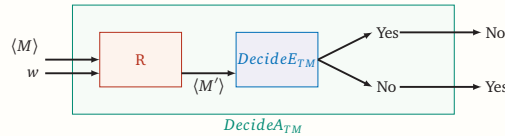


Prove that the following languages are undecidable.

$$1. E_{TM} := \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$$

**Solution:**  $E_{TM}$  is the problem of determining whether the language of a TM is empty. We will reduce  $DecideA_{TM}$  to  $DecideE_{TM}$ .



$\begin{aligned} \underline{M'(x)}: \\ \text{if } x \neq w \\ \quad \text{REJECT} \\ \text{else} \\ \quad \text{Run } M \text{ on input } w \text{ and accept iff } M \text{ accepts } w \end{aligned}$
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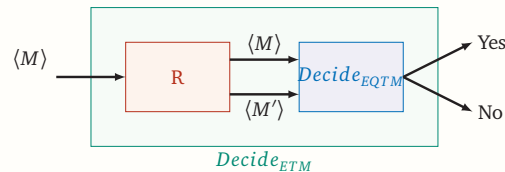
$\begin{aligned} \underline{DecideA_{TM}(\langle M, w \rangle)}: \\ \text{Construct } M' \text{ using } M \text{ and } w \\ \text{Run } DecideE_{TM} \text{ on } \langle M' \rangle \\ \text{if } DecideE_{TM}(\langle M' \rangle) \\ \quad \text{reject} \\ \text{else} \\ \quad \text{accept} \end{aligned}$
--

If  $DecideE_{TM}$  were a Decider for  $E_{TM}$ , then  $DecideA_{TM}$  is a Decider on  $A_{TM}$ . But a decider for  $A_{TM}$  can not exist, and hence  $E_{TM}$  is undecidable. ■

$$2. EQ_{TM} := \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$$

**Solution:**  $EQ_{TM}$  is the problem of determining whether the languages of two TMs are the same. Let us assume that one of the languages is  $\emptyset$ , we end up with the problem of determining whether the language of the other machine is empty—that is, problem 1 ( $E_{TM}$ ). Let's do a reduction from  $E_{TM}$ .

The reduction is as follows. Let  $Decide_{EQ_{TM}}$  decide  $EQ_{TM}$  and we construct  $Decide_{E_{TM}}$  to decide  $E_{TM}$  as follows.



$Decide_{ETM}(\langle M \rangle)$ :

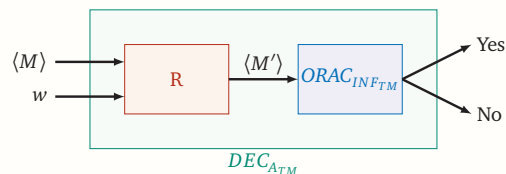
Let  $M'$  be a TM that rejects all inputs ( $L(M') = \emptyset$ ).  
 if  $Decide_{EQTM}(\langle M, M' \rangle)$   
   return TRUE  
 else  
   return FALSE

If  $Decide_{EQTM}$  decides  $EQTM$ ,  $Decide_{ETM}$  decides  $E_{TM}$ . But  $E_{TM}$  is undecidable as we proved in problem 1, so  $EQTM$  also must be undecidable. ■

3.  $INF_{TM} := \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is an infinite language} \}$

**Solution:** Let's do a reduction from the accept language:

$$A_{TM} \Rightarrow INF_{TM}$$



The reduction is as follows. On input  $\langle M, w \rangle$  we encode the following machine:

$M'(x)$ :

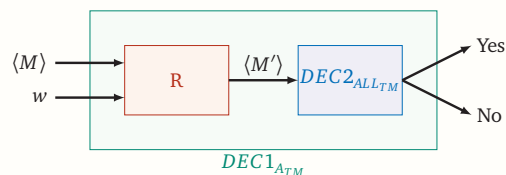
run  $M$  on input  $w$  and return TRUE if  $M$  accepts  $w$   
 otherwise return false

In this case, if  $ORAC_{INF_{TM}}$  output yes, you know that the language  $M'$  represents is infinite which is only possible if  $M$  accepts  $w$ . If the oracle returns not true, you know  $M$  must not accept  $w$ . ■

4.  $ALL_{TM} := \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^* \}$

**Solution:** Let's do a reduction from  $A_{TM}$ .

$$A_{TM} \Rightarrow ALL_{TM}$$



The reduction is as follows. On input  $\langle M, w \rangle$  we encode the following machine:

$DEC1_{ATM}(w)$ :

```

Let  $M'$  be a TM that runs  $w$  on  $M$  and returns TRUE if  $M$  accepts  $w$ 
if  $DEC2_{ALLTM}(\langle M' \rangle)$ 
  return TRUE
else
  return FALSE

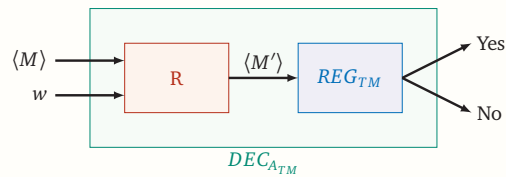
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If  $DEC2_{ALLTM}$  outputs yes,  $M$  accepts  $w$  and  $L(M') = \Sigma^*$  and decides for  $ALLTM$ . If  $DEC1_{ALLTM}$  decides  $ALLTM$ , then  $DEC2_{ATM}$  decides  $A_{TM}$ . But  $A_{TM}$  is undecidable, so  $DEC1_{ALLTM}$  cannot exist and hence  $ALLTM$  also must be undecidable. ■

5.  $REG_{TM} := \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language} \}$

**Solution:** Let's do a reduction from the accept language:

$$A_{TM} \Rightarrow REG_{TM}$$



The reduction is as follows. On input  $\langle M, w \rangle$  we encode the following machine:

$M'(x)$ :

```

if  $x$  is of the form  $0^n 1^n$ 
  accept  $x$ 
elseif  $M$  accepts  $w$ 
  accept  $x$ 
else
  reject  $x$ 

```

This means: If the original  $M$  accepts  $w$ , then  $M'$  will accept every string, this is regular. If the original  $M$  rejects  $w$ , then  $M'$  will only accept strings  $0^n 1^n$ , this is not regular.

So on the input  $\langle M' \rangle$ , if  $REG_{TM}$  returns TRUE then  $M$  accepts  $w$  and if  $REG_{TM}$  returns FALSE then  $M$  rejects  $w$ .

Therefore  $REG_{TM}$  must be undecidable. ■