Design Turing machines $M = (Q, \Sigma, \Gamma, \delta, \text{start, accept, reject})$ for each of the following tasks, either by listing the states $Q$, the tape alphabet $\Gamma$, and the transition function $\delta$ (in a table), or by drawing the corresponding labeled graph.

Each of these machines uses the input alphabet $\Sigma = \{1, \#\}$; the tape alphabet $\Gamma$ can be any superset of $\{1, \#, \Box, \triangleright\}$ where $\Box$ is the blank symbol and $\triangleright$ is a special symbol marking the left end of the tape. Each machine should reject any input not in the form specified below.

1. On input $1^n$, for any non-negative integer $n$, write $1^n\#1^n$ on the tape and accept.

2. On input $\#^n1^m$, for any non-negative integers $m$ and $n$, write $1^m$ on the tape and accept. In other words, delete all the $\#$s and shift the $1$s to the start of the tape.

3. On input $\#1^n$, for any non-negative integer $n$, write $1^{2n}$ on the tape and accept. [Hint: Modify the Turing machine from problem 1.]

4. On input $1^n$, for any non-negative integer $n$, write $1^{2^n}$ on the tape and accept. [Hint: Use the three previous Turing machines as subroutines.]

Questions to ponder:

- Think of a simple problem for which a 2-tape TM seems to offer much better efficiency than a 1-tape TM. Can you argue that 2-tape machine can be simulated by a 1-tape machine with only a quadratic slow down?

- Can you think about why having more than 2 tapes does not buy a lot of speed up? Can you argue why a $k$-tape TM can be simulated by a 2-tape TM with a slow down that has only a poly-logarithmic overhead?

- How many bits does each word in your laptop/desktop have? How many bits did a desktop have 10 years ago, 20 years ago and 30 years ago? How does it limit the data you can work with?

- Suppose you want to multiply two $n$ bit integers where $n = 10,000$. How would you write a program for it? What would be the time complexity?

- You may know about cryptography and RSA. The current RSA public key is 512 bits. Can you think of an algorithm to check if a given 512 bit number is a prime number? How many steps will it take?

- How can a RAM model with say 64 bits per word be simulated by a $k$-tape TM? What would be the slow down?