This lab is on reductions. The first problem emphasizes the care one needs in making sure that a reduction is correct. The second one is about the notion of self-reductions; how one can reduce search and optimization problems to decision versions in many settings.

1. Let $G = (V, E)$ be a graph. A set of edges $M \subseteq E$ is said to be a matching if no two edges in M intersect at a vertex. A matching M is *perfect* if every vertex in V is incident to some edge in M; alternatively M is perfect if $|M| = |V|/2$ (which in particular implies $|V|$ is even). See Wikipedia article for some example graphs and further background.

   The PerfectMatching problem is the following: does the given graph G have a perfect matching? This can be solved in polynomial time which is a fundamental result in combinatorial optimization with many applications in theory and practice. It turns out that the PerfectMatching problem is easier to solve in *bipartite* graphs. A graph $G = (V, E)$ is bipartite if its vertex set V can be partitioned into two sets L, R (left and right say) such that all edges are between L and R (in other words L and R are independent sets). Here is an attempted reduction from general graphs to bipartite graphs.

   Given a graph $G = (V, E)$ create a bipartite graph $H = (V \times \{1, 2\}, E_H)$ as follows. Each vertex u is made into two copies $(u, 1)$ and $(u, 2)$ with $V_1 = \{(u, 1) \mid u \in V\}$ as one side and $V_2 = \{(u, 2) \mid u \in V\}$ as the other side. Let $E_H = \{((u, 1), (v, 2)) \mid (u, v) \in E\}$. In other words we add an edge betwen $(u, 1)$ and $(v, 2)$ iff (u, v) is an edge in E. Note that $((u, 1), (u, 2))$ is not an edge in H for any $u \in V$ since there are no self-loops in G.

   Is the preceding reduction correct? To prove it is correct we need to check that H has a perfect matching if and only if G has one.

   - Prove that if G has perfect matching then H has a perfect matching.
   - Consider G to be K_3 the complete graph on 3 vertices (a triangle). Show that G has no perfect matching but H has a perfect matching.
   - Extend the previous example to obtain a graph G with an even number of vertices such that G has no perfect matching but H has.

   Thus the reduction is incorrect although one of the directions is true.

2. The traveling salesman problem can be defined in two ways:

   - The Traveling Salesman Problem
     - Input: A weighted graph G
     - Output: Which tour $(v_1, v_2, \ldots, v_n)$ minimizes $\sum_{i=1}^{n-1} (d[v_i, v_i + 1]) + d[v_n, v_1]$
   - The Traveling Salesman *Decision* Problem
     - Input: A weighted graph G and an integer k
     - Output: Does there exist and TSP tour with cost $\leq k$

   Suppose we are given an algorithm that can solve the traveling salesman decision problem in (say) linear time. Give an efficient algorithm to find the actual TSP tour by making a polynomial number of calls to this subroutine.

3. An ***independent set*** in a graph G is a subset S of the vertices of G, such that no two vertices in S are connected by an edge in G. Suppose you are given a magic black box that somehow answers the following decision problem *in polynomial time*:

   - INPUT: An undirected graph G and an integer k.
   - OUTPUT: TRUE if G has an independent set of size k, and FALSE otherwise.

   (a) Using this black box as a subroutine, describe algorithms that solves the following *optimization* problem *in polynomial time*:

      - INPUT: An undirected graph G.
      - OUTPUT: The *size* of the largest independent set in G.

   (b) Using this black box as a subroutine, describe algorithms that solves the following *search* problem *in polynomial time*:

      - INPUT: An undirected graph G.
      - OUTPUT: An independent set in G of maximum size.

To think about later:

4. Formally, a ***proper coloring*** of a graph $G = (V, E)$ is a function $c\colon V \to \{1, 2, \ldots, k\}$, for some integer k, such that $c(u) \neq c(v)$ for all $uv \in E$. Less formally, a valid coloring assigns each vertex of G a color, such that every edge in G has endpoints with different colors. The ***chromatic number*** of a graph is the minimum number of colors in a proper coloring of G.

   Suppose you are given a magic black box that somehow answers the following decision problem *in polynomial time*:

   - INPUT: An undirected graph G and an integer k.
   - OUTPUT: TRUE if G has a proper coloring with k colors, and FALSE otherwise.

   Using this black box as a subroutine, describe an algorithm that solves the following ***coloring problem*** *in polynomial time*:

   - INPUT: An undirected graph G.
   - OUTPUT: A valid coloring of G using the minimum possible number of colors.

   *[Hint: You can use the magic box more than once. The input to the magic box is a graph and **only** a graph, meaning **only** vertices and edges.]*