Pre-lecture brain teaser

Write a (very simple) recursive algorithm that calcuates the Fibonnacci n^{th} number.

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = 0, F_1 = 1$$

ECE-374-B: Lecture 13 - Dynamic Programming I

Instructor: Nickvash Kani

March 02, 2023

University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

Write a (very simple) recursive algorithm that calcuates the Fibonnacci n^{th} number.

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = 0, F_1 = 1$$

Recursion and Memoization

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting properties. A journal The Fibonacci Quarterly[1]!

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting properties. A journal The Fibonacci Quarterly[1]!

- Binet's formula: $F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \approx \frac{1.618^n - (-0.618)^n}{\sqrt{5}} \approx \frac{1.618^n}{\sqrt{5}}$
  φ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
- $\lim_{n\to\infty} F(n+1)/F(n) = \varphi$

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

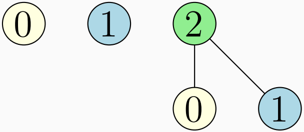
$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$: $T(n) = \Theta(\varphi^n)$.

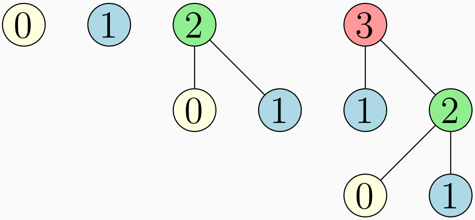
The number of additions is exponential in n. Can we do better?

Recursion tree for the Recursive Fibonacci

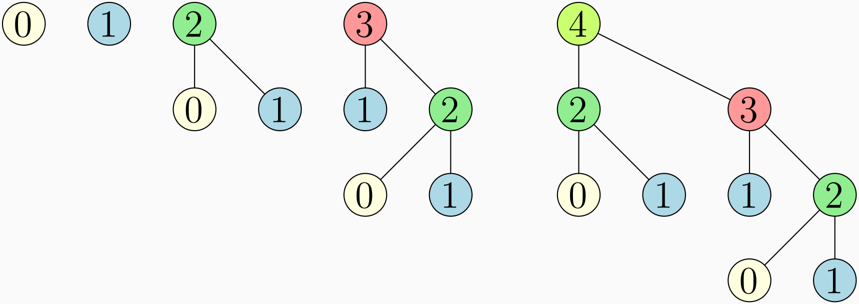
Recursion tree for the Recursive Fibonacci



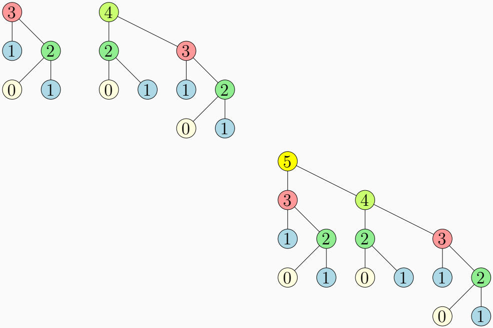
Recursion tree for the Recursive Fibonacci



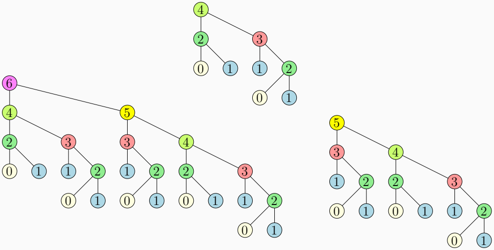
Recursion tree for the Recursive Fibonacci



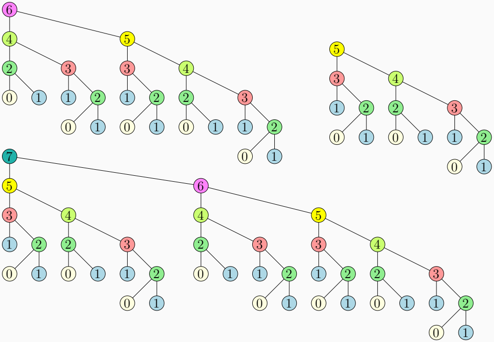
Recursion tree for the Recursive Fibonacci



Recursion tree for the Recursive Fibonacci



Recursion tree for the Recursive Fibonacci



An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

What is the running time of the algorithm?

An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

What is the running time of the algorithm? $O(n)$ additions.

What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value.

What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

Dynamic Programming: Finding a recursion that can be effectively/efficiently memorized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

Automatic/implicit memorization

Automatic Memorization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

Automatic Memorization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n − 1) + Fib(n − 2)
```

Automatic Memorization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n − 1) + Fib(n − 2)
```

How do we keep track of previously computed values?

Automatic Memorization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n − 1) + Fib(n − 2)
```

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure)

Automatic implicit memorization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (n is already in D)
        return value stored with n in D
    val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Use hash-table or a map to remember which values were already computed.

Explicit memorization (not automatic)

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \dots, n$.

Explicit memorization (not automatic)

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.
- Resulting code:

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (M[n] ≠ −1) // M[n]: stored value of Fib(n)
        return M[n]
    M[n] ⇐ Fib(n − 1) + Fib(n − 2)
    return M[n]
```

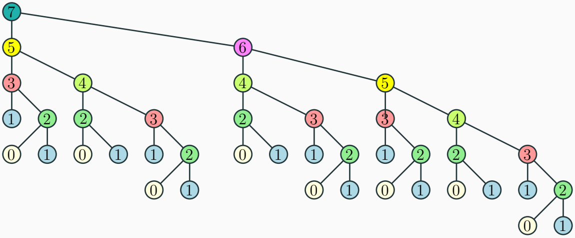
Explicit memorization (not automatic)

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.
- Resulting code:

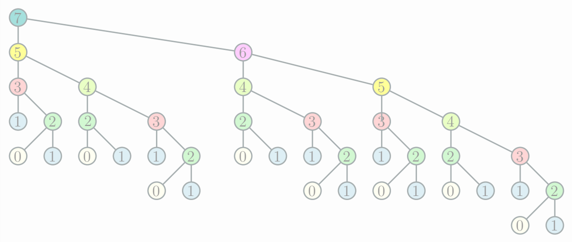
```
Fib(n):
        if (n = 0)
            return 0
        if (n = 1)
            return 1
        if (M[n] ≠ −1) // M[n]: stored value of Fib(n)
            return M[n]
        M[n] ⇐ Fib(n − 1) + Fib(n − 2)
        return M[n]
```

- Need to know upfront the number of sub-problems to allocate memory.

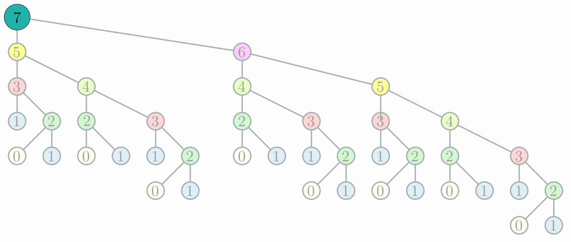
Recursion tree for the memorized Fib...



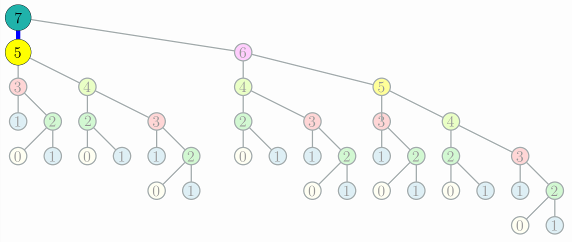
Recursion tree for the memorized Fib...



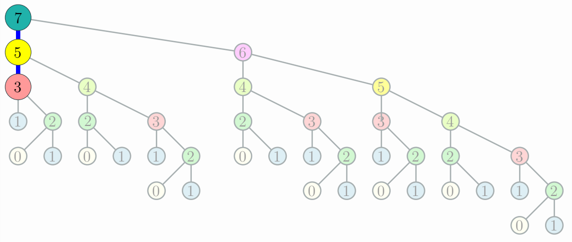
Recursion tree for the memorized Fib...



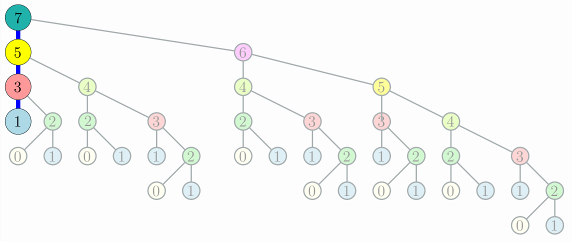
Recursion tree for the memorized Fib...



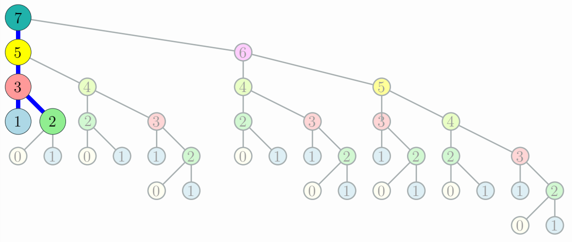
Recursion tree for the memorized Fib...



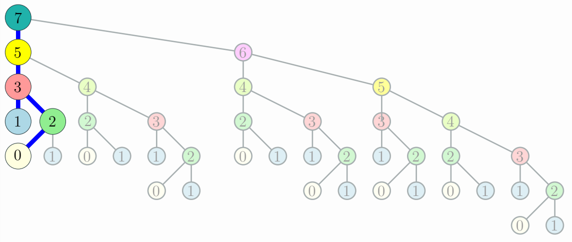
Recursion tree for the memorized Fib...



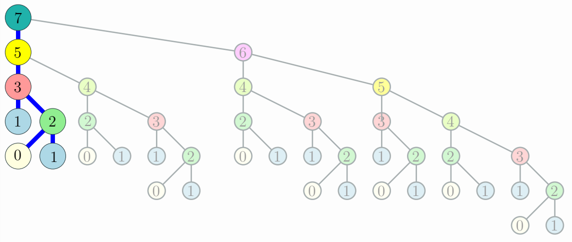
Recursion tree for the memorized Fib...



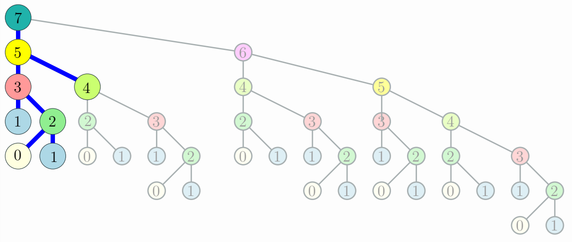
Recursion tree for the memorized Fib...



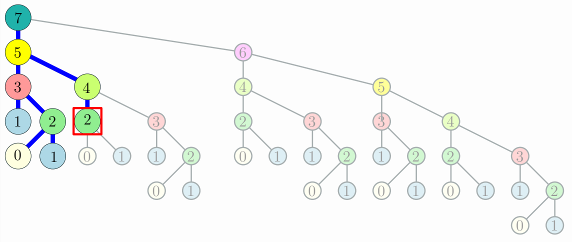
Recursion tree for the memorized Fib...



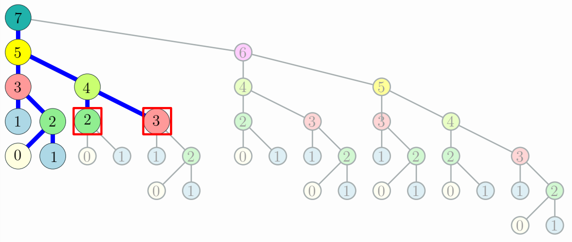
Recursion tree for the memorized Fib...



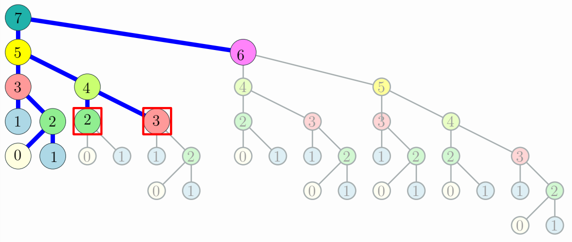
Recursion tree for the memorized Fib...



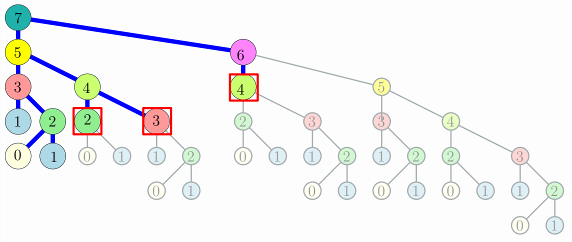
Recursion tree for the memorized Fib...



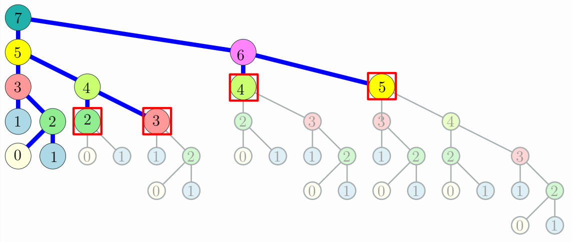
Recursion tree for the memorized Fib...



Recursion tree for the memorized Fib...



Recursion tree for the memorized Fib...



Automatic (Implicit) Memorization

- Recursive version:

```
f(x_1, x_2, ..., x_d):
        CODE
```

- Recursive version with memoization:

```
g(x_1, x_2, ..., x_d):
        if f already computed for (x_1, x_2, ..., x_d) then
            return value already computed
        NEW_CODE
```

- NEW_CODE:
  - Replaces any "**return** α" with
  - Remember "$f(x_1, \ldots, x_d) = \alpha$"; **return** α.

Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
  - analyze problem ahead of time

Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.

Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.
- Implicit (automatic) memoization:
  - problem structure or algorithm is not well understood.

Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.
- Implicit (automatic) memoization:
  - problem structure or algorithm is not well understood.
  - Need to pay overhead of data-structure.

Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.
- Implicit (automatic) memoization:
  - problem structure or algorithm is not well understood.
  - Need to pay overhead of data-structure.
  - Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.

Explicit/implicit memorization for Fibonacci

```
Init: M[i] = −1, i = 0, . . . , n.

Fib(k):
    if (k = 0)
        return 0
    if (k = 1)
        return 1
    if (M[k] ≠ −1)
        return M[n]
    M[k] ⇐ Fib(k − 1) + Fib(k − 2)
    return M[k]
```

Explicit memorization

```
Init: Init dictionary D

Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (n is already in D)
        return value stored with n in D
        val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Implicit memorization

Dynamic programming

Removing the recursion by filling the table in the right order

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (M[n] ≠ −1)
        return M[n]
    M[n] ⇐ Fib(n − 1) + Fib(n − 2)
    return M[n]
```

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

Dynamic programming: Saving space!

Saving space. Do we need an array of n numbers? Not really.

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n do
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp

    return prev1
```

Dynamic programming – quick review

Dynamic Programming is smart recursion

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memorization

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memorization

+ filling the table in right order

+ removing recursion.

Analyzing memorized recursive function

Suppose we have a recursive program $foo(x)$ that takes an input x.

- On input of size n the number of distinct sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

Analyzing memorized recursive function

Suppose we have a recursive program *foo*(x) that takes an input x.

- On input of size n the number of distinct sub-problems that *foo*(x) generates is at most $A(n)$
- *foo*(x) spends at most $B(n)$ time not counting the time for its recursive calls.

Suppose we memorize the recursion.
Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Analyzing memorized recursive function

Suppose we have a recursive program $foo(x)$ that takes an input x.

- On input of size n the number of distinct sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

Suppose we memorize the recursion.
Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of memorized version of $foo(x)$ if $|x| = n$?

Analyzing memorized recursive function

Suppose we have a recursive program $foo(x)$ that takes an input x.

- On input of size n the number of distinct sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

Suppose we memorize the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of memorized version of $foo(x)$ if $|x| = n$? $O(A(n)B(n))$.

Fibonacci numbers are big – corrected running time analysis

Back to Fibonacci Numbers

T Is the iterative algorithm a polynomial time algorithm? Does it take $O(n)$ time?

- input is n and hence input size is $\Theta(\log n)$
- output is $F(n)$ and output size is $\Theta(n)$. Why?
- Hence output size is exponential in input size so no polynomial time algorithm possible!
- Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?

Longest Increasing Sub-sequence Revisited

Sequences

Definition
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

Definition
$a_{i_1}, \ldots, a_{i_k}$ is a sub-sequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Sequences - Example...

Example

- Sequence: $6, 3, 5, 2, 7, 8, 1$
- Subsequence of above sequence: $5, 2, 1$
- Increasing sequence: $3, 5, 9, 17, 54$
- Decreasing sequence: $34, 21, 7, 5, 1$
- Increasing subsequence of the first sequence: $2, 7, 8$.
- *Longest* Increasing subsequence of the first sequence: $3, 5, 7, 8$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

Goal Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

Longest Increasing Subsequence Problem

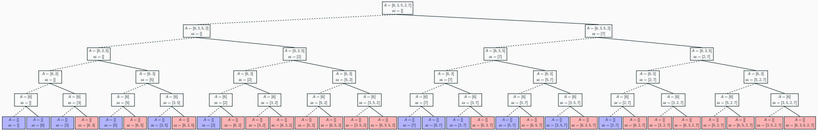
Input A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

Goal Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

Naive Recursion Enumeration - State Tree



- This is just for [6,3,5,2,7]! (Tikz won't print larger trees)
- How many leafs are there for the full [6,3,5,2,7, 8, 1] sequence
- What is the running time?

Naive Recursion Enumeration - Code

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Backtracking Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[0..n-1]$):

Backtracking Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

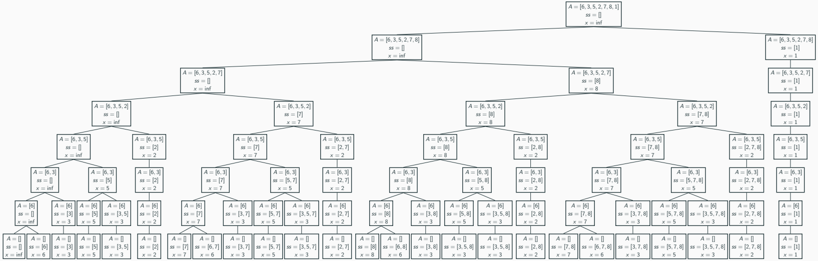
LIS($A[0..n-1]$):

- Case 1: Does not contain $A[n-1]$ in which case LIS($A[0..n-1]$) = LIS($A[0..(n-1)]$)
- Case 2: contains $A[n-1]$ in which case LIS($A[0..n-1]$) is not so clear.

Observation
For second case we want to find a subsequence in $A[1..(n-2)]$ that is restricted to numbers less than $A[n-1]$. This suggests that a more general problem is **LIS_smaller**($A[0..n-1], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Example

Sequence: $A[0..6] = 6, 3, 5, 2, 7, 8, 1$



Recursive Approach

$LIS(A[1..n])$: the length of longest increasing subsequence in A

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

Recursive Approach

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate?

Recursive Approach

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$

Recursive Approach

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memorize recursion?

Recursive Approach

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memorize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.

Recursive Approach

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memorize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.
- How much space for memorization?

Recursive Approach

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memorize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.
- How much space for memorization? $O(n^2)$

Naming sub-problems and recursive equation

After seeing that number of sub-problems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

$LIS(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

Naming sub-problems and recursive equation

After seeing that number of sub-problems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n + 1$)

$LIS(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

Base case: $LIS(0, j) = 0$ for $1 \leq j \leq n + 1$
Recursive relation:

- $LIS(i, j) = LIS(i - 1, j)$ if $A[i] \geq A[j]$
- $LIS(i, j) = \max\{LIS(i - 1, j), 1 + LIS(i - 1, i)\}$ if $A[i] < A[j]$

Output: $LIS(n, n + 1)$.

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0									
[6]	1									
[6,3]	2									
[6,3,5]	3									
[6,3,5,2]	4									
[6,3,5,2,7]	5									
[6,3,5,2,7,8]	6									
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Sequence:

$$A[1\dots 7] = [6,3,5,2,7,8,1]$$

Recursive relation:

$$LIS(i,j) = \begin{cases} 0 & i = 0 \\ LIS(i-1,j) & A[i] \geq A[j] \\ \max\begin{cases} LIS(i-1,j) \\ 1 + LIS(i-1,i) \end{cases} & A[i] < A[j] \end{cases}$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1									
[6,3]	2									
[6,3,5]	3									
[6,3,5,2]	4									
[6,3,5,2,7]	5									
[6,3,5,2,7,8]	6									
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Sequence:

$$A[1\dots7] = [6,3,5,2,7,8,1]$$

Recursive relation:

$$LIS(i,j) = \begin{cases} 0 & i = 0 \\ LIS(i-1,j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1,j) \\ 1 + LIS(i-1,i) \end{cases} & A[i] < A[j] \end{cases}$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2									
[6,3,5]	3									
[6,3,5,2]	4									
[6,3,5,2,7]	5									
[6,3,5,2,7,8]	6									
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Sequence:

$$A[1\ldots7] = [6,3,5,2,7,8,1]$$

Recursive relation:

$$LIS(i,j) =$$

$$\begin{cases} 0 & i = 0 \\ LIS(i-1,j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1,j) \\ 1 + LIS(i-1,i) \end{cases} & A[i] < A[j] \end{cases}$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3									
[6,3,5,2]	4									
[6,3,5,2,7]	5									
[6,3,5,2,7,8]	6									
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Sequence:

$$A[1 \dots 7] = [6, 3, 5, 2, 7, 8, 1]$$

Recursive relation:

$$LIS(i, j) = \begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4									
[6,3,5,2,7]	5									
[6,3,5,2,7,8]	6									
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Recursive relation:

Sequence:

$$A[1 \dots 7] = [6, 3, 5, 2, 7, 8, 1]$$

$$LIS(i, j) = \begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5									
[6,3,5,2,7,8]	6									
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Recursive relation:

Sequence:

$$A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1]$$

$$LIS(i, j) = \begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5						3	0	3	
[6,3,5,2,7,8]	6									
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Recursive relation:

$$LIS(i,j) =$$

$$\begin{cases} 0 & i = 0 \\ LIS(i-1,j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1,j) \\ 1 + LIS(i-1,i) \end{cases} & A[i] < A[j] \end{cases}$$

Sequence:

$$A[1\ldots7] = [6,3,5,2,7,8,1]$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5						3	0	3	
[6,3,5,2,7,8]	6							0	4	
[6,3,5,2,7,8,1]	7									
Represents sub-array	i									

Sequence:

$$A[1\dots 7] = [6, 3, 5, 2, 7, 8, 1]$$

Recursive relation:

$$LIS(i, j) =$$

$$\begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5						3	0	3	
[6,3,5,2,7,8]	6							0	4	
[6,3,5,2,7,8,1]	7								4	
Represents sub-array	i									

Sequence:

$$A[1 \dots 7] = [6, 3, 5, 2, 7, 8, 1]$$

Recursive relation:

$$LIS(i, j) = \begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

Iterative algorithm

The dynamic program for longest increasing subsequence

LIS-Iterative($A[1..n]$):
    $A[n+1] = \infty$
    `int` $LIS[0..n-1, 0..n]$
    for $j = 0 \ldots n$) **if** `A[i]` $\leq$ `A[j]` **then** $LIS[0][j] = 1$

    for $i = 1 \ldots n-1$ **do**
        for $j = i \ldots n-1$ **do**
            if $(A[i] \geq A[j])$
                $LIS[i,j] = LIS[i-1,j]$
            else
                $LIS[i,j] = \max(LIS[i-1,j], 1 + LIS[i-1,i])$

    `Return` $LIS[n, n+1]$

Running time: $O(n^2)$
Space: $O(n^2)$

Iterative algorithm

The dynamic program for longest increasing subsequence

LIS-Iterative($A[1..n]$):
  $A[n+1] = \infty$
  `int` $LIS[0..n-1, 0..n]$
  for $j = 0 \ldots n$) **if** `A[i]` $\leq$ `A[j]` **then** $LIS[0][j] = 1$

  for $i = 1 \ldots n-1$ **do**
    for $j = i \ldots n-1$ **do**
      if $(A[i] \geq A[j])$
        $LIS[i,j] = LIS[i-1,j]$
      else
        $LIS[i,j] = \max(LIS[i-1,j], 1 + LIS[i-1,i])$

  `Return` $LIS[n, n+1]$

Running time: $O(n^2)$
Space: $O(n^2)$ Can be done in linear space. How?

Two comments

Question: Can we compute an optimum solution and not just its value?

Two comments

Question: Can we compute an optimum solution and not just its value?

Yes! See notes.

Finding the sub-sequence

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5						3	0	3	
[6,3,5,2,7,8]	6							0	4	
[6,3,5,2,7,8,1]	7								4	
Represents sub-array	i									

Sequence:

$$A[1 \dots 7] = [6, 3, 5, 2, 7, 8, 1]$$

We know the LIS length (4) but how do we find the LIS itself?

$$LIS = [3, 5, 7, 8]$$

Recursive relation:

$$LIS(i, j) = \begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

Finding the sub-sequence

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5						3	0	3	
[6,3,5,2,7,8]	6							0	4	
[6,3,5,2,7,8,1]	7								4	
Represents sub-array	i									

Sequence:

$$A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1]$$

We know the LIS length (4) but how do we find the LIS itself?

$$LIS = [3, 5, 7, 8]$$

Recursive relation:

$$LIS(i, j) = \begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

Two comments

Question: Can we compute an optimum solution and not just its value?
Yes!

Question: Is there a faster algorithm for LIS?

Two comments

Question: Can we compute an optimum solution and not just its value?
Yes!

Question: Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $O(n \log n)$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.

How to come up with dynamic programming algorithm: summary

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.

Dynamic Programming

- Find a "smart" recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.
- Come up with an explicit memorization algorithm for the problem.

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.
- Come up with an explicit memorization algorithm for the problem.
- Eliminate recursion and find an iterative algorithm.

Dynamic Programming

- Find a "smart" recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.
- Come up with an explicit memorization algorithm for the problem.
- Eliminate recursion and find an iterative algorithm.
- ...need to find the right way or order the sub-problems evaluation. Th is leads to an a dynamic programming algorithm.

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.
- Come up with an explicit memorization algorithm for the problem.
- Eliminate recursion and find an iterative algorithm.
- ...need to find the right way or order the sub-problems evaluation. Th is leads to an a dynamic programming algorithm.
- Optimize the resulting algorithm further

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.
- Come up with an explicit memorization algorithm for the problem.
- Eliminate recursion and find an iterative algorithm.
- ...need to find the right way or order the sub-problems evaluation. Th is leads to an a dynamic programming algorithm.
- Optimize the resulting algorithm further
- ...

Dynamic Programming

- Find a "smart" recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.
- Come up with an explicit memorization algorithm for the problem.
- Eliminate recursion and find an iterative algorithm.
- ...need to find the right way or order the sub-problems evaluation. Th is leads to an a dynamic programming algorithm.
- Optimize the resulting algorithm further
- ...
- Get rich!