Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

(Hint) Write a recurrence to analyze the algorithm's running time if we choose a list of size k.

ECE-374-B: Lecture 12 - Backtracking and memorization

Instructor: Nickvash Kani

February 28, 2023

University of Illinois at Urbana-Champaign

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We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

(Hint) Write a recurrence to analyze the algorithm's running time if we choose a list of size k.

Assume k is odd

Review linear time selection

Given an array $A = [0, ..., n-1]$ of n numbers and an index i, where $0 \leq i \leq n-1$, find the i^{th} smallest element of A.

For instance, assume $n = 20$ and $i = 10$.

4	3	15	7	1	17	9	10	14	13	8	18	11	2	12	16	6	19	5	20

The smallest element of rank 10 would be 11. But how do we figure that out

Do median of medians.....

Call Median-of-Medians(A, 10)

Review linear time selection

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First thing we need to do is find the pivot!

Review linear time selection

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The smallest element of rank 10 would be 11. But how do we figure that out

Do median of medians.....

Call Median-of-Medians(A, 10)

First thing we need to do is find the pivot!

Review linear time selection

First we reorganize:

4	17	8	16
3	9	18	6
15	10	11	19
7	14	2	5
1	13	12	20

Review linear time selection

First we reorganize:

4	17	8	16
3	9	18	6
15	10	11	19
7	14	2	5
1	13	12	20

Then we sort each column:

1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

$n/5 \cdot c$

Review linear time selection

First we reorganize:

4	17	8	16
3	9	18	6
15	10	11	19
7	14	2	5
1	13	12	20

Then we sort each column:

1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

Still need the pivot. Find median of medians

Review linear time selection

1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

Review linear time selection

1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

- Call Median-of-Medians([4,13,11,16], floor(len/2) = 2)
- Can sort this in linear time.
- Get back 13.
- **13** is our new pivot!

Review linear time selection

Back to our original array! Use the pivot (=13) to break it up into two.

4	3	15	7	1	17	9	10	14	13	8	18	11	2	12	16	6	19	5	20

4	3	7	1	9	10	8	11	2	12	6	5

13

15	17	14	18	16	19	20

We know the following:

- $\text{len}(A_{Lower}) = 12$
- $\text{len}(A_{Upper}) = 7$
- Want $k = 10$

Review linear time selection

Back to our original array! Use the pivot (=13) to break it up into two.

4	3	15	7	1	17	9	10	14	13	8	18	11	2	12	16	6	19	5	20

4	3	7	1	9	10	8	11	2	12	6	5

13

15	17	14	18	16	19	20

We know the following:

- $\text{len}(A_{Lower}) = 12$
- $\text{len}(A_{Upper}) = 7$
- Want $k = 10$

Call Median-of-Medians$(A_{Lower}, 10)$

Review linear time selection

Then we do this again:

4	3	7	1	9	10	8	11	2	12	6	5

Review linear time selection

Then we do this again:

4	3	7	1	9	10	8	11	2	12	6	5

First we reorganize:

4	10	
3	8	6
7	11	5
1	2	
9	12	

Review linear time selection

Then we do this again:

4	3	7	1	9	10	8	11	2	12	6	5

First we reorganize:

4	10	
3	8	6
7	11	5
1	2	
9	12	

Then we sort each column:

1	2	
3	8	5
4	10	6
7	11	
9	12	

Review linear time selection

1	2	
3	8	5
4	10	6
7	11	
9	12	

Review linear time selection

1	2	
3	8	5
4	10	6
7	11	
9	12	

- Call Median-of-Medians([4,10,6], floor(n/2) = 10)
- Can sort this in linear time.
- Get back 6.
- **6** is our new pivot!

Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

4	3	7	1	9	10	8	11	2	12	6	5

4	3	1	2	5

6

7	9	10	8	11	12

k is somewhere here

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 6$
- Want $k = 10$ (pivot is of rank 6)

Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

4	3	7	1	9	10	8	11	2	12	6	5

4	3	1	2	5

6

7	9	10	8	11	12

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 6$
- Want $k = 10$ (pivot is of rank 6)

Call Median-of-Medians(A_{Upper}, $10 - 6 = 4$)

Review linear time selection

Then we do this again:

7	9	10	8	11	12

Review linear time selection

Then we do this again:

7	9	10	8	11	12

First we reorganize:

7	
9	
10	12
8	
11	

Review linear time selection

Then we do this again:

7	9	10	8	11	12

First we reorganize:

7	
9	
10	12
8	
11	

Then we sort each column:

7	
8	
9	12
10	
11	

Review linear time selection

7	
8	
9	12
10	
11	

Review linear time selection

7	
8	
9	12
10	
11	

- Call Median-of-Medians([9,12], floor(len/2) = 1)
- Can sort this in linear time.
- Get back 12.
- **12** is our new pivot!

Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

7	9	10	8	11	12

7	9	10	8	11		12

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 0$
- Want $k = 4$ (pivot is of rank 5)

Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

7	9	10	8	11	12

7	9	10	8	11

12

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 0$
- Want $k = 4$ (pivot is of rank 5)

Call Median-of-Medians(A_{Lower}, 4)

Review linear time selection

Final Step!

7	9	10	8	11

Can sort in linear time!

7	8	9	10	11

Return $\textit{Sorted}(A[4]) = 11$

Median of medians time analysis

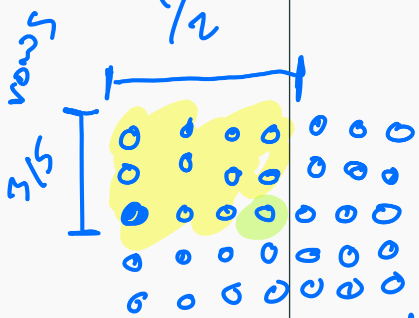
```
Median-of-medians(A, i):
    sublists = [A[j:j+5] for j ∈range(0, len(A), 5)]
    medians = [sorted (sublist)[len (sublist)/2] for sublist ∈sublists]

    // Base Case
    if len (A) ≤ 5 return sorted (a)[i]

    // Find median of medians
    if len (medians) ≤ 5
        pivot = sorted (medians)[len (medians)/2]
    else
        pivot = Median-of-medians (medians, len/2)

    // Partitioning Step
    low = [j for j ∈A if j < pivot]
    high = [j for j ∈A if j > pivot]

    k = len (low)
    if i < k
        return Median-of-medians (low, i)
    elseif i > k
        return Median-of-medians (low, i-k-1)
    else
    return pivot
```



$$T(n) = T(\tfrac{7}{10}n) + T(n/5) + O(n)$$

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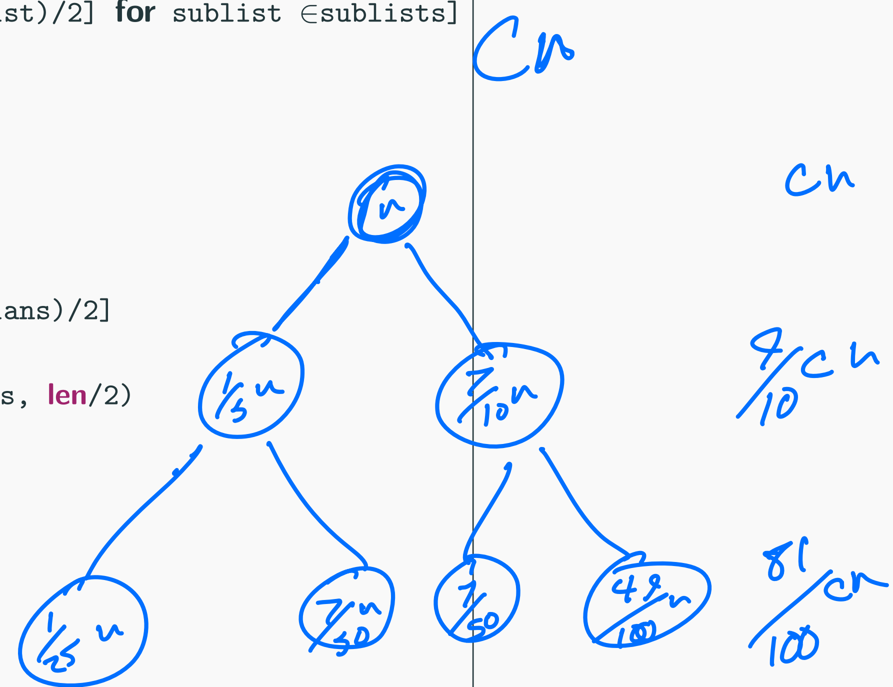


$$T(n) = T(\frac{1}{5}n) + T(\frac{7}{10}n) + cn = O(n)$$

Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

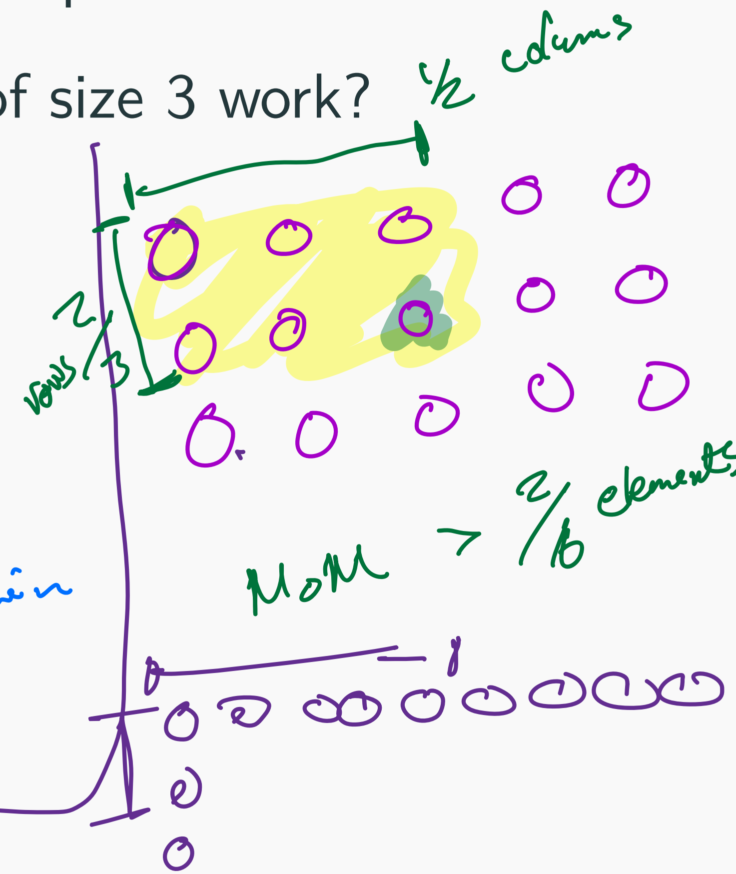
for |subarray| = 5

$$T(n) = T(\tfrac{1}{5}n) + T(\tfrac{7}{10}n) + cn$$

Find median of median

recurse on the main function

sort subarrays



for |subarray| = 3

$$T(n) = T(\tfrac{1}{3}n) + T(\tfrac{2}{3}n) + cn$$

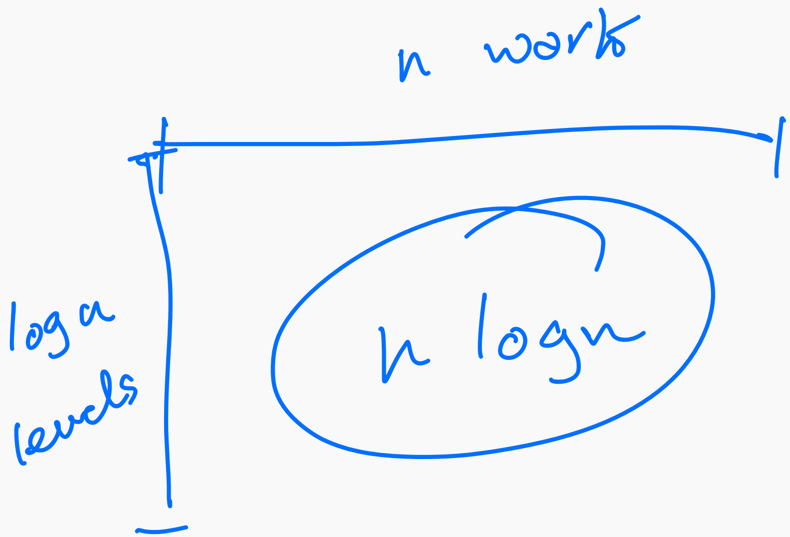
Pre-lecture brain teaser

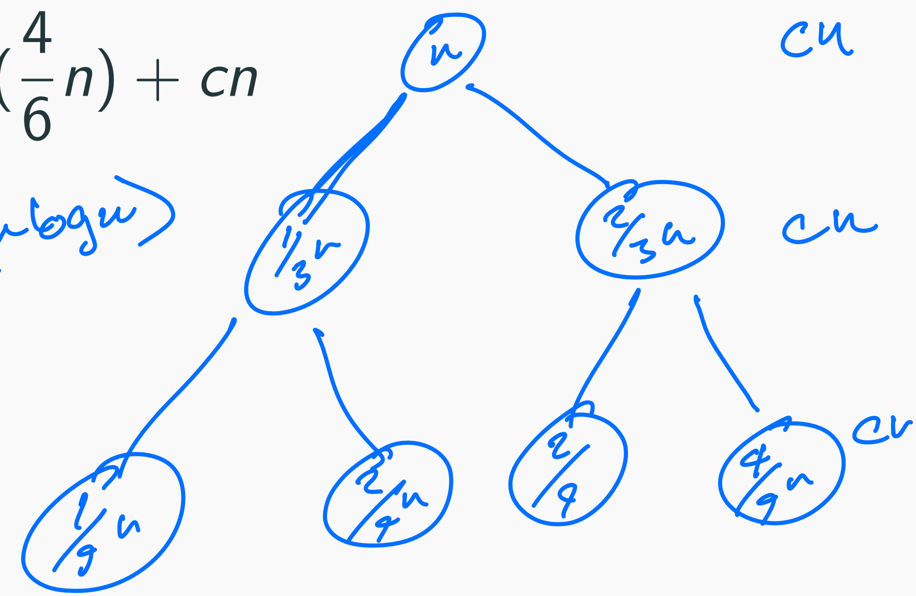
We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

$= O(n \log n)$





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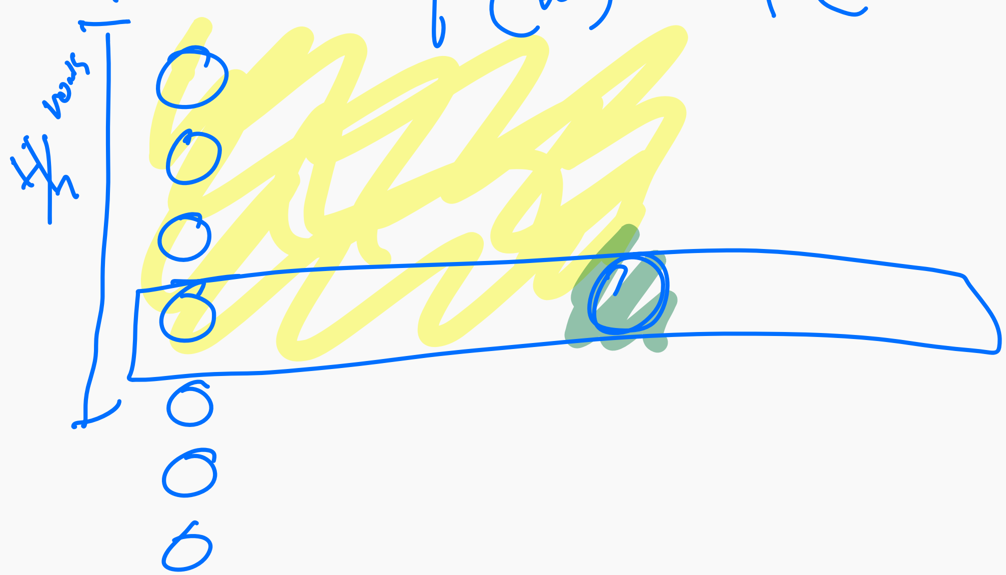
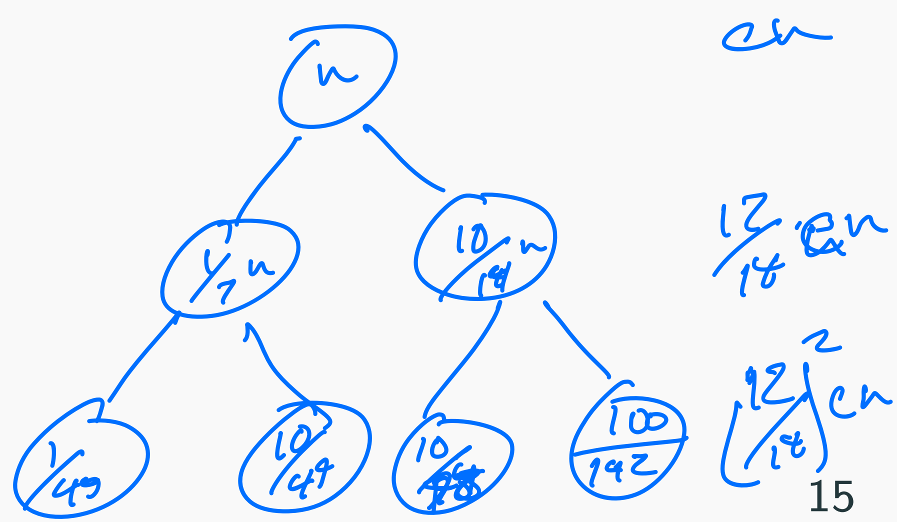
Why did we choose lists of size 5? Will lists of size 3 work?

$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

What about $k = 7$?

$$T(n) = T(\frac{1}{7}n) + T(\frac{10}{14}n) + cn = O(n)$$





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What about $k = 7$?

$$T(n) = T(\frac{1}{7}n) + T(\frac{10}{14}n) + cn$$

On different techniques for recursive algorithms

Recursion

Reduction: Reduce one problem to another

Recursion
A special case of reduction

- reduce problem to a smaller instance of itself
- self-reduction

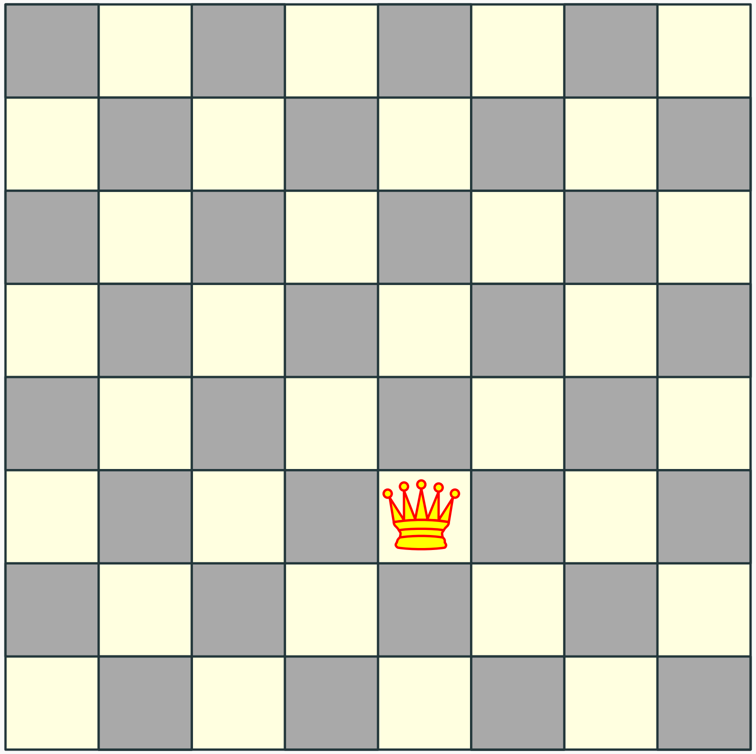
- Problem instance of size n is reduced to one or more instances of size $n - 1$ or less.
- For termination, problem instances of small size are solved by some other method as base cases.

Recursion in Algorithm Design

- Tail Recursion: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.
  Examples: Interval scheduling, MST algorithms....
- Divide and Conquer: Problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem.
  Examples: Closest pair, median selection, quick sort.
- Backtracking: Refinement of brute force search. Build solution incrementally by invoking recursion to try all possibilities for the decision in each step.
- Dynamic Programming: problem reduced to multiple (typically) dependent or overlapping sub-problems. Use memorization to avoid recomputation of common solutions leading to iterative bottom-up algorithm.

Search trees and backtracking

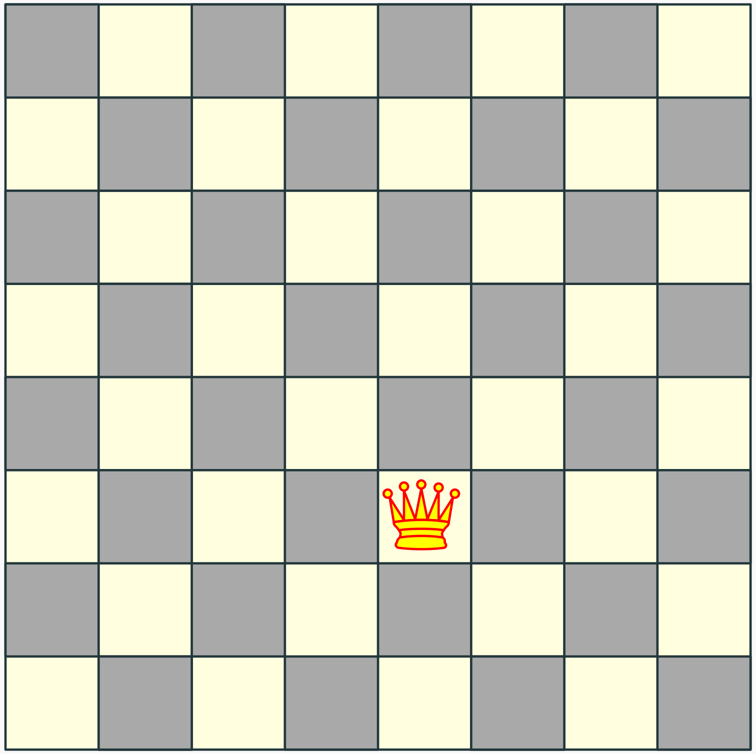
The queens problem



Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board?

The queens problem



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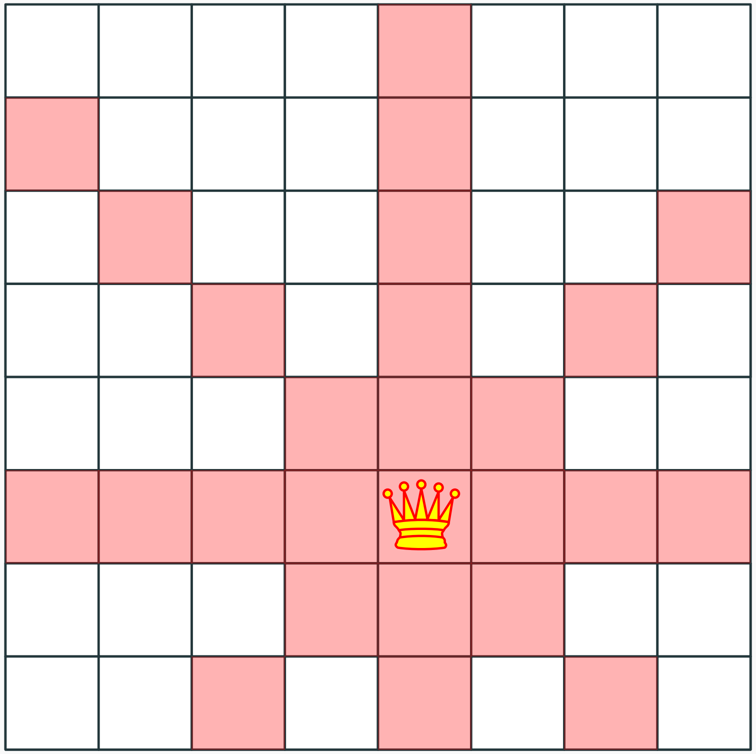

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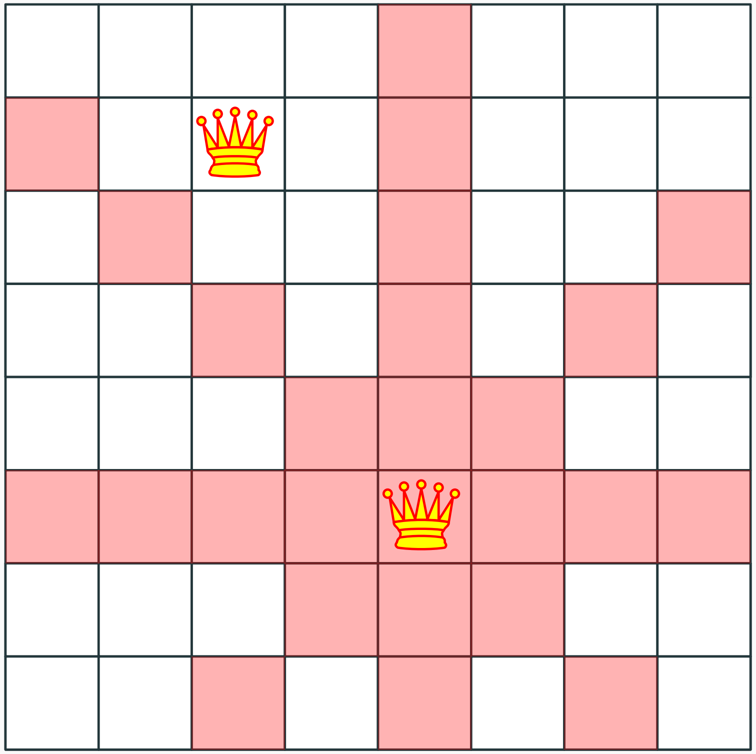
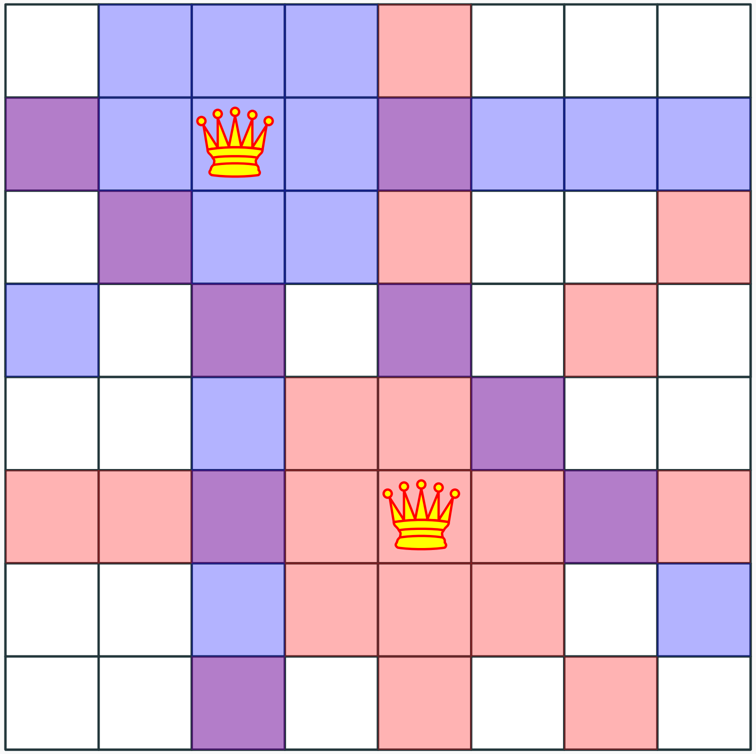


The queens problem



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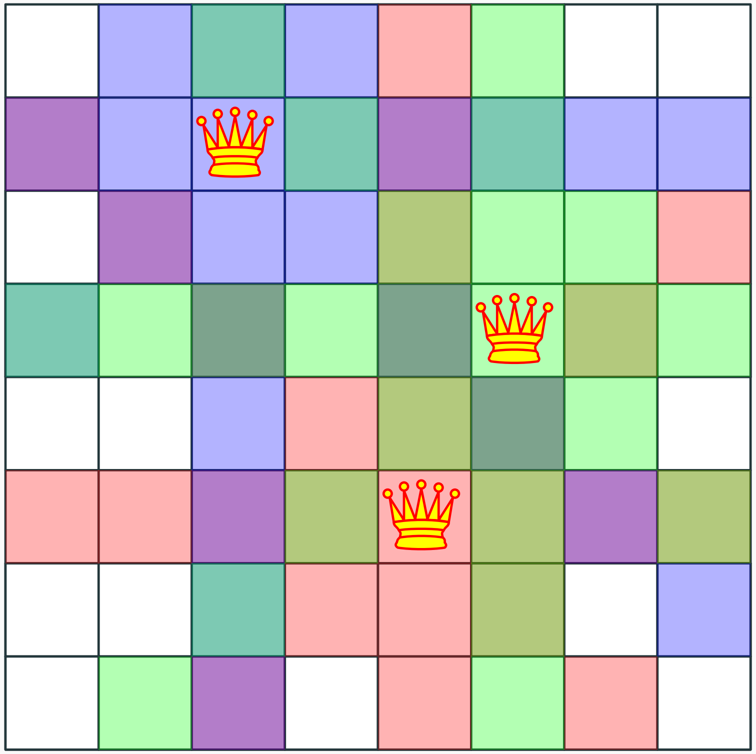

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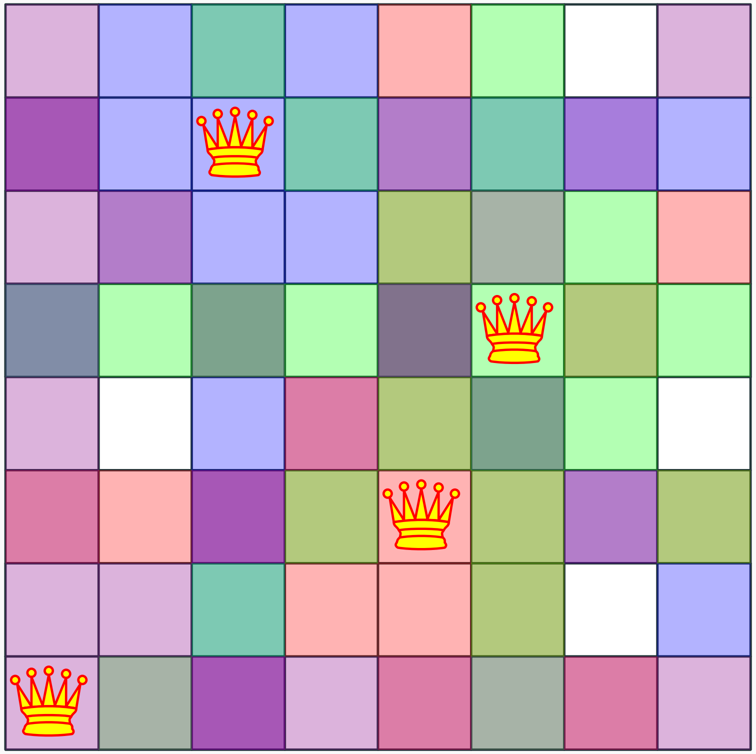

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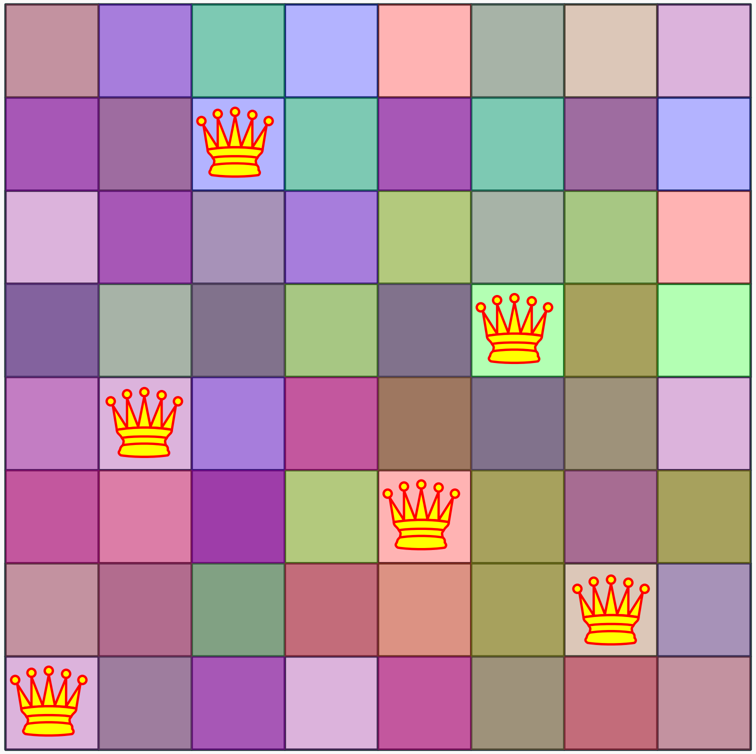

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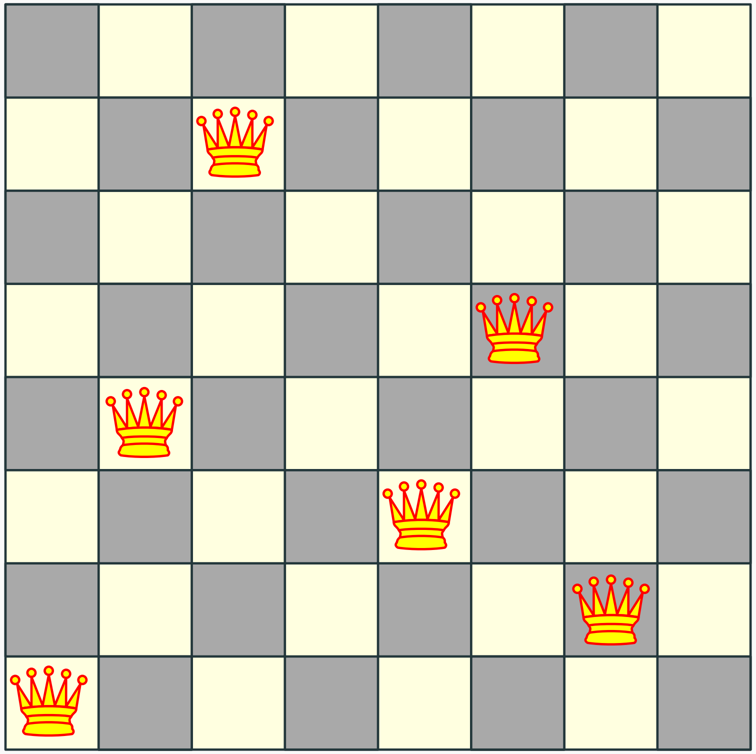
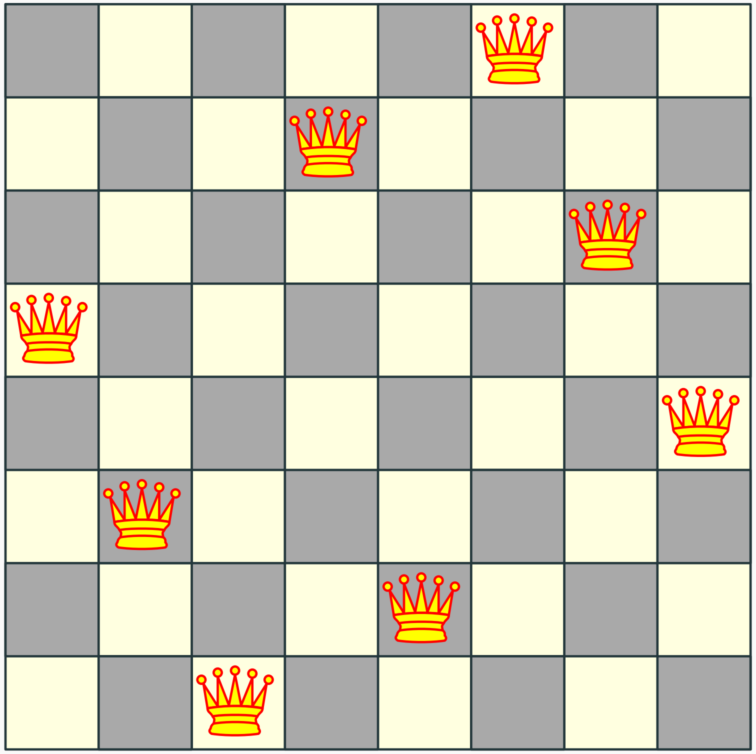


Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board? How many permutations?

The eight queens puzzle

Problem published in 1848, solved in 1850.



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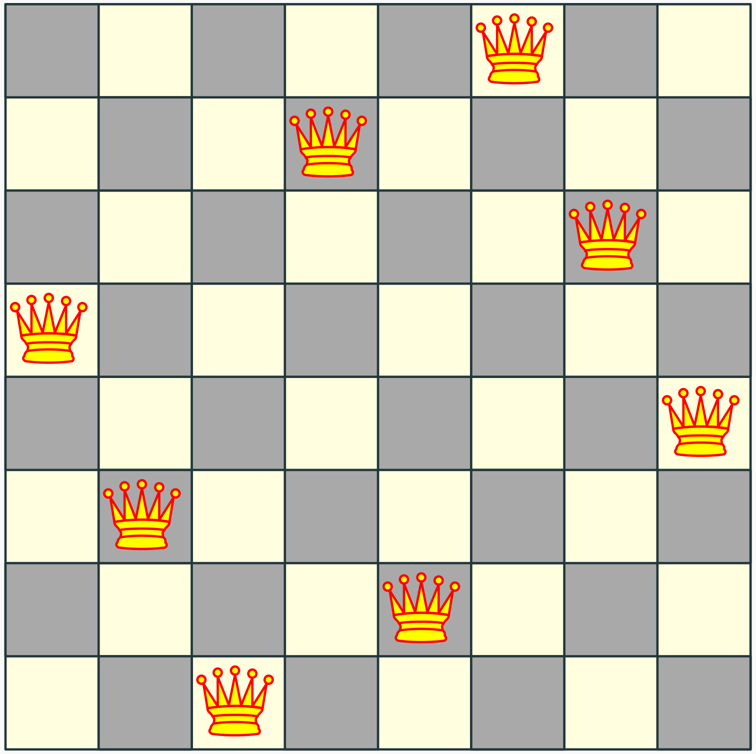
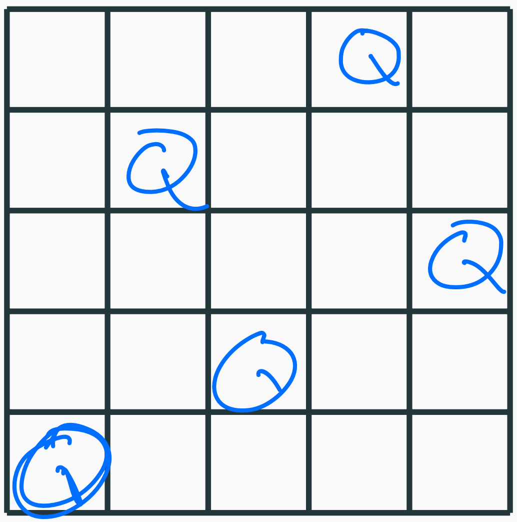


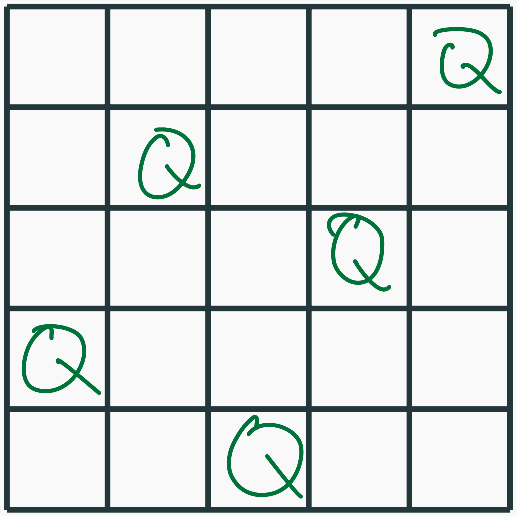
Q: How to solve problem for general n?

Introducing concept of state tree



What if we attempt to find all the possible permutations and then check?

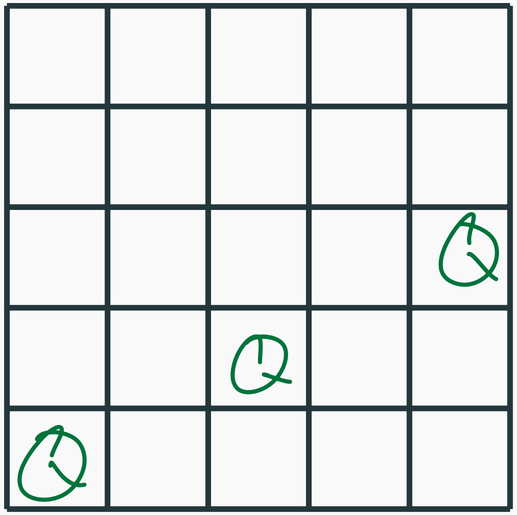
Search tree for 5 queens

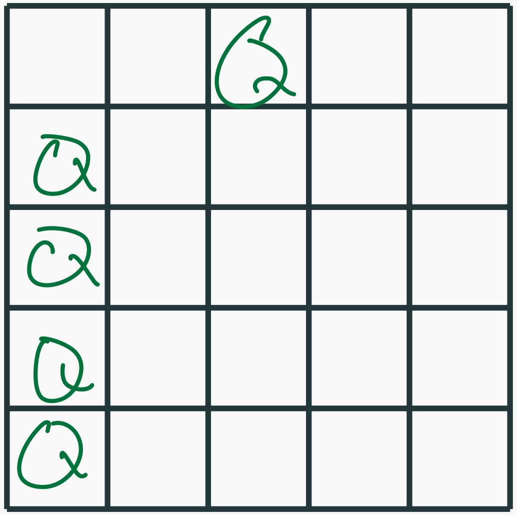


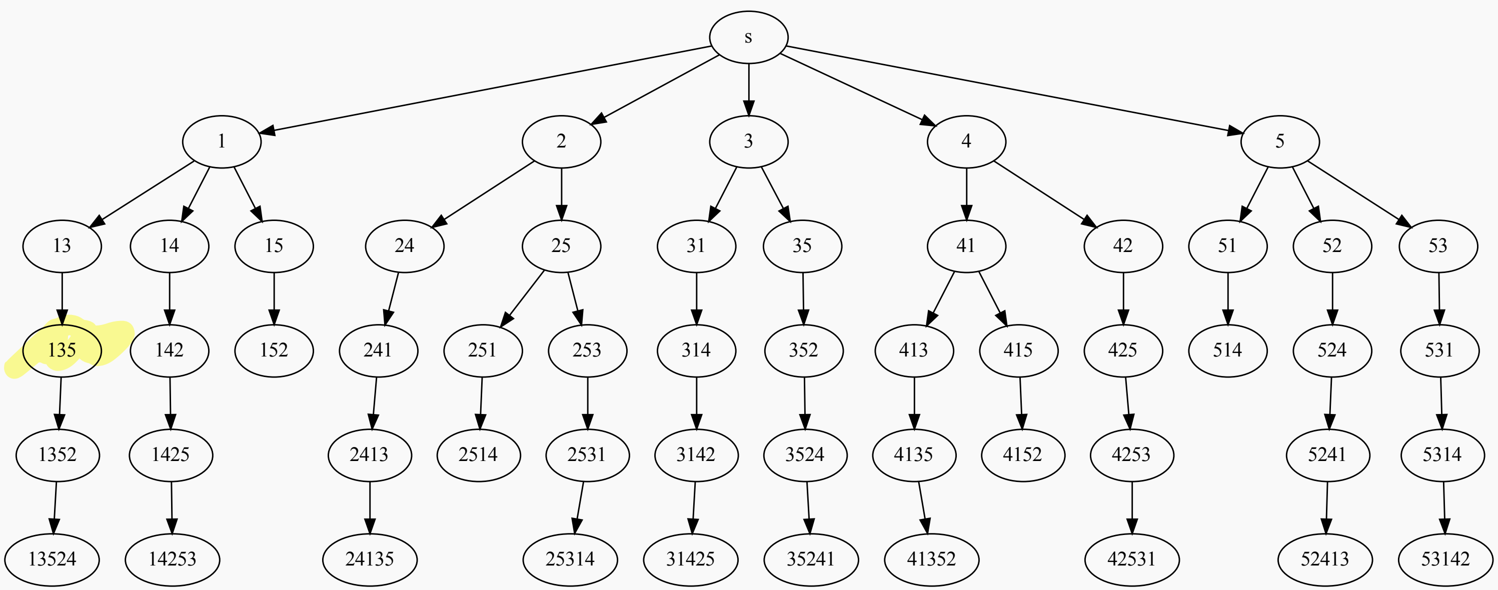
Let's be a bit smarter and recognize that:

- Queens can't be on the same row, column or diagonal
- Can have *n* queens max.

Search tree for 5 queens







Backtracking: Informal definition

Recursive search over an implicit tree, where we “backtrack” if certain possibilities do not work.

n queens C++ code

```
void  generate_permutations( int  * permut, int  row, int  n )
{
   if  ( row == n ) {
      print_board( permut, n );
      return;
   }

   for  ( int  val = 1; val <= n; val++ )
      if  ( isValid( permut, row, val ) ) {
         permut[ row ] = val;
         generate_permutations( permut, row + 1, n );
      }
}

generate_permutations( permut, 0, 8 );
```

Quick note: n queens - number of solutions

N	Number of Solutions	Number of Unique Solutions
1	1	1
2	0	0
3	0	0
4	2	1
5	10	2
6	4	1
7	40	6
8	92	12
9	352	46
10	724	92
11	2,680	341
12	14,200	1,787
13	73,712	9,233
14	365,596	45,752
15	2,279,184	285,053

Longest Increasing Sub-sequence

Sequences

Definition
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

Definition
$a_{i_1}, \ldots, a_{i_k}$ is a subsequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

1 2 3 4 5 6
1 2 3 4 5 6
2 4 5 .
1 3 6

Definition
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Sequences - Example...

Example

- Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
- Subsequence of above sequence: $5, 2, 1$
- Increasing sequence: $3, 5, 9, 17, 54$
- Decreasing sequence: $34, 21, 7, 5, 1$
- Increasing subsequence of the first sequence: $2, 7, 9$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

6,

Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

algLISNaive($A[1..n]$):
  $max = 0$
  for each subsequence B of A **do** ← 2^n subsequences
    if B is increasing and $|B| > max$ **then** n
      $max = |B|$

  Output max

Running time: $O(n 2^n)$

Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

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        if B is increasing and |B| > max then
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    Output max
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Recursive Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

A = 6 3 5 2 7 8 1

LIS($A[1..n]$):

{6 3 . . . . . .}

Recursive Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[1..n]$):

- Case 1: Does not contain $A[n]$ in which case LIS($A[1..n]$) = LIS($A[1..(n-1)]$)
- Case 2: contains $A[n]$ in which case LIS($A[1..n]$) is

Recursive Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

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- Case 1: Does not contain $A[n]$ in which case LIS($A[1..n]$) $=$ LIS($A[1..(n-1)]$)
- Case 2: contains $A[n]$ in which case LIS($A[1..n]$) is not so clear.

Recursive Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

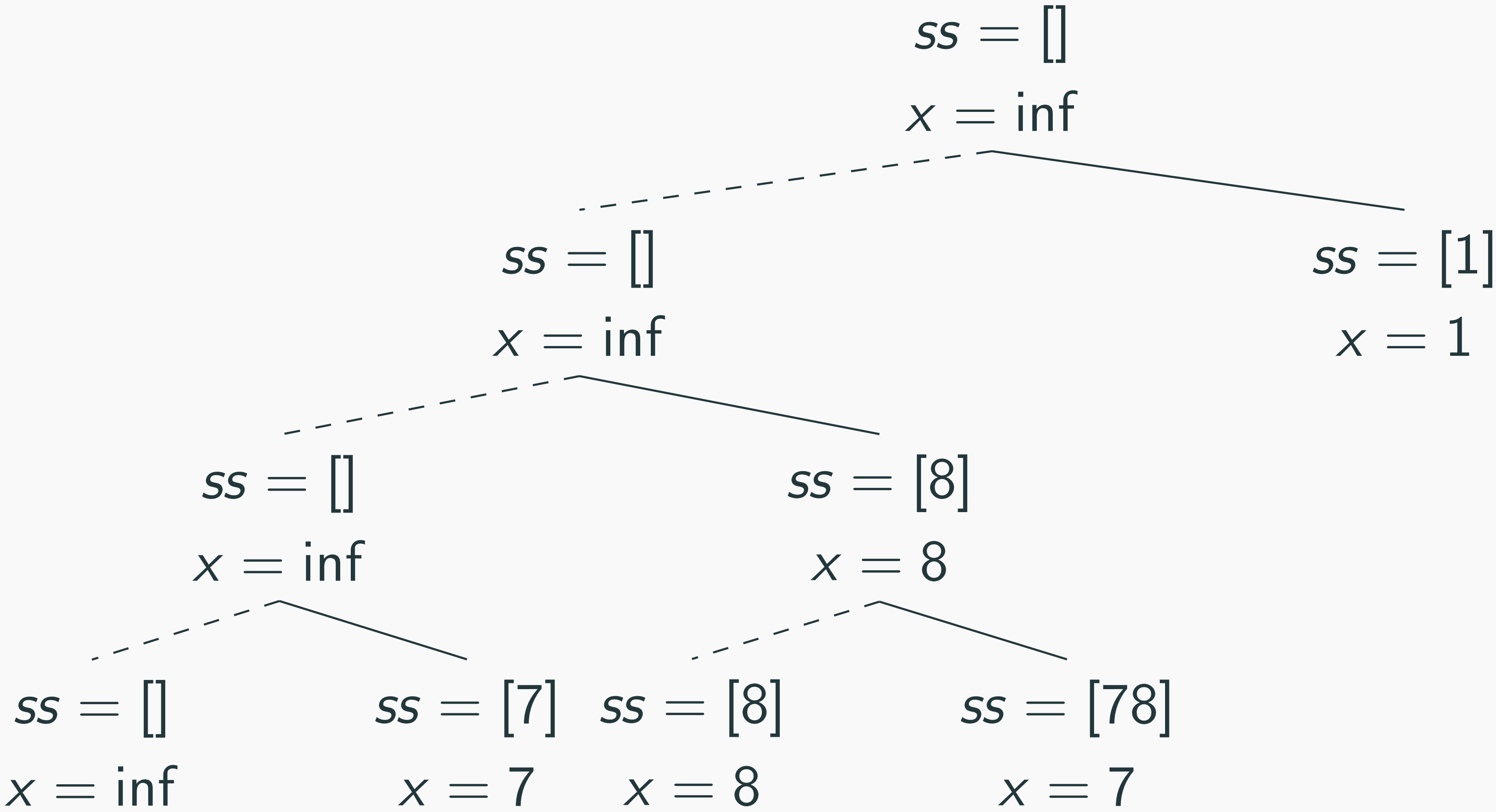
LIS($A[1..n]$):

- Case 1: Does not contain $A[n]$ in which case LIS($A[1..n]$) = LIS($A[1..(n-1)]$)
- Case 2: contains $A[n]$ in which case LIS($A[1..n]$) is not so clear.

Observation
For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS_smaller**($A[1..n], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Example

Sequence: $A[1..5] = 5, 9, 7, 8, 1$



Recursive Approach

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
```

```
LIS(A[1..n]):
        return LIS_smaller(A[1..n], ∞)
```

Running time analysis

Running time of LIS([1..n])

LIS_smaller$(A[1..n], x)$:
  if $(n = 0)$ **then return** 0
  $m =$ **LIS_smaller**$(A[1..(n-1)], x)$
  if $(A[n] < x)$ **then**
    $m = max(m, 1 +$ **LIS_smaller**$(A[1..(n-1)], A[n]))$
  `Output` m

LIS$(A[1..n])$:
  return LIS_smaller$(A[1..n], \infty)$

A = [1, 2, 3, 4, 5, 6]

Running time of LIS([1..n])

Lemma
LIS_smaller *runs in* $O(2^n)$ *time.*

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Improvement: From $O(n2^n)$ to $O(2^n)$.

....one can do much better using memorization!