Context-sensitive and decidable languages

Sides based on material by Kani, Erickson, Chekuri, et. al.

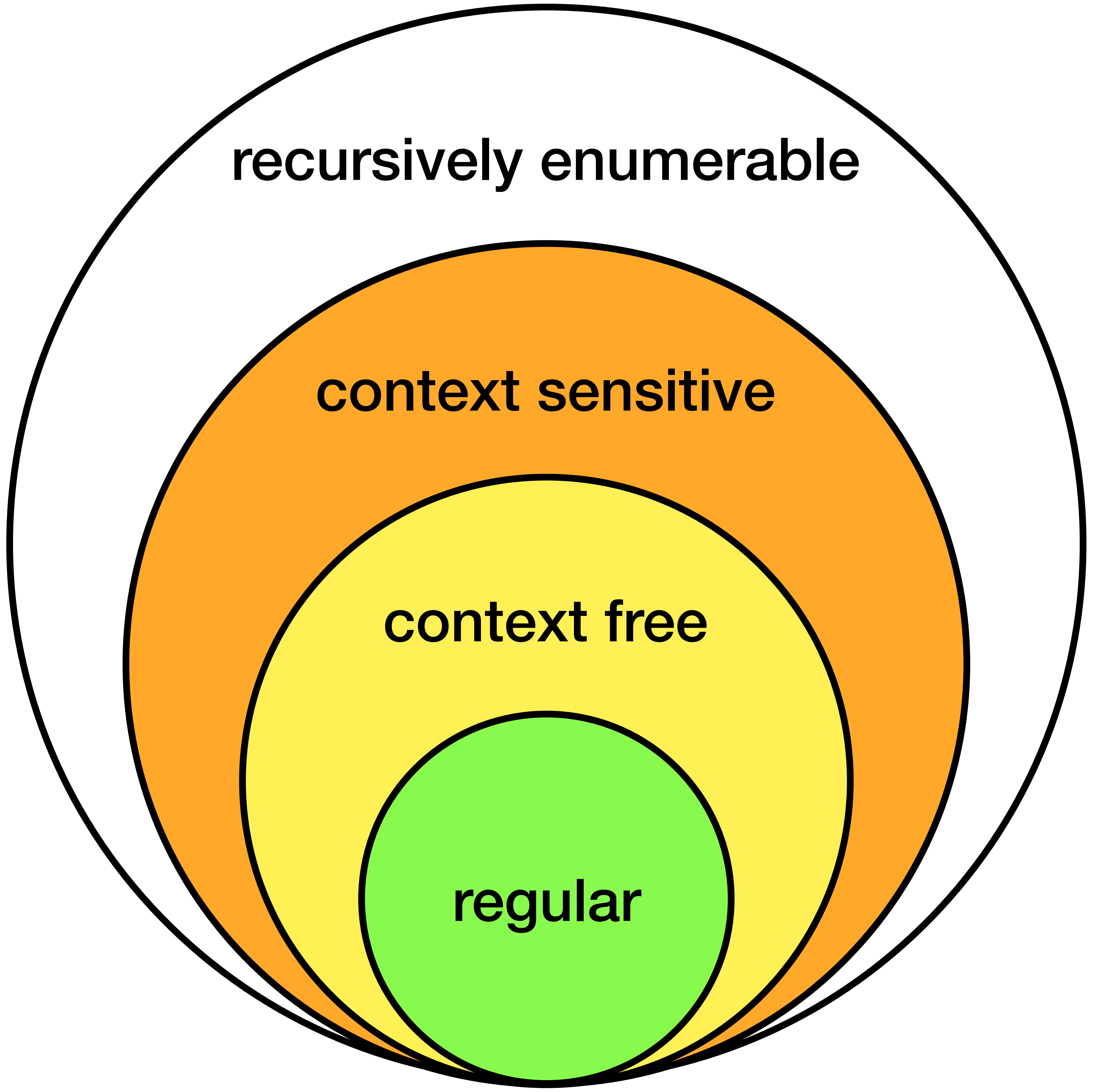
All mistakes are my own! - Ivan Abraham (Fall 2024)

Image by ChatGPT (probably collaborated with DALL-E)

Context Sensitive Language

Definition

A language L is said to be context-sensitive if there exists a context-sensitive grammar G, such that $L = L(G)$.



Context Sensitive Grammar (CSG)

Definition

A ~~CFG~~ is a quadruple $G = (V, T, P, S)$

$V \rightarrow$ finite set of non-terminals

$T \rightarrow$ " " " terminals

$P \rightarrow \quad P \subseteq V \times (V \cup T)^*$

rules have the form

$V \rightarrow \alpha \quad , \quad \alpha \in (V \cup T)^*$

$S \rightarrow$ start symbol.

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- V is a finite set of *non-terminal symbols.*
- T is a finite set of *terminal symbols* (alphabet).
- P is a finite set of *productions*, each of the form $A \to \alpha$ where $A \in V$ and α is a string in $(V \cup T)^*$.
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Context Sensitive Grammar (CSG)

Example formally for completeness

$$L = \{a^n b^n c^n \mid n \geq 1\}$$

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$$T = \{a, b, c\}$$

$S \to abc \mid aAbC$

$Ab \to bA$

$Ac \to Bbcc$

$bB \to Bb$

$aB \to aa \mid$ ____

abAc.

abBbcc

aBbbcc

aabbcc

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$$G = \left(\{S, A, B\}, \{a, b, c\}, \left\{ \begin{aligned} S &\to abc \mid aAbc, \\ Ab &\to bA, \\ Ac &\to Bbcc, \\ bB &\to Bb, \\ aB &\to aa \mid aaA \end{aligned} \right\}, S \right)$$

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$S \rightsquigarrow aAbc \rightsquigarrow abAc \rightsquigarrow abBbcc \rightsquigarrow aBbbcc$

$\rightsquigarrow aaAbbcc \rightsquigarrow aabAbcc \rightsquigarrow aabbAcc$

$\rightsquigarrow aabbBbccc \rightsquigarrow aabBbbccc$

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$$L = \{a^n b^n c^n \mid n \geq 1\}$$

Context Sensitive Language

Another solution

$S \to aTU \mid aSTU$

$aT \to ab$

$bU \to bc$

$UT \to TU$

$bT \to bb$

$bU \to bc$

$cU \to cc$

Apologies for the "colour coding", Hard to do it on the fly but leaving it unchanged so it matches the recording

$aTU \to abU \to abc$

$aSTU \to aaTUTU$

$\downarrow$

$aabUTU$

$\downarrow$

$aabTUU$

$aabbUU$

$aabbcU$

$aabbcc$

Recursively enumerable (Turing Recognizable) language

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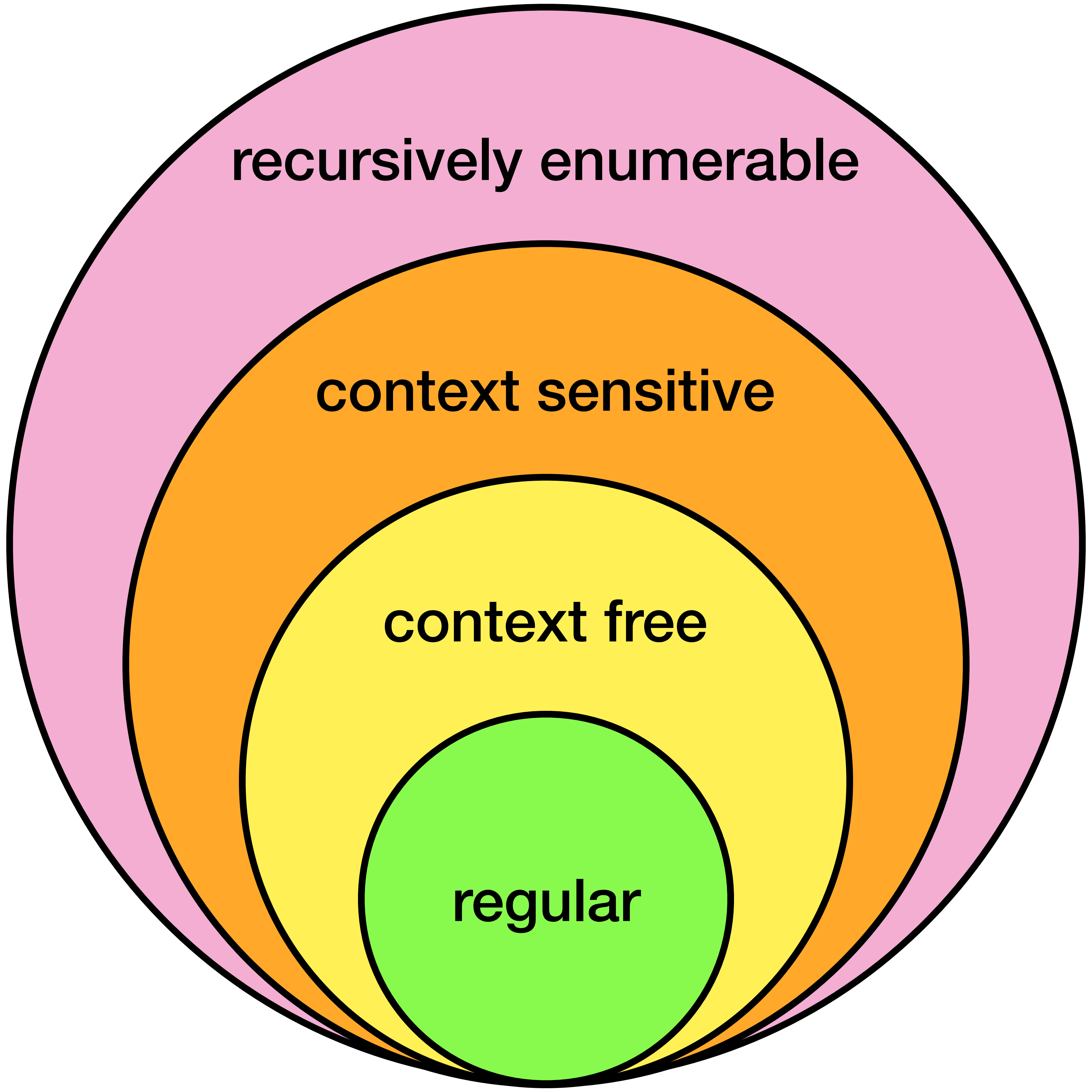

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There is a gap b/w # of computer programs and all problems.

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- **Conclusion: There are languages for which there are no programs.**

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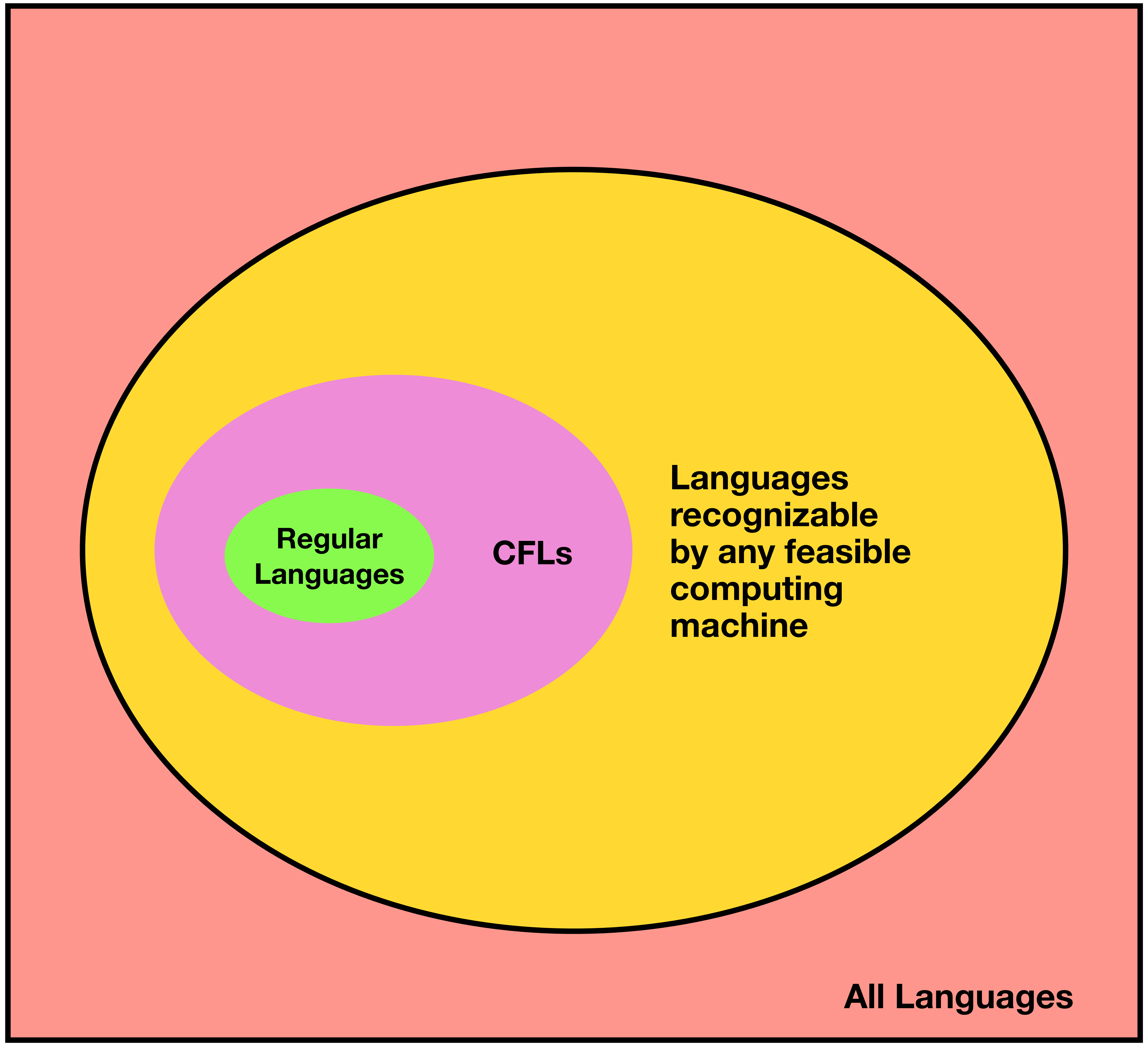

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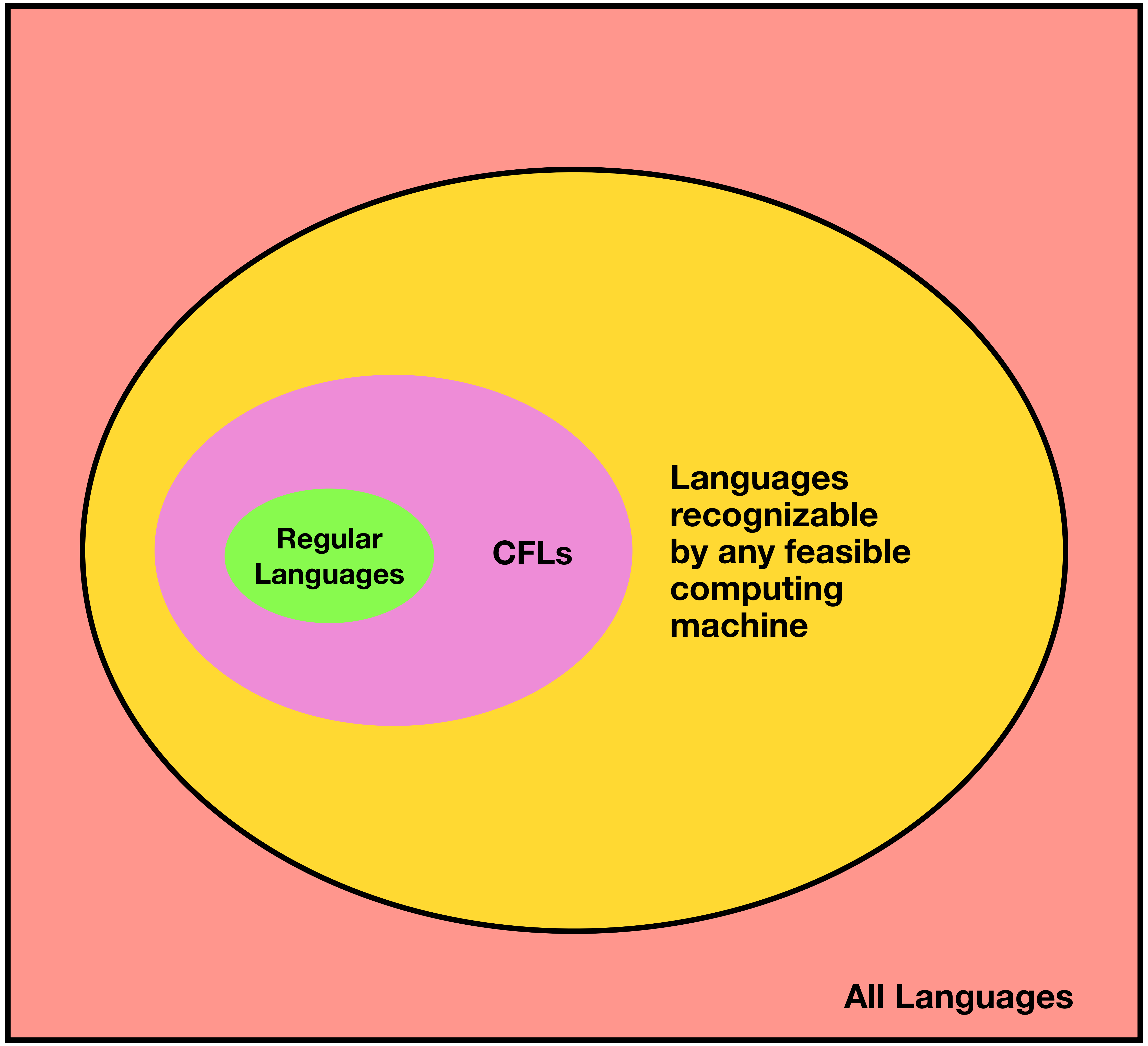

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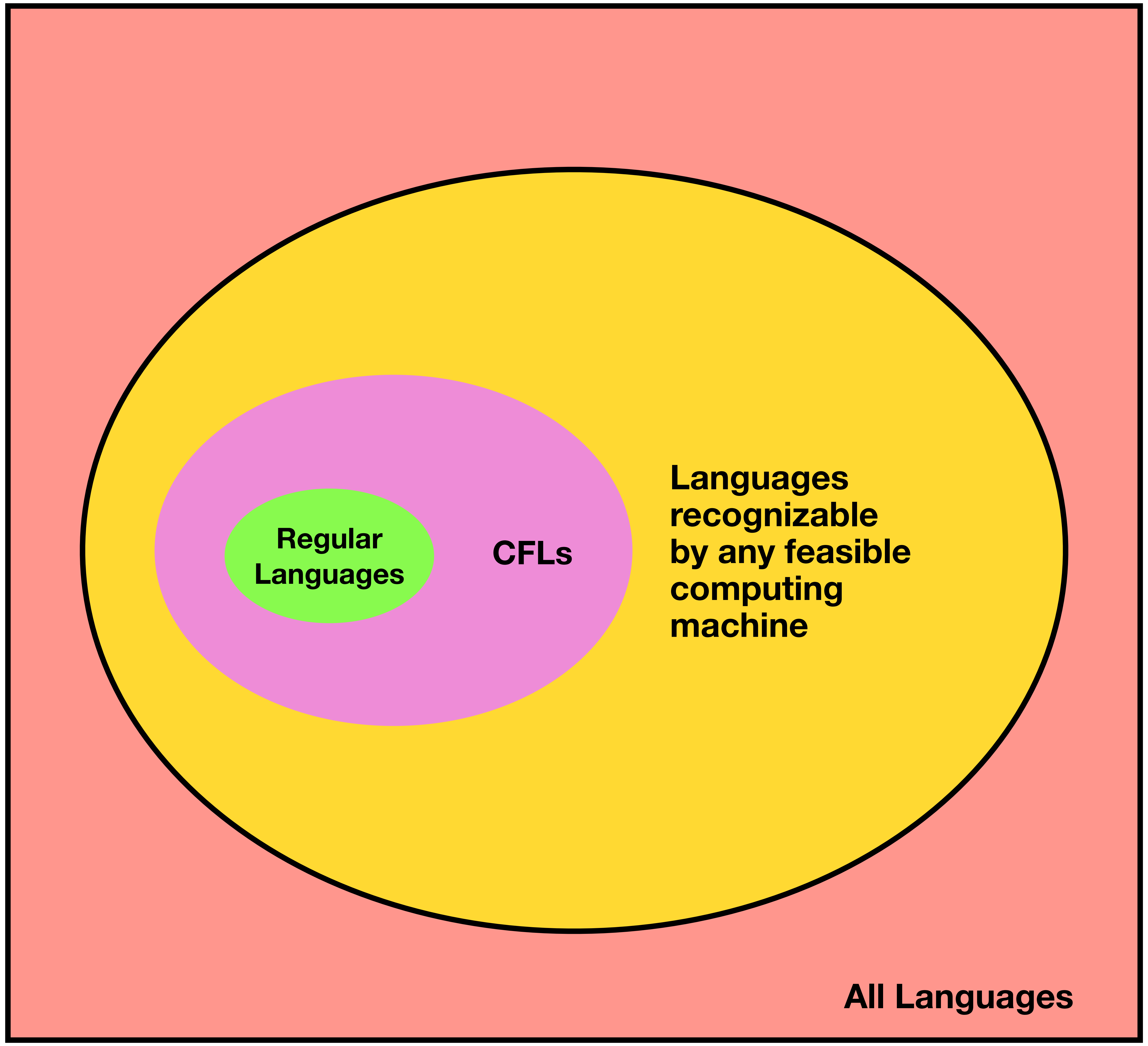

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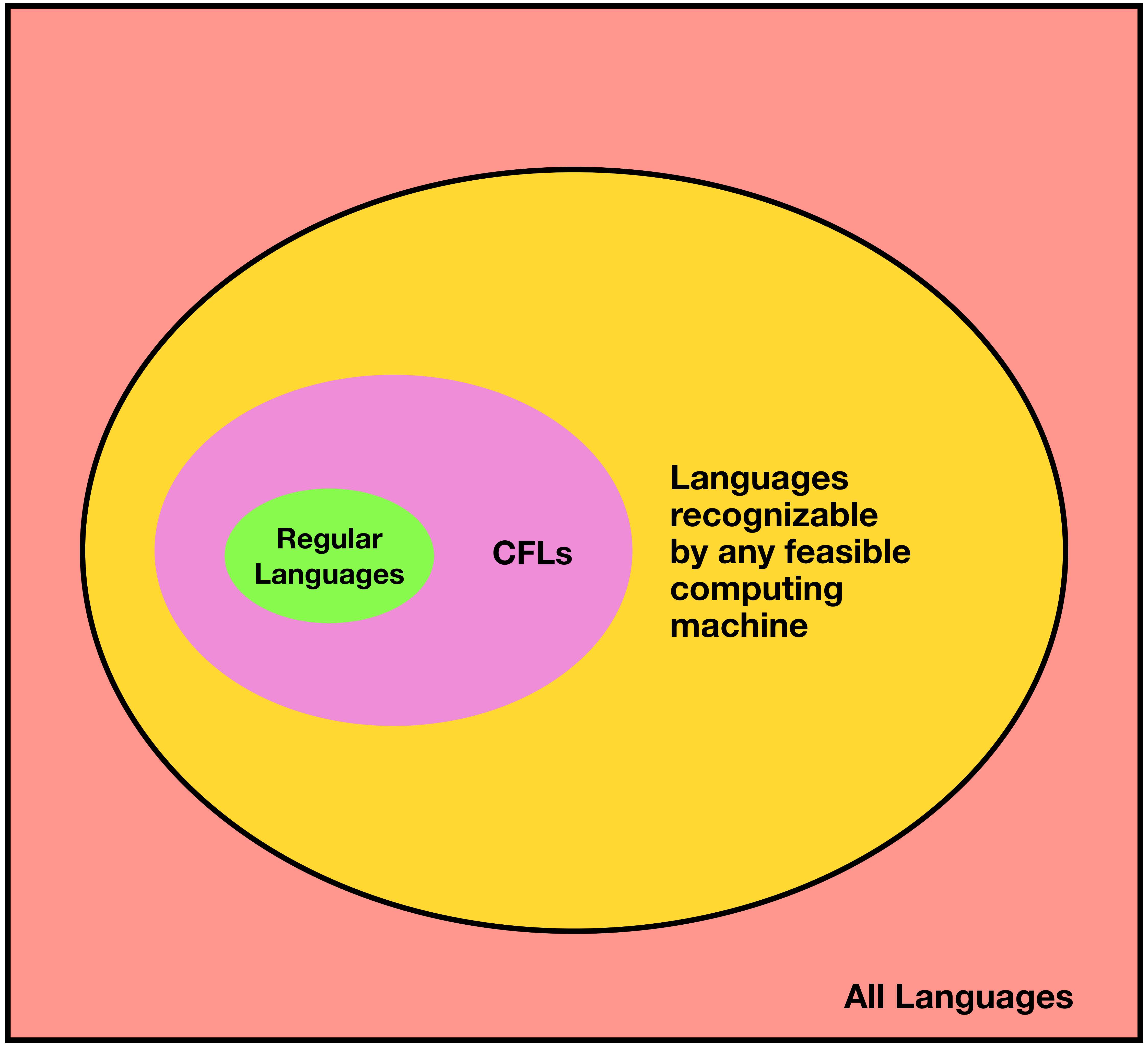

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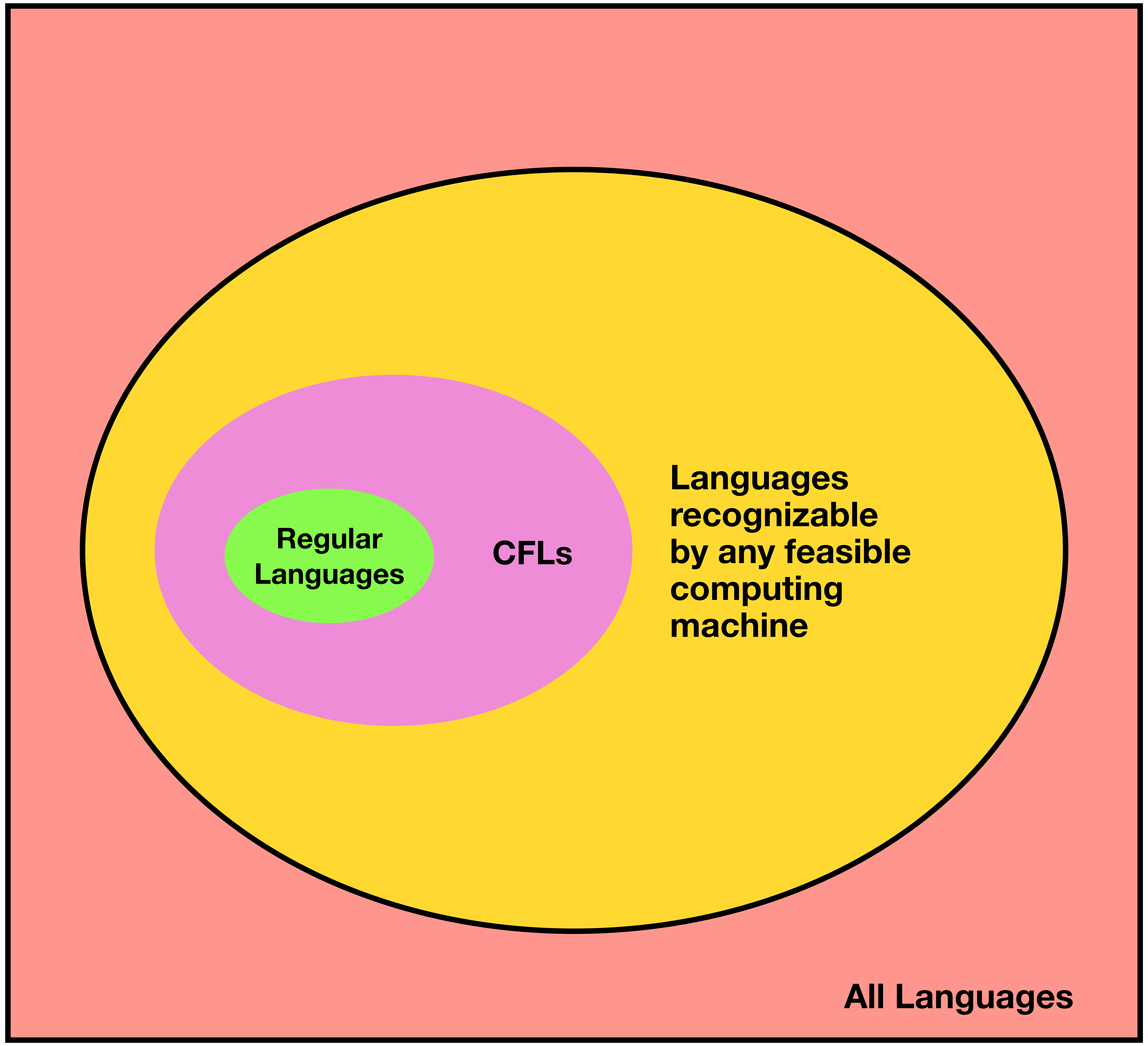
How do we model a computing device that can recognize as many languages as possible?



Turing Machine

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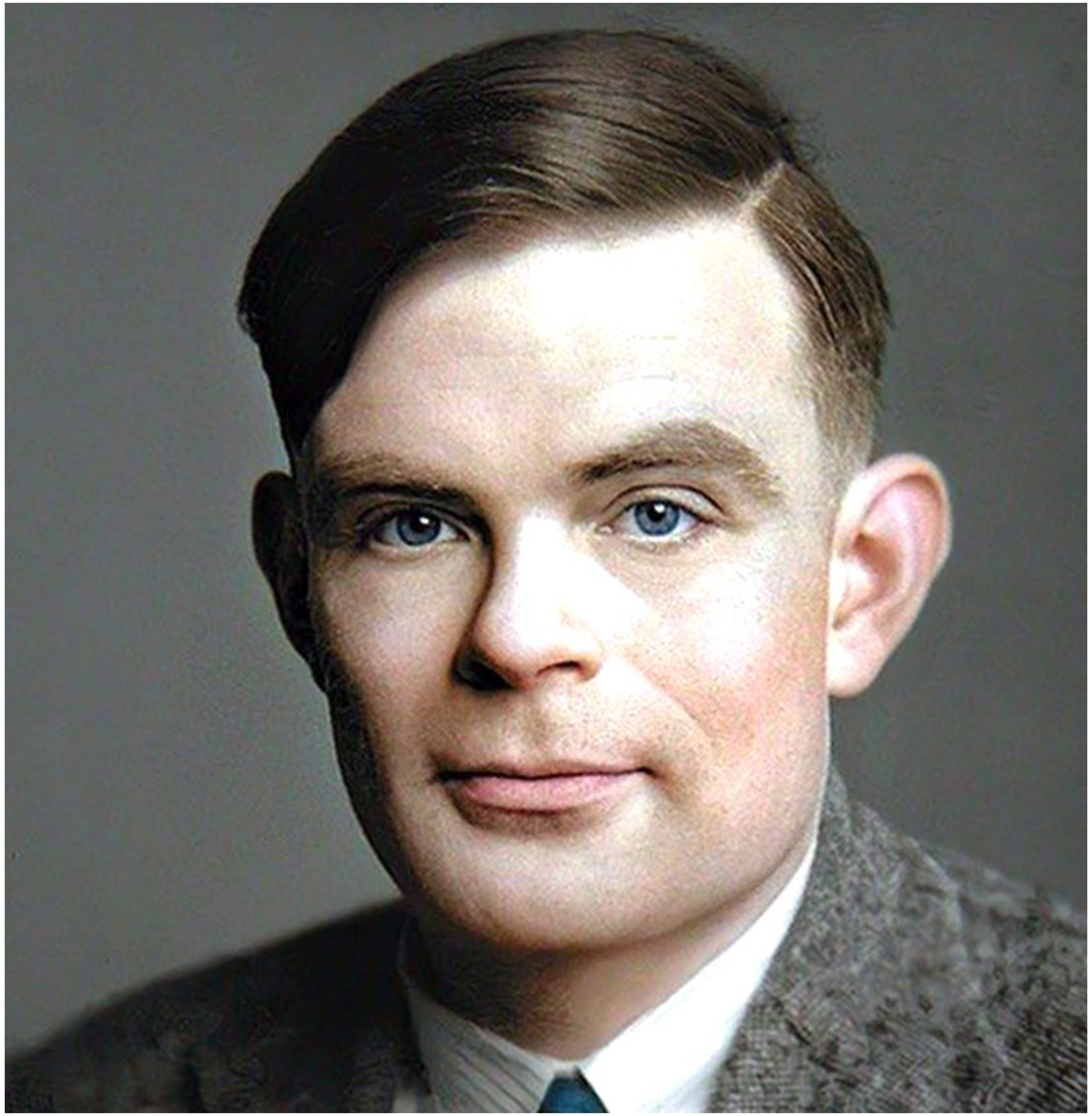

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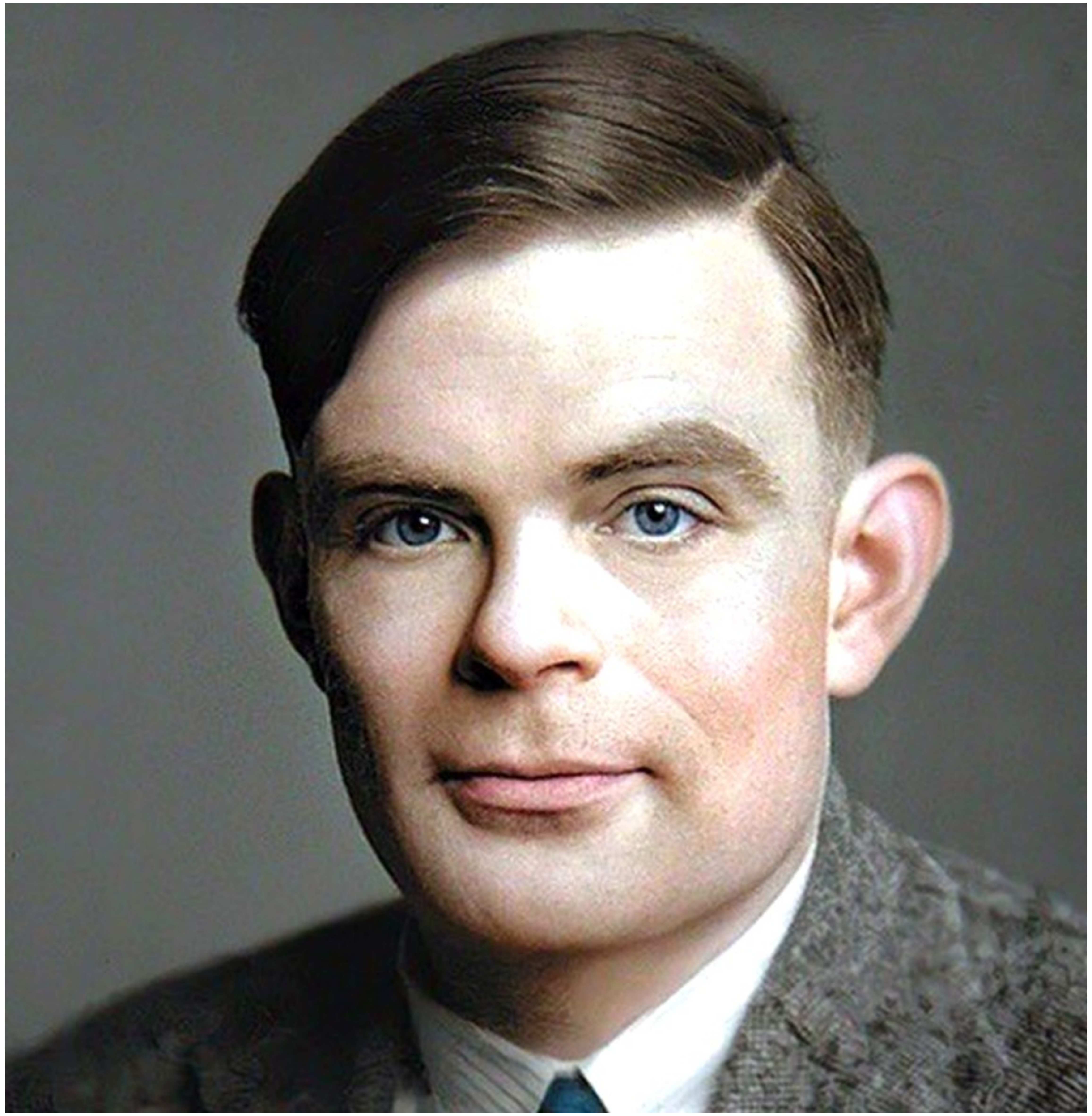

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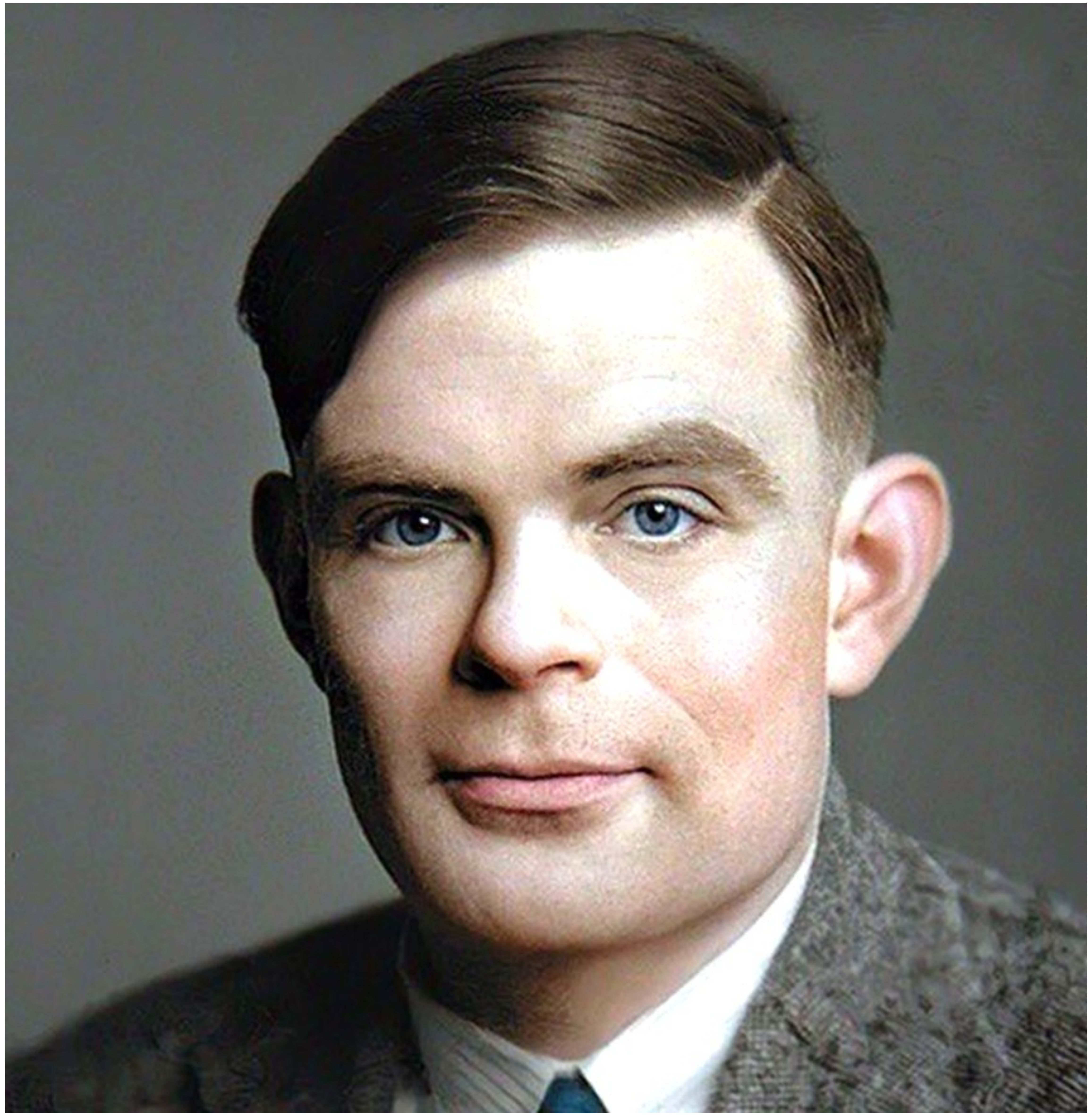
- Watch: The Imitation Game!



Turing Machine

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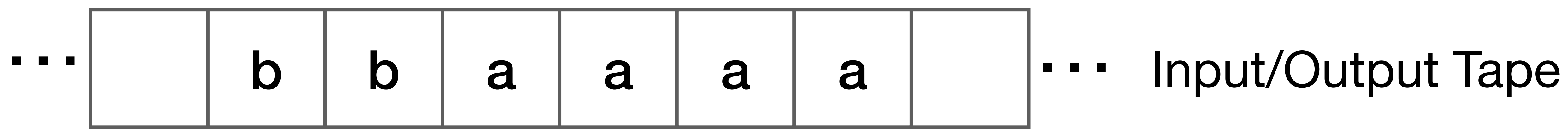
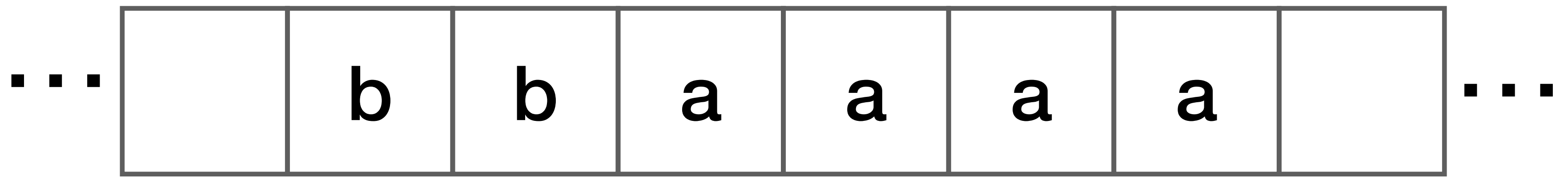


- Input written on (infinite) one sided tape.

Turing Machine

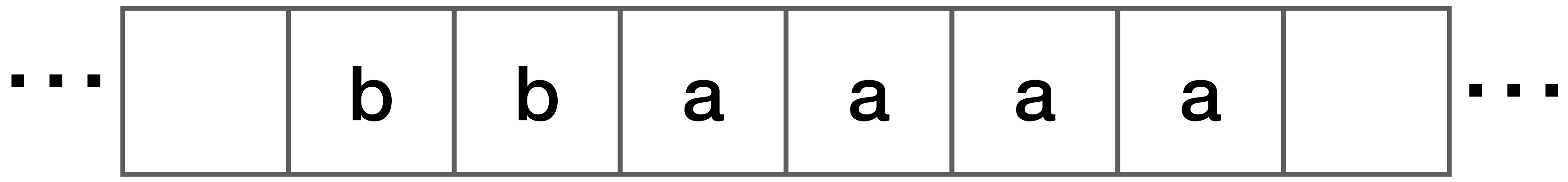


Input/Output Tape

allow blanks ("erase") on memory/tape.

- Input written on (infinite) one sided tape.
- Special blank characters.

Turing Machine



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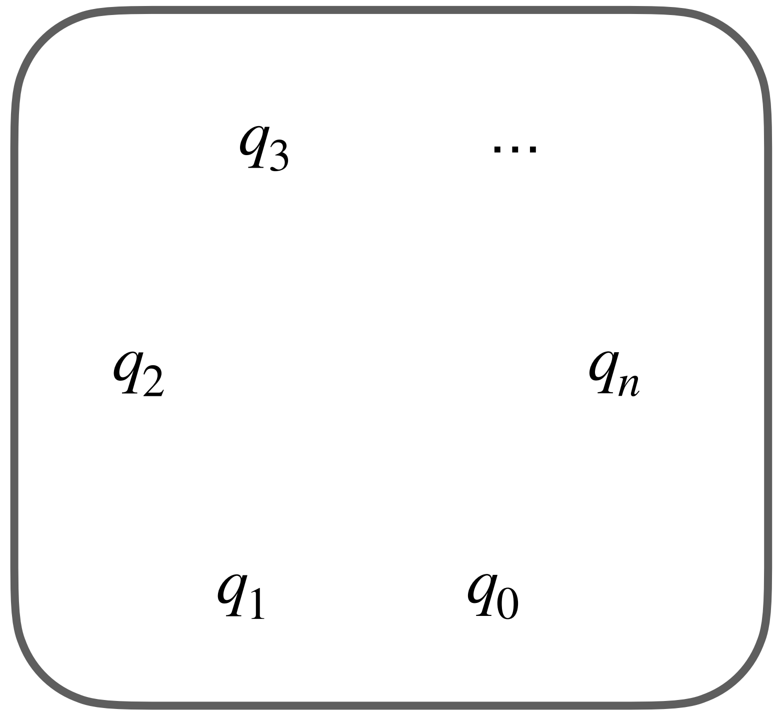
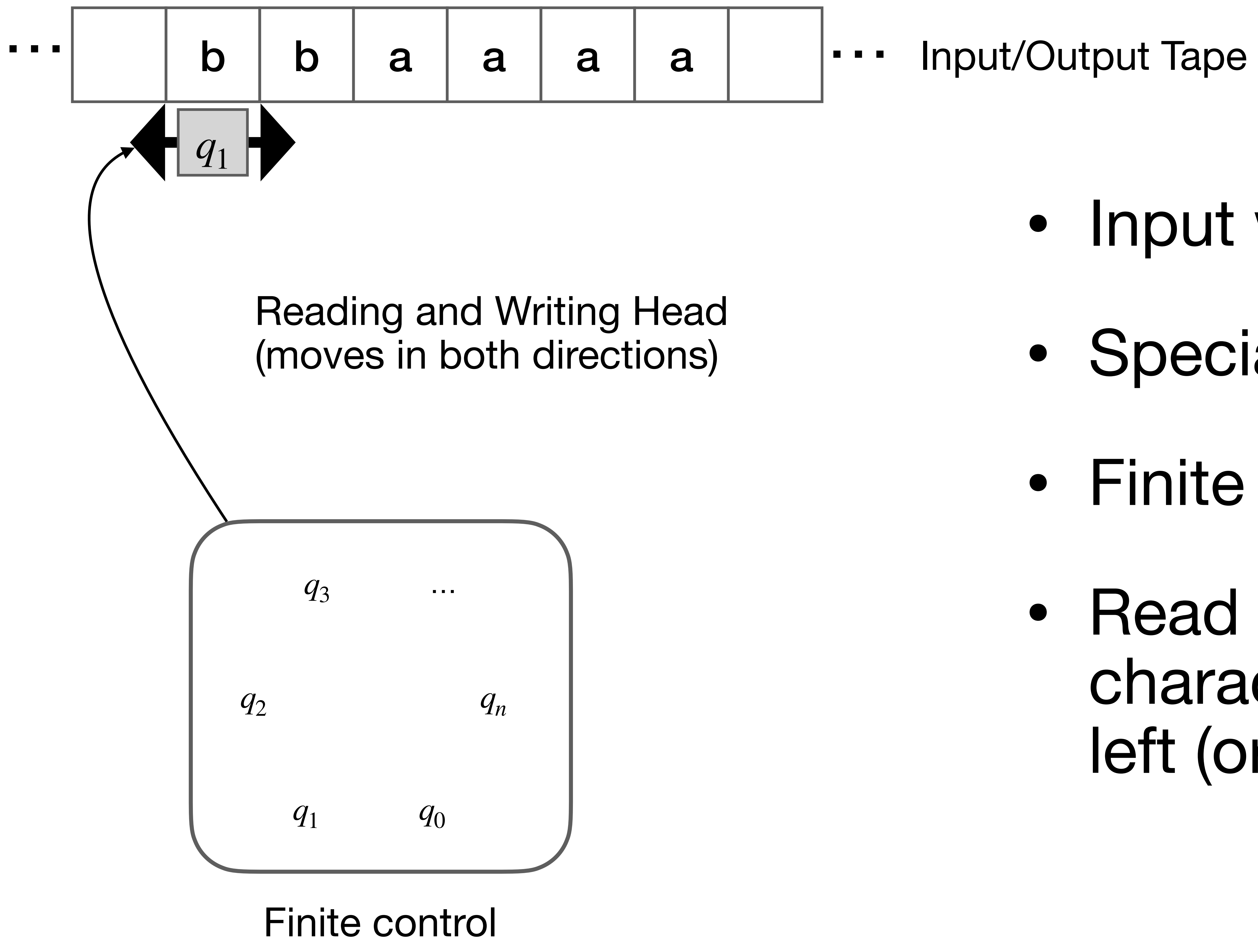


Finite control

Turing Machine



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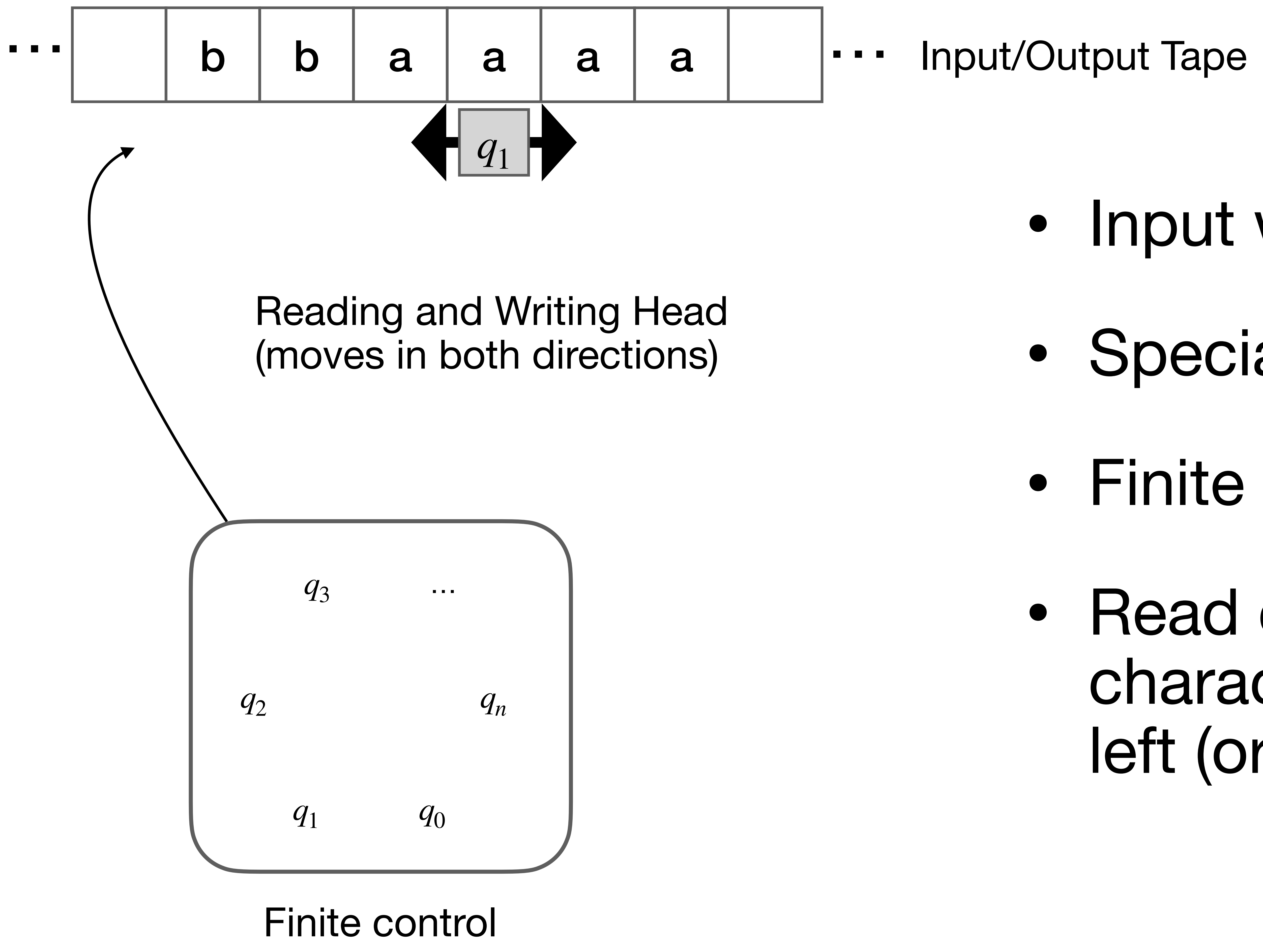

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Turing Machine
High level goals

- Church-Turing Thesis: TMs are the most general computing devices. So far, no “counter-example.”

Turing Machine

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Not a theorem

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Turing Machine

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- Every TM can be represented as a string.
- The existence of a Universal Turing Machine, which is the model/inspiration for stored program computing. UTM can simulate any TM!
- Implications for what can be computed and what cannot be computed

Example

Computers exist because we are lazy… so Stanford's CS 103 to the rescue.

Turing Machine

Formal definition

A Turing machine is a 7-tuple $\left(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}\right)$ where

- Q is a finite set of states
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- $\sqcup$ a special symbol for blank on the tape

Turing Machine

Transition function

current symbol on tape

new symbol on tape

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

current state

new state

direction to move head one step.

Turing Machine

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current state — symbol scanned — new state — symbol to write — Direction to move on tape

From state q, on reading a:

- go to state p
- write b
- move head *Left*
- missing transitions lead to BSOD

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$$\delta(q, a) = (p, b, L)$$

aka CRASH!!

More example(s)

Same link as last time again

Languages defined by a Turing machine

Recursive vs. Recursively Enumerable

- TMs have a new behavior - it may never halt.

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  - $L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}$

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Recursive vs. Recursively Enumerable

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Languages defined by a Turing machine

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More on these closer to end of semester.

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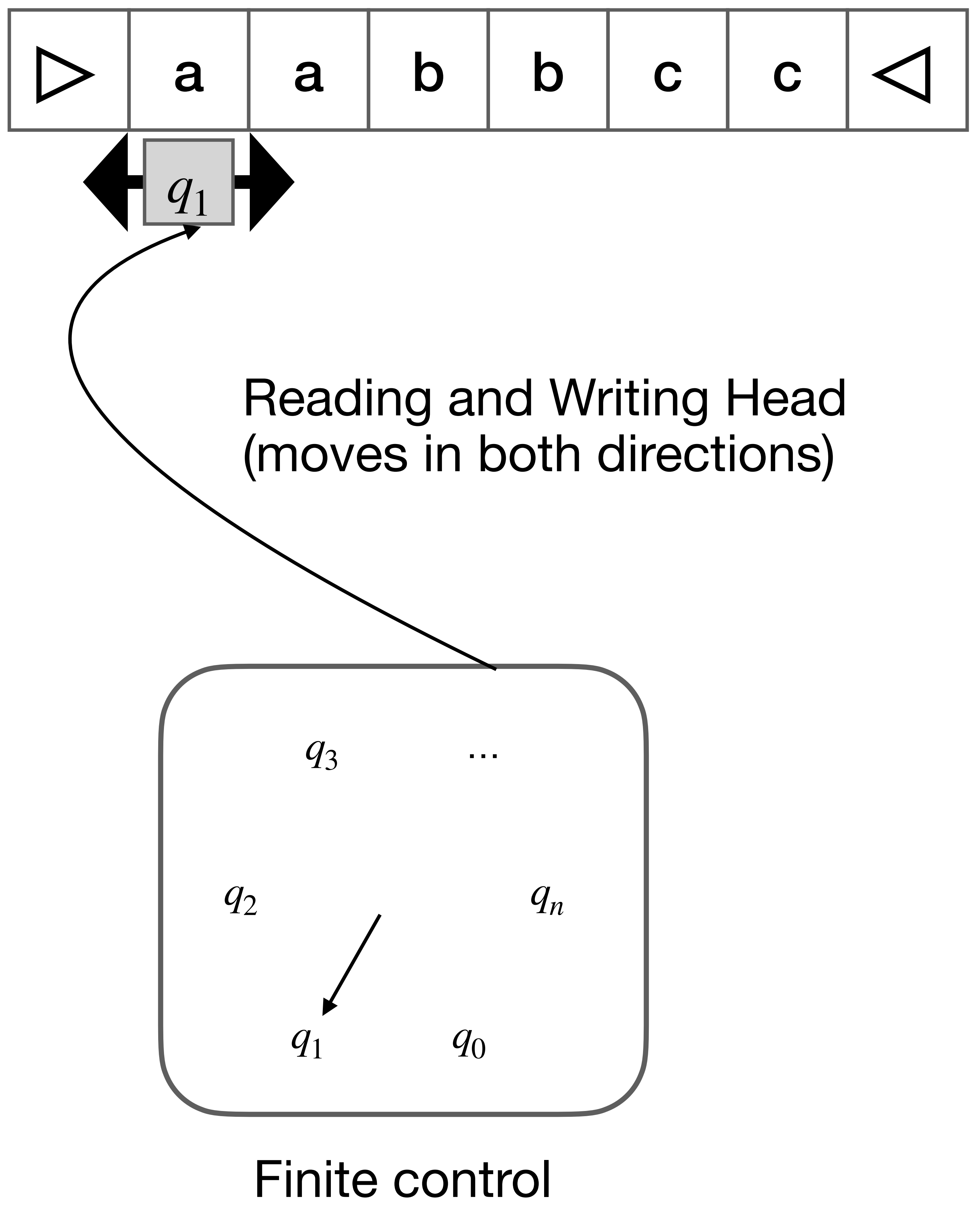
Linear Bounded Automata

Relation to TMs



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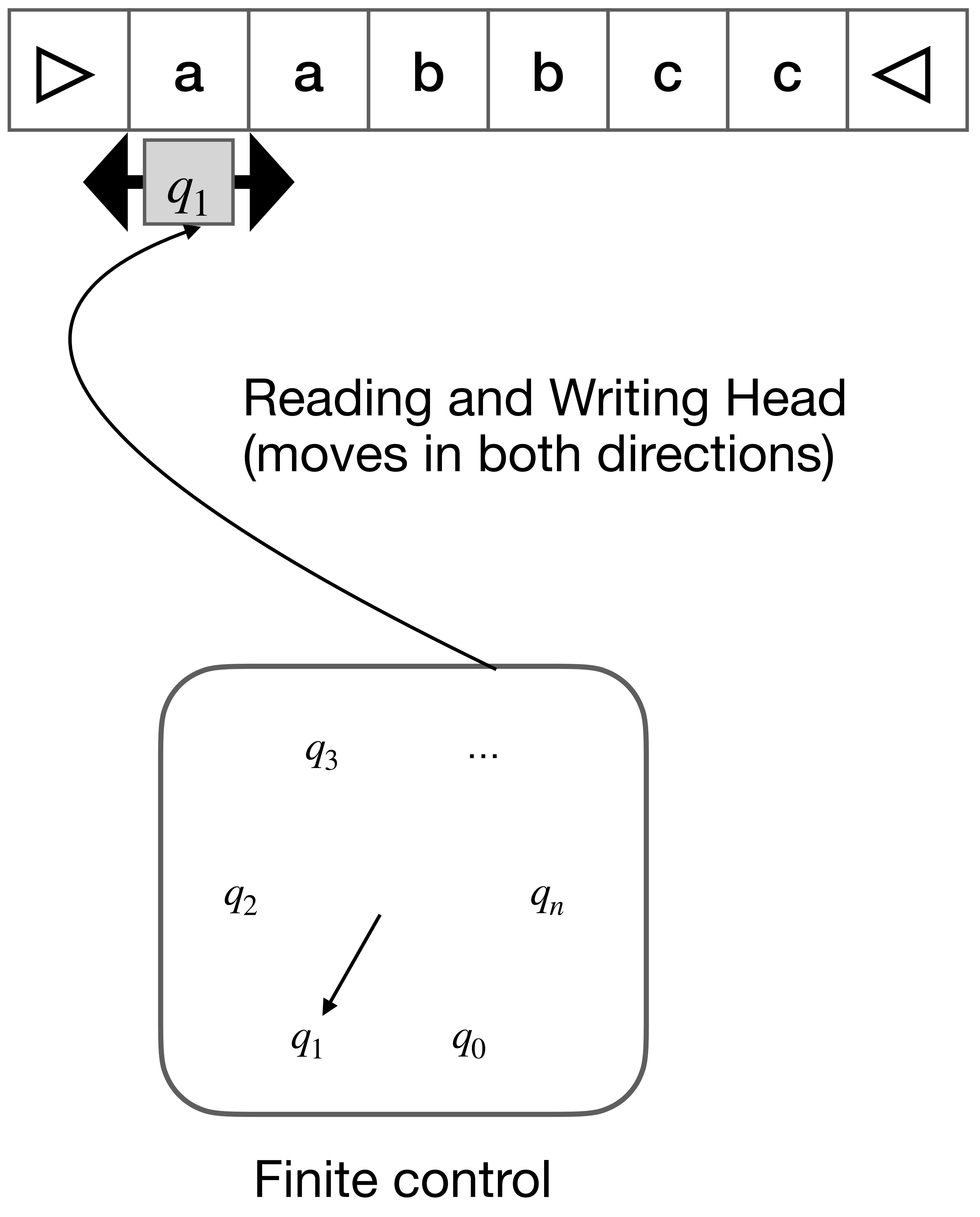

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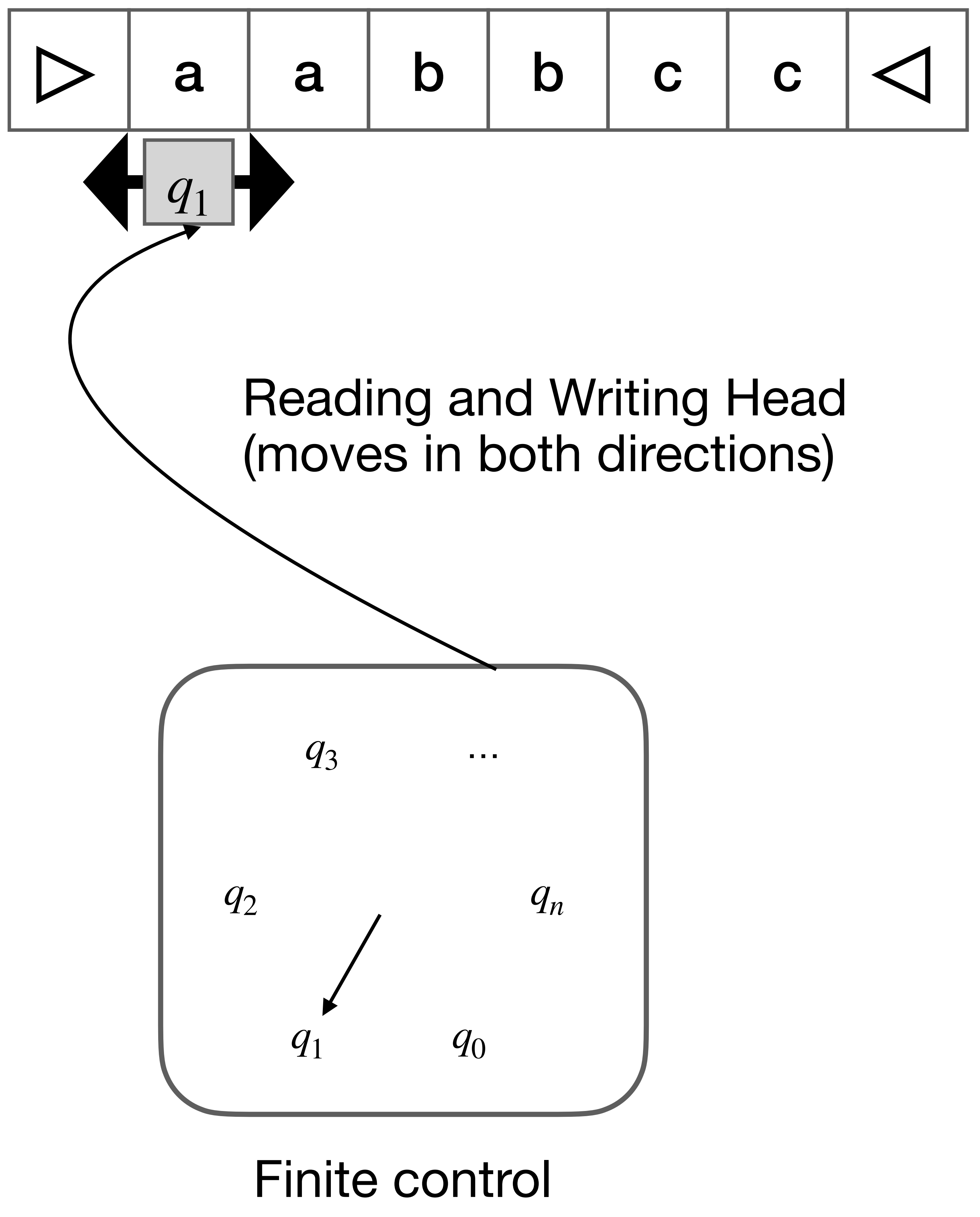
    - Tape used grows *linearly* with size of input

Linear Bounded Automata

Relation to TMs



- We skipped LBAs
  - They can be thought of as *restricted* Turing Machines.
  - Tape used grows *linearly* with size of input
- (Nondeterministic) LBA can recognize all context-sensitive languages.

Wrap up …

… the four-week tour of Models of Computation.

Chomsky's terminology

Grammar	Languages	Production Rules	Automation	Examples
Type-0	Turing ~~machine~~ recognizable	$\gamma \rightarrow \alpha$ (no constraints)	Turing Machines	L = { w | w is a TM which halts }
Type-1	Context-sensitive	$\alpha A \beta \rightarrow \alpha \gamma \beta$	Linear Bounded Automata	$L = \{ a^n b^n c^n \mid n > 0 \}$
Type-2	Context-free	$A \rightarrow \alpha$	Pushdown Automata	$L = \{ a^n b^n \mid n > 0 \}$
Type-3	Regular	$A \rightarrow aB$	Non-determinstic Finite Automata	$L = \{ a^n \mid n > 0 \}$

same as DFAs or RegExes

Next class

Universal Turing Machines

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What is the language of a UTM?