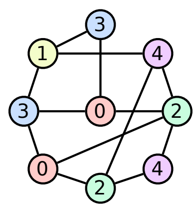
Prove that each of the following problems is NP-hard.

1. Given an undirected graph G, does G contain a simple path that visits all but 374 vertices?

2. Given an undirected graph G, does G have a spanning tree in which every node has degree at most 374?

3. Given an undirected graph G, does G have a spanning tree with at most 374 leaves?

4. Recall that a 5-coloring of a graph G is a function that assigns each vertex of G a "color" from the set $\{0, 1, 2, 3, 4\}$, such that for any edge uv, vertices u and v are assigned different "colors". A 5-coloring is ***careful*** if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard. *[Hint: Reduce from the standard 5Color problem.]*



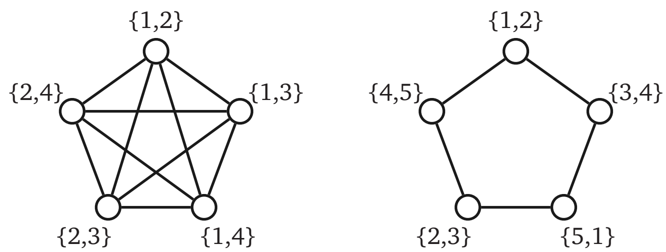
A careful 5-coloring.

5. Prove that the following problem is NP-hard: Given an undirected graph G, find *any* integer $k > 374$ such that G has a proper coloring with k colors but G does not have a proper coloring with $k - 374$ colors.

6. **To think about later:** A ***bicoloring*** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

   (a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

   (b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.



Left: A weak bicoloring of a 5-clique with four colors.
Right A strong bicoloring of a 5-cycle with five colors.