**Prologue** [by Nickvash Kani]: This is a problem/solution posited in Lab 22 since I began teaching ECE374 (over 4 years ago). In the Summer of 2024 **Eric Carl Roth** sent me a solution to this problem which I believe is correct. Eric was an amazing student, an exceptional course assistant and this proof solving a problem so many others (including professors) were not able to solve is a testament to his love of computer science.

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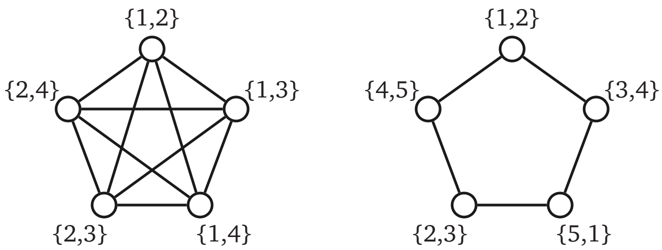
**Problem (This is a variation of a problem from Lab 21 - NP-Hardness-II):**

A ***bicoloring*** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

Prove that finding a five-strong-bicoloring of a given graph is NP-hard.




Left: A weak bicoloring of a 5-clique with four colors.
Right A strong bicoloring of a 5-cycle with five colors.

**Solution:** *Intuition*: Since this is a coloring problem, it's natural to assume we want to reduce from a coloring problem. An observation is that there are $\binom{5}{2} = 10$ possible bicolors, so maybe 10Color is a good problem to reduce from. However, we see the strong bicoloring problem is stricter since neighboring nodes cannot share a common color in their biclorings. To be specific, if we have a node arbitrarily bicolored, there are 3 possible ways to bicolor a neighboring node. So, how do we somehow map this restriction onto the 10Color problem? This motivates the creation of an edge gadget, which is seen and proved to work below.

*Lemma*: The following edge gadget is strongly 5 bicolorable if and only if the ends are different bicolors.


*Proof*: Let the set of five colors be $A = \{1, 2, 3, 4, 5\}$ and any two element subset be a bicolor. We must prove both directions of the lemma.

←: We choose to prove this direction via contrapositive. Suppose that the ends $a, d$ are the same bicolor and WLOG call it $X = \{1, 2\}$. We color $b$ with any valid, neighboring bicolor, so WLOG color it $Y = \{3, 4\}$. It's easily seen that there is no bicolor which can be neighbors to $X$ and $Y$ since the only color not in $X$ and $Y$ is 5. Therefore, the edge gadget is NOT strongly 5 bicolorable.

→: Suppose the ends are different bicolors. There are two cases to consider. *Case 1*: The colors are disjoint. WLOG color $a$ with $X = \{1, 2\}$ and $d$ with $Y = \{3, 4\}$. We can color $b$ with $Y$ and $c$ with $X$ and it's a valid strong 5 bicoloring.

*Case 2*: The colors share a common element and WLOG say it's 2. WLOG color $a$ with $X = \{1, 2\}$ and $d$ with $Y = \{2, 3\}$. Color $b$ with $Z = \{3, 4\}$ and color $c$ with $\{1, 5\}$. This is a valid strong 5 bicoloring.

We've shown both directions of the statement. □

*Proposition*: The strong bicoloring problem for 5 colors is NP-Hard.

*Proof*: Let $G = (V, E)$ be an arbitrary, undirected graph. We build a new graph $H = (V', E')$ such that $V' = V \cup \{v_{xy}^1, v_{xy}^2 \mid xy \in E\}$ and $E' = \{xv_{xy}^1, v_{xy}^1 v_{xy}^2, v_{xy}^2 y \mid xy \in E\}$. In essence, we've replaced each edge by a path of length 3. I claim that $H$ has a strong 5 bicoloring if and only if $G$ has a 10 coloring.

$\leftarrow$: Suppose $G$ has a valid 10 coloring and assign each color a unique 5 bicolor. Fix a valid 10 coloring and in the transformed graph $H$ assign the vertices $v \in V$ the bicolor corresponding to their respective color in $G$. By the lemma above, each path of length 3 in $H$ is strongly 5 bicolorable since the endpoints are different bicolors. Therefore, $H$ is strongly 5 bicolorable.

$\rightarrow$: Suppose $H$ has a valid strong 5 bicoloring. Then, each edge gadget must have differently bicolored endpoints from to the lemma above. Assign each bicolor a unique 10 color and in $G$ assign each $v \in V$ their respective color in $H$. This is a valid 10 coloring.

It's clear we can construct $H$ in polynomial time and thus the problem is NP-Hard.

□ ■