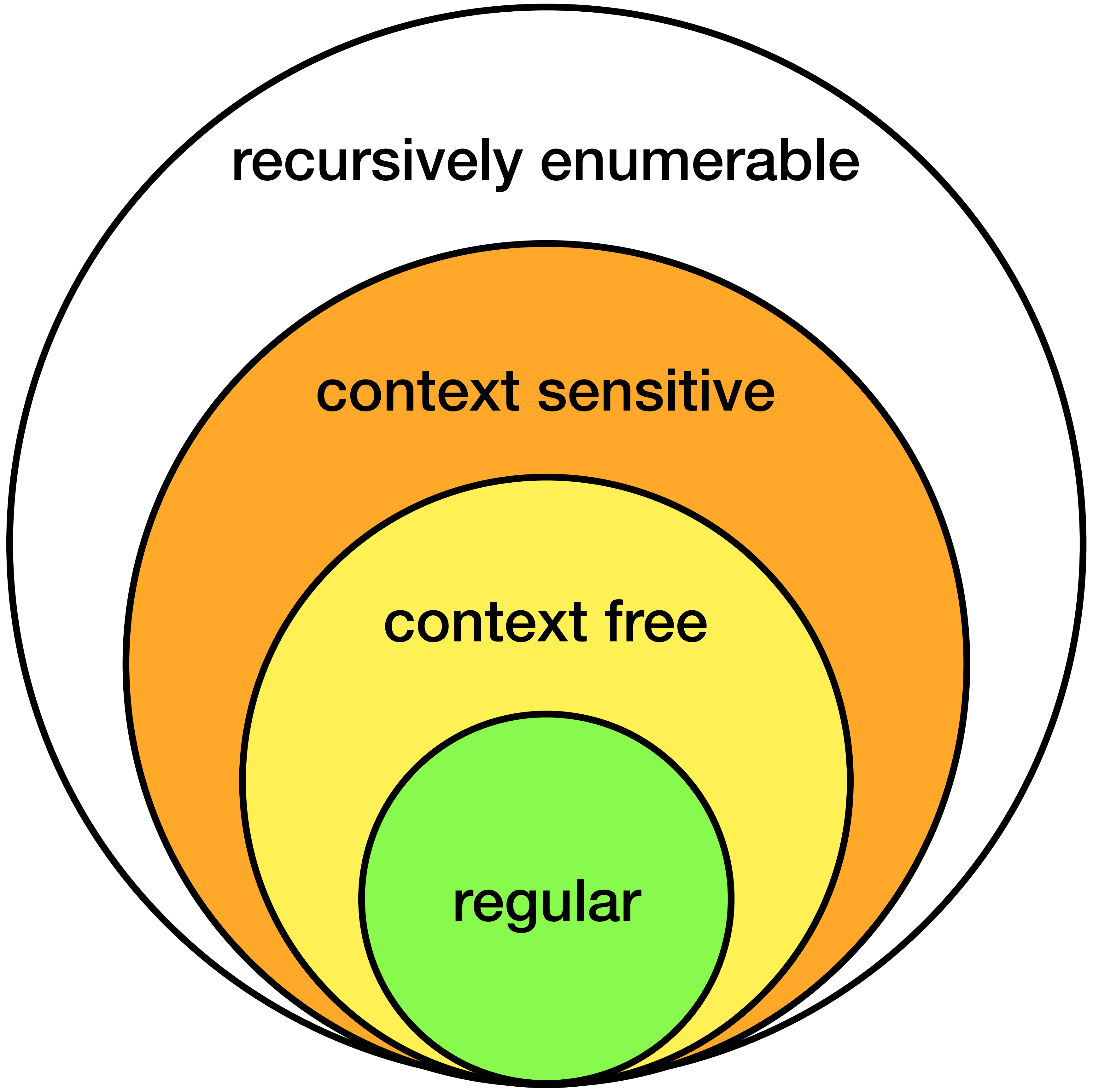
All mistakes are my own! - Ivan Abraham (Fall 2024)

Image by ChatGPT (probably collaborated with DALL-E)

Context Sensitive Language

Definition

A language L is said to be context-sensitive if there exists a context-sensitive grammar G, such that $L = L(G)$.



Context Sensitive Grammar (CSG)

Definition

A is a quadruple $G = (V, T, P, S)$

- V is a finite set of *non-terminal symbols.*
- T is a finite set of *terminal symbols* (alphabet).
- P is a finite set of *productions*, each of the form $A \to \alpha$ where $A \in V$ and α is a string in $(V \cup T)^*$.
- $S \in V$ is a *start symbol.*

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Context Sensitive Grammar (CSG)

Example formally for completeness

$$L = \{a^n b^n c^n \mid n \geq 1\}$$

$$V = \{S, A, B\}$$

$$T = \{a, b, c\}$$

$$P = \left\{ \begin{aligned} S \to abc \mid aAbc, \\ Ab \to bA, \\ Ac \to Bbcc, \\ bB \to Bb, \\ aB \to aa \mid aaA \end{aligned} \right\}$$

$$G = \left(\{S, A, B\}, \{a, b, c\}, \left\{ \begin{aligned} S \to abc \mid aAbc, \\ Ab \to bA, \\ Ac \to Bbcc, \\ bB \to Bb, \\ aB \to aa \mid aaA \end{aligned} \right\}, S \right)$$

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Context Sensitive Language

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$V = \{S, A, B\}$

$T = \{a, b, c\}$

$$P = \left\{ \begin{array}{r} S \rightarrow abc \mid aAbc, \\ Ab \rightarrow bA, \\ Ac \rightarrow Bbcc, \\ bB \rightarrow Bb, \\ aB \rightarrow aa \mid aaA \end{array} \right\}$$

$$S \rightsquigarrow aAbc \rightsquigarrow abAc \rightsquigarrow abBbcc \rightsquigarrow aBbbcc$$

$$\rightsquigarrow aaAbbcc \rightsquigarrow aabAbcc \rightsquigarrow aabbAcc$$

$$\rightsquigarrow aabbBbccc \rightsquigarrow aabBbbccc$$

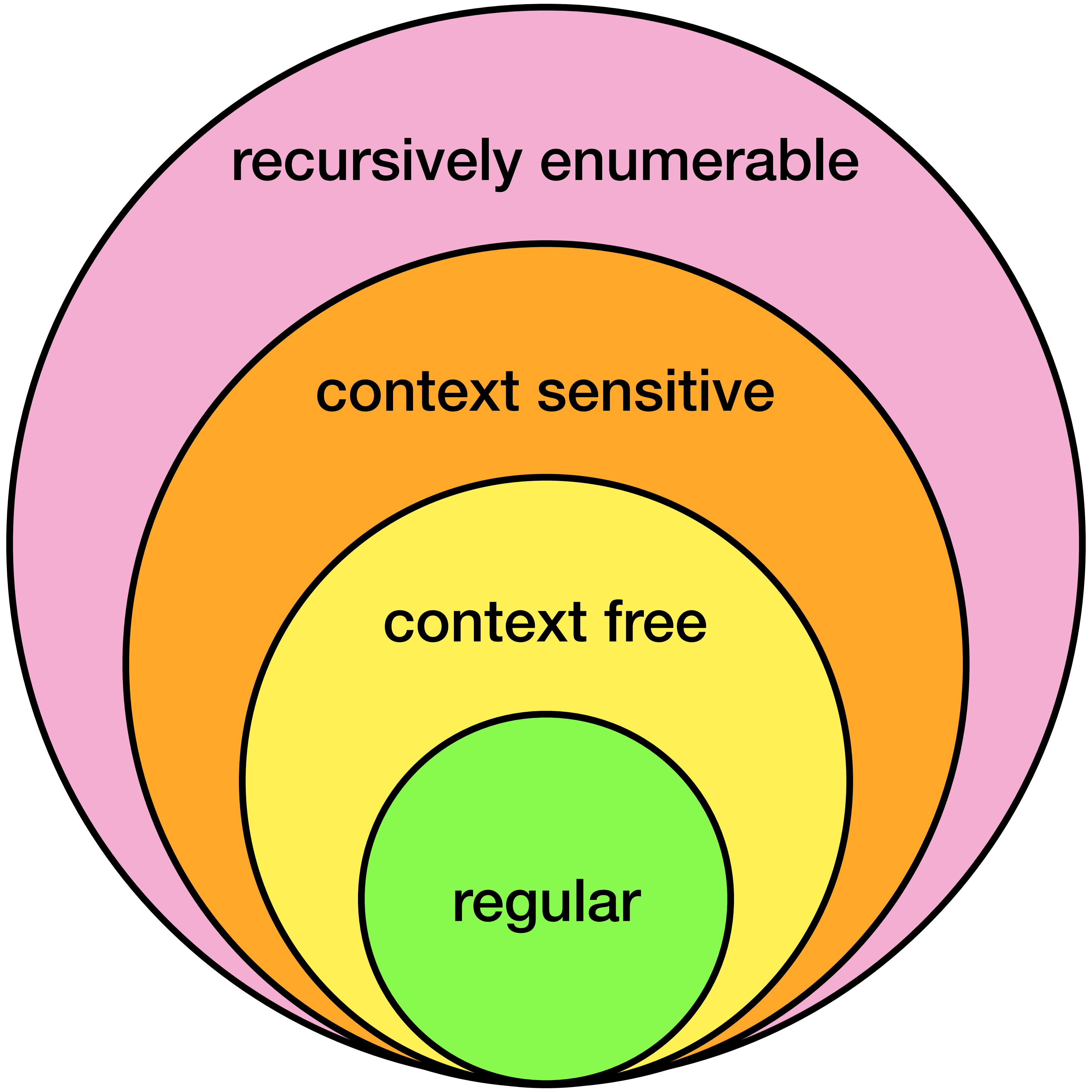
$$\rightsquigarrow aaBbbbccc \rightsquigarrow aaabbbccc$$

$$L = \{a^n b^n c^n \mid n \geq 1\}$$

Context Sensitive Language

Another solution

Recursively enumerable (Turing Recognizable) language



“Most general” computer

- Is there a kind of computer that can accept any language or compute any function?

- Recall counting argument. Set of all languages: $\{L \mid L \subseteq \{0,1\}^*\}$ is

(a) countably infinite   (b) uncountably infinite

- Set of all programs: $\{P \mid P \text{ is a finite length computer program}\}$ is

(a) countably infinite   (b) uncountably infinite

- **Conclusion: There are languages for which there are no programs.**

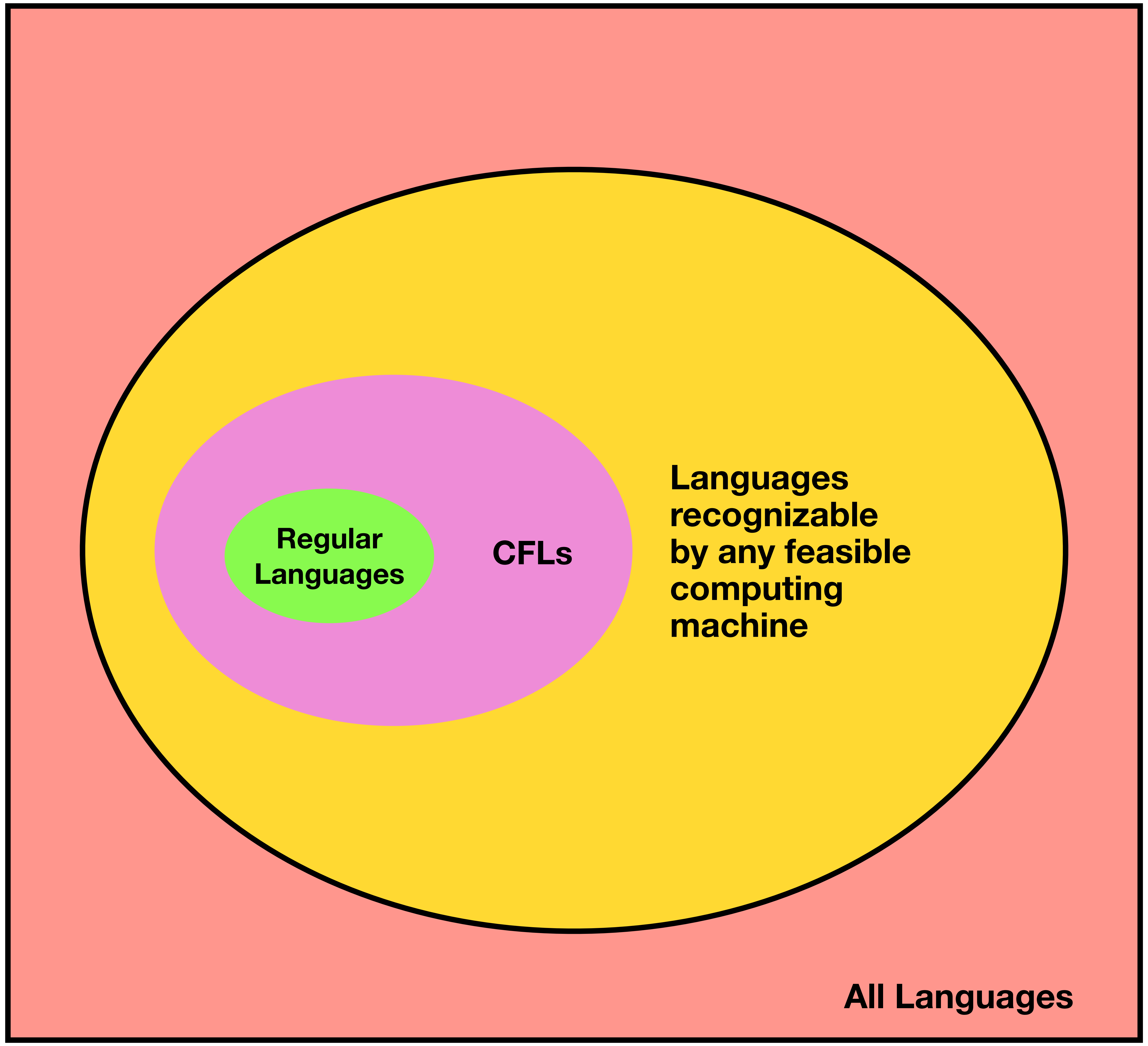
“Most general” computer

- We may need unbounded memory to recognize context-free languages.
  - E.g. $\{a^n b^n \mid n \in \mathbb{N}\}$ requires unbounded counting.

How do we model a computing device that has unbounded memory?

OR

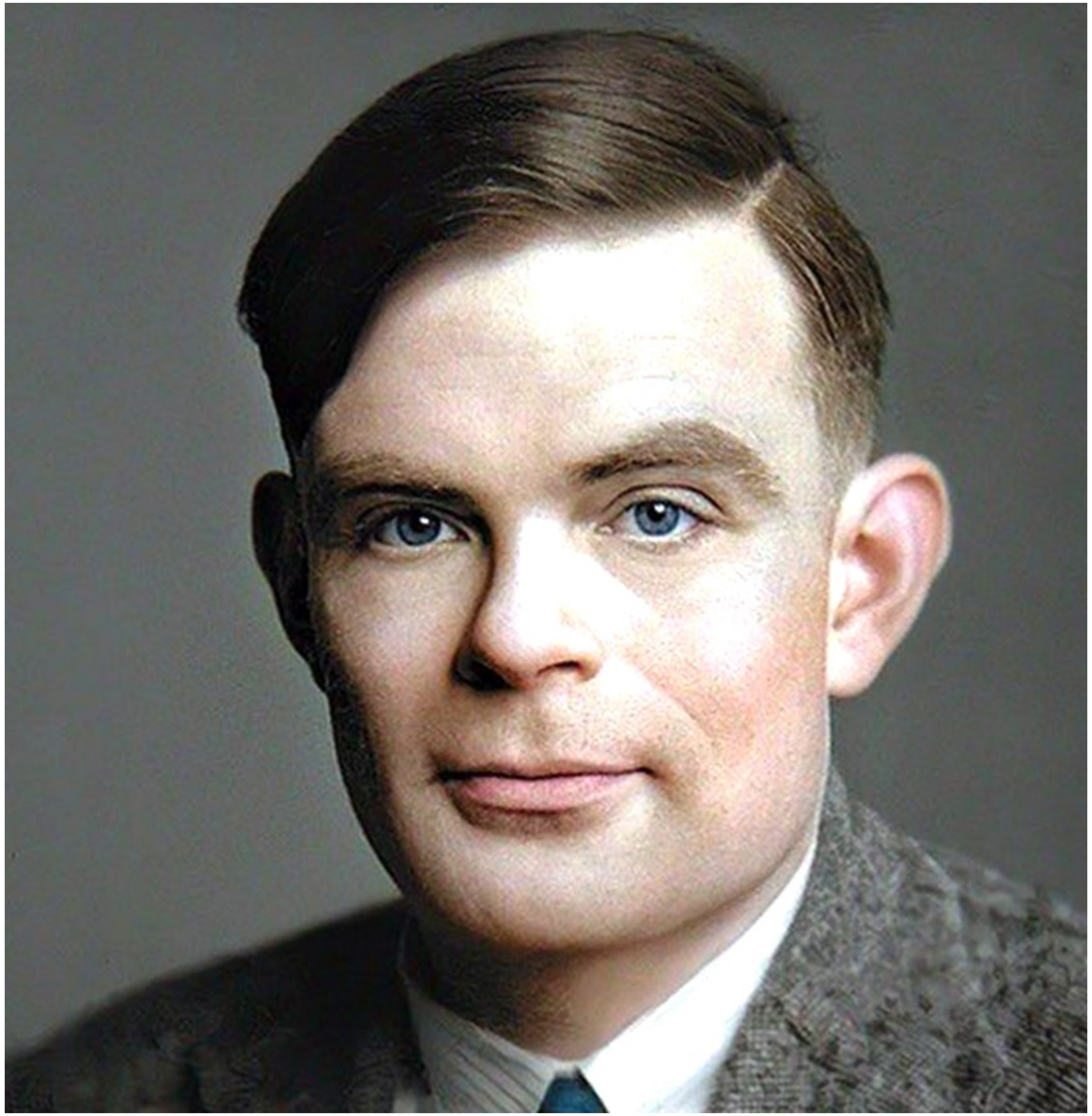
How do we model a computing device that can recognize as many languages as possible?



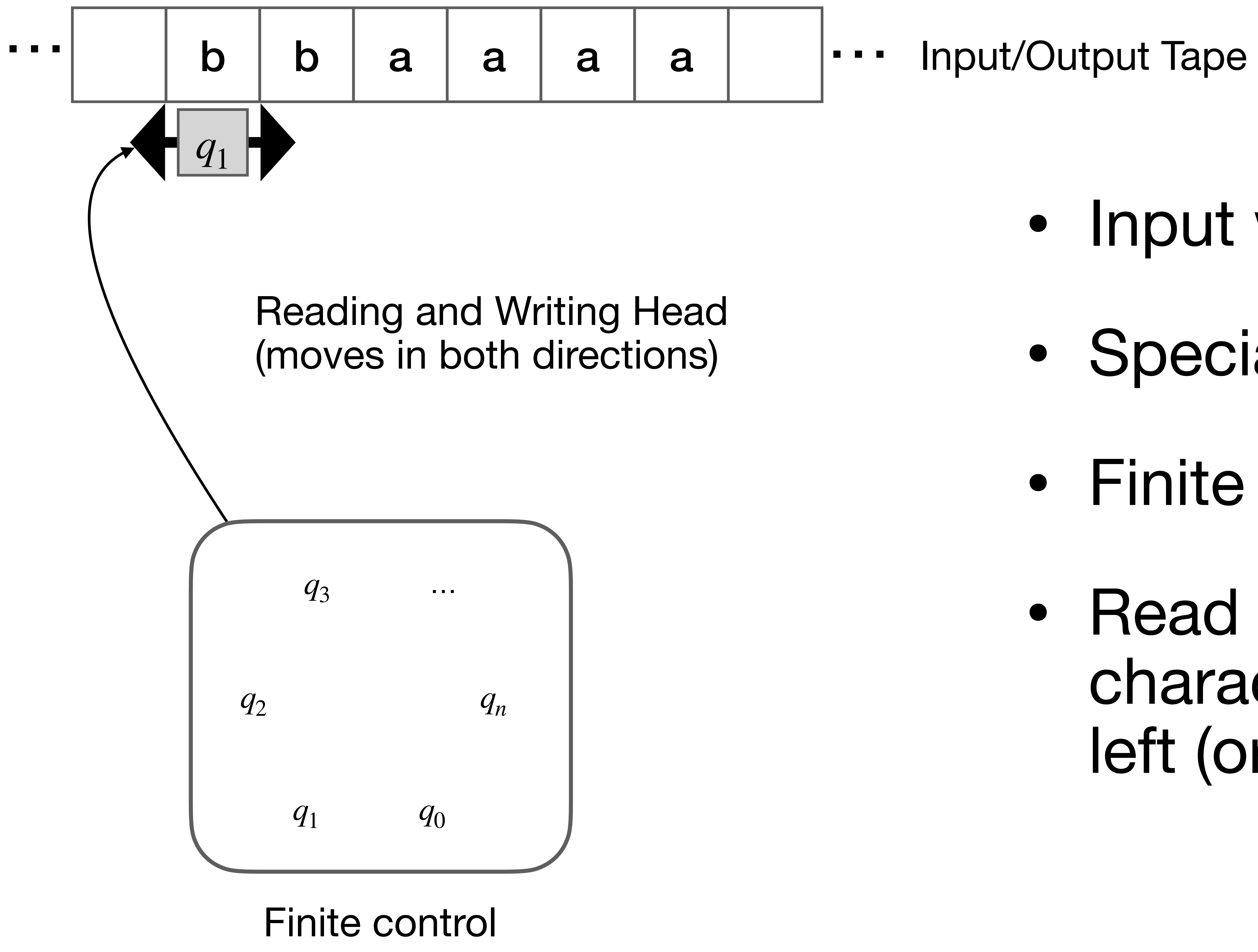
Turing Machine

A brief history

- In March 1936, Alan Turing (aged 23!) published a paper detailing the **a-machine** (for **automatic machine**), an automaton for computing on real numbers.
- They're now more popularly referred to as **Turing machines** in his honor.
- Watch: The Imitation Game!



Turing Machine



- Input written on (infinite) one sided tape.
- Special blank characters.
- Finite state control (similar to DFA).
- Read character under head, write character out, move the head right or left (or stay).

Turing Machine

High level goals

- Church-Turing thesis: TMs are the most general computing devices. So far, no counter-example.
- Every TM can be represented as a string.
- The existence of a Universal Turing Machine, which is the model/inspiration for stored program computing. UTM can simulate any TM!
- Implications for what can be computed and what cannot be computed

Example

Computers exist because we are lazy… so Stanford's CS 103 to the rescue.

Turing Machine

Formal definition

A Turing machine is a 7-tuple $\left(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}\right)$ where

- Q is a finite set of states
- Σ is a finite set called the input alphabet
- Γ is a finite set called the tape alphabet
- δ: $Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$: Transition function.
- $q_0 \in Q$ is called the initial state.
- $q_{acc} \in Q$, $q_{rej} \in Q$ are the accepting state and rejecting state, respectively.
- $\sqcup$ a special symbol for blank on the tape

Turing Machine

Transition function

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

current state | symbol scanned | new state | symbol to write | Direction to move on tape

From state q, on reading a:

- go to state p
- write b
- move head *Left*
- missing transitions lead to BSOD

$$\delta(q, a) = (p, b, L)$$

More example(s)

Same link as last time again

Languages defined by a Turing machine

Recursive vs. Recursively Enumerable

- TMs have a new behavior - it may never halt.
  - Need to distinguish cases, related to “deciders” (partial and total).
- *Recursively enumerable* (aka RE) languages
  - $L = \{L(M) \mid M \text{ some Turing machine}\}$
- *Recursive/decidable* languages
  - $L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}$

Languages defined by a Turing machine

Recursive vs. Recursively Enumerable

- A **total decider** is a TM which will always halt in an accept or reject state.
    - *Recursive languages* are called decidable language precisely because they have total deciders.
- A **partial decider** is a TM, which if given string in its language, will reach an accept state.
    - If given a string that is not in its language, it could loop forever.
    - *Recursively enumerable* languages are ones for which a partial decider exists.

Languages defined by a Turing machine

Recursive vs. Recursively Enumerable

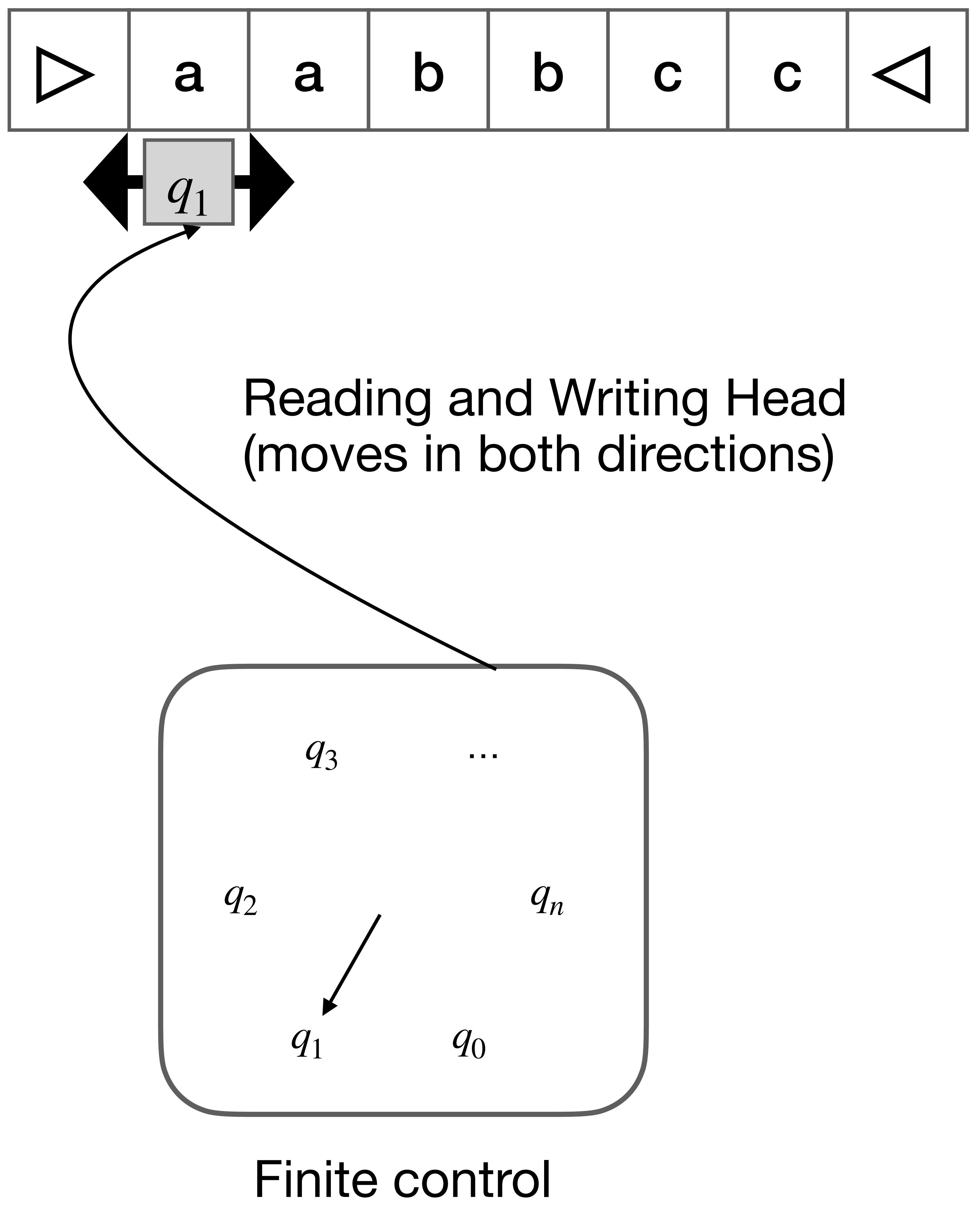
- Fundamental questions:
  - What languages are RE?
  - Which are recursive?
  - What is the difference?
  - What makes a language decidable?

More on these closer to end of semester.

- A ***semi-decidable*** problem (equivalent of recursively enumerable) could be:
  - **Decidable** - equivalent of recursive (**TM** always accepts or rejects).
  - **Undecidable** - Problem is not recursive
- There are also undecidable problems that are not recursively enumerable!

Linear Bounded Automata

Relation to TMs



- We skipped LBAs
  - They can be thought of as *restricted* Turing Machines.
  - Tape used grows *linearly* with size of input
- (Nondeterministic) LBA can recognize all context-sensitive languages.

Wrap up …

… the four-week tour of Models of Computation.

Grammar	Languages	Production Rules	Automation	Examples
Type-0	Turing machine	$\gamma \rightarrow \alpha$ (no constraints)	Turing Machines	L = { w | w is a TM which halts }
Type-1	Context-sensitive	$\alpha A \beta \rightarrow \alpha \gamma \beta$	Linear Bounded Automata	$L = \{ a^n b^n c^n \mid n > 0 \}$
Type-2	Context-free	$A \rightarrow \alpha$	Pushdown Automata	$L = \{ a^n b^n \mid n > 0 \}$
Type-3	Regular	$A \rightarrow aB$	Non-determinstic Finite Automata	$L = \{ a^n \mid n > 0 \}$

Next class

Universal Turing Machines

- **Theorem (Turing, 1936):** There is a Turing Machine, called the Universal Turing Machine, that, when run on an input of the form $\langle M, w \rangle$ where M is a TM and w is a string, simulates M running on w and does whatever M does on w — accepts, rejects, loops.
- Question for the weekend: Recall that the language of a TM is the set of strings it accepts.

What is the language of a UTM?