Pre-lecture brain teaser

Write a (very simple) recursive algorithm that calcuates the Fibonnacci n^{th} number.

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = 0, F_1 = 1$$

ECE-374-B: Lecture 13 - Dynamic Programming I

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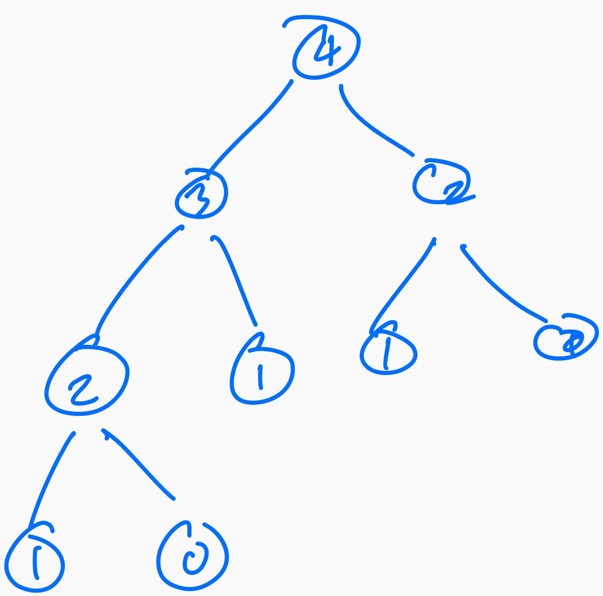
$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = 0, F_1 = 1$$

```
Fib(n)
    if n=0 || 1        //Base Case
        return n
    else
        return Fib(n-1) + Fib(n-2)
```



Recursion and Memoization

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting properties. A journal The Fibonacci Quarterly[1]!

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- Binet's formula: $F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \approx \frac{1.618^n - (-0.618)^n}{\sqrt{5}} \approx \frac{1.618^n}{\sqrt{5}}$
  φ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
- $\lim_{n\to\infty} F(n+1)/F(n) = \varphi$

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
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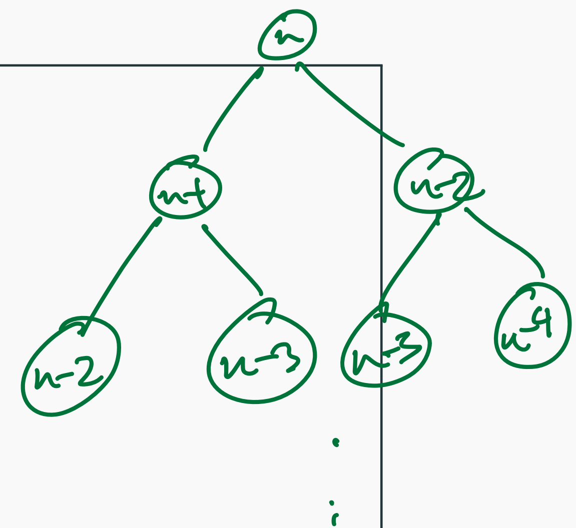

1

2

4

2^n

Running time? Let $T(n)$ be the number of additions in Fib(n). $T(0)=0$

$T(n) = T(n-1) + T(n-2) + O(1)$ $T(1)=1$

$= O(2^n)$

Fib = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, . . .

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

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Fib(n):
    if (n = 0)
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Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

$T(n) = T(n-1) + T(n-1) + 1 = O(2^n)$

Roughly same as $F(n)$: $T(n) = \Theta(\varphi^n)$. $\leq O(2^n)$

The number of additions is exponential in n. Can we do better?

Recursion tree for the Recursive Fibonacci

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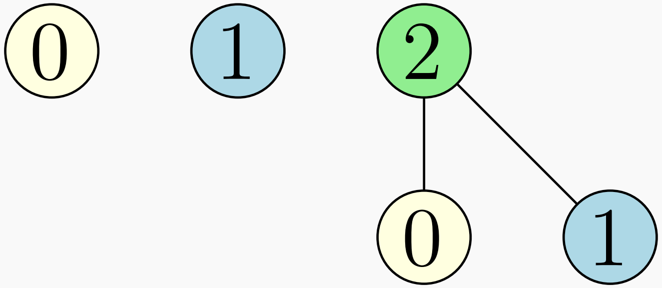

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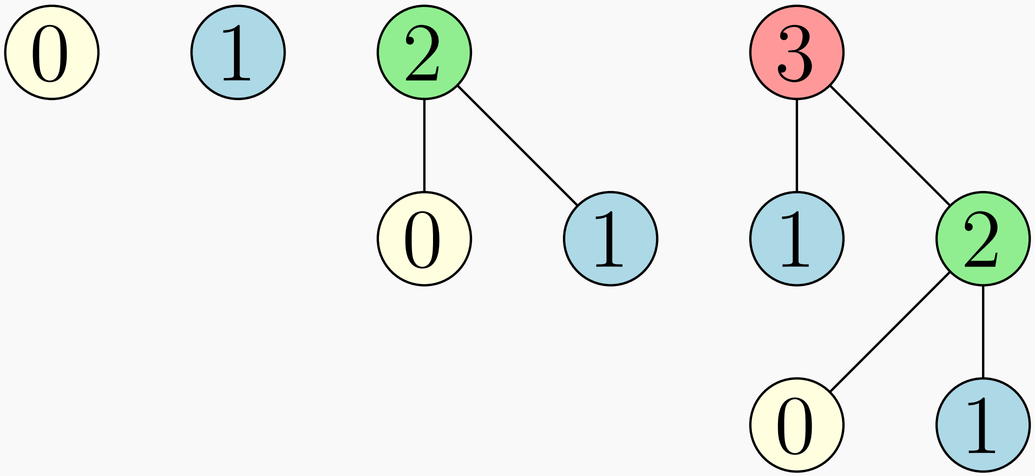

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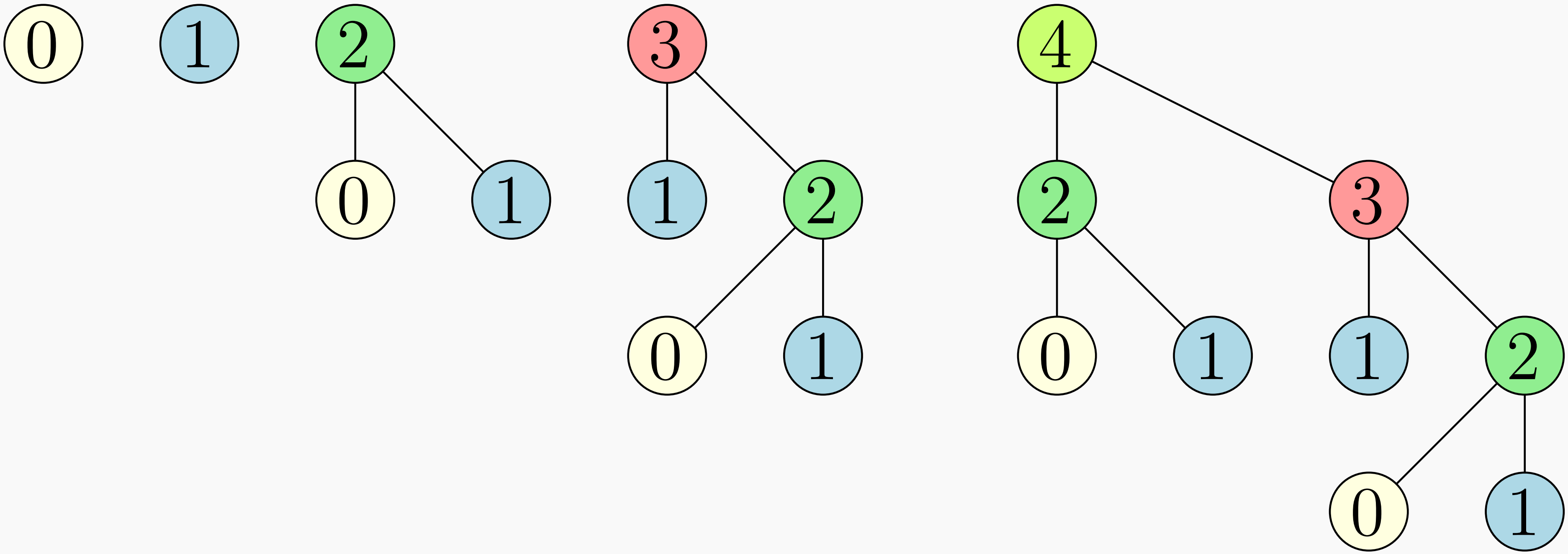

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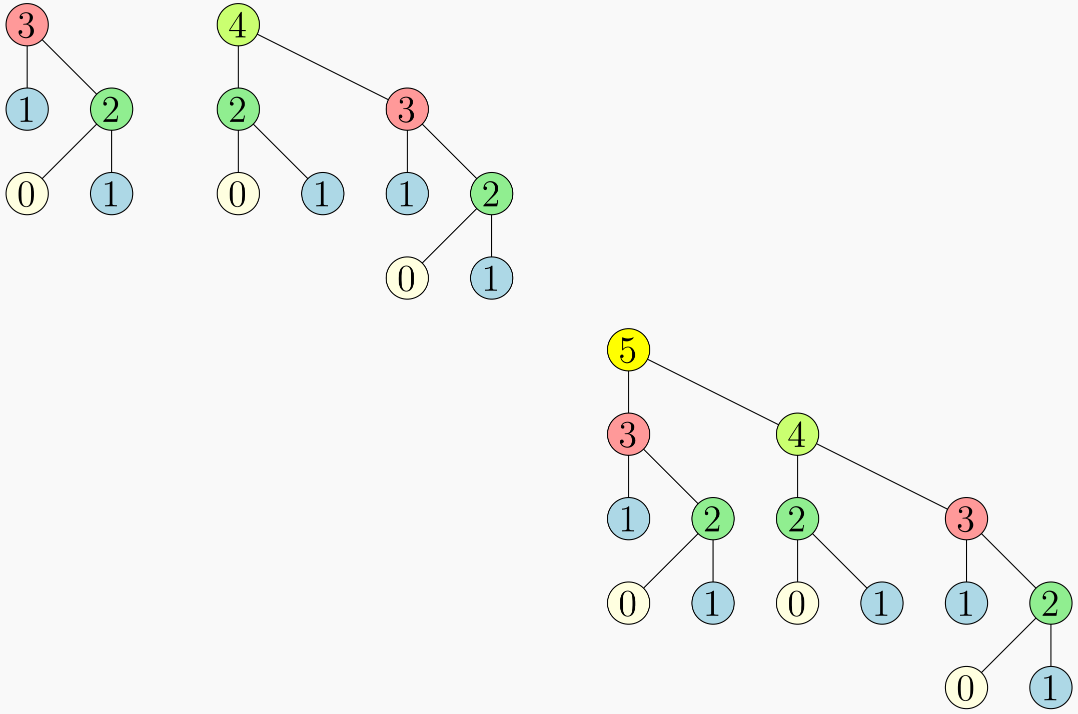

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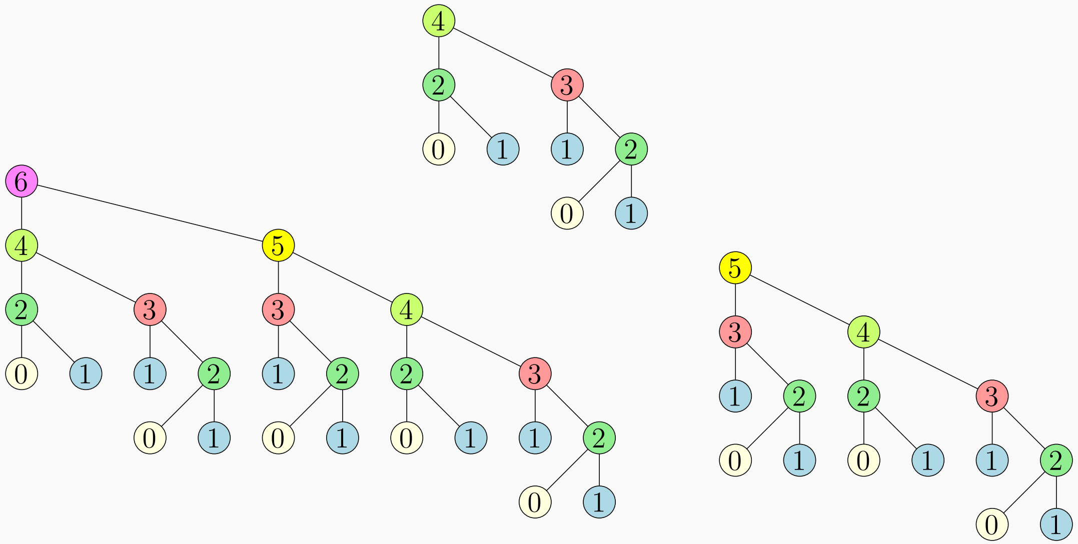

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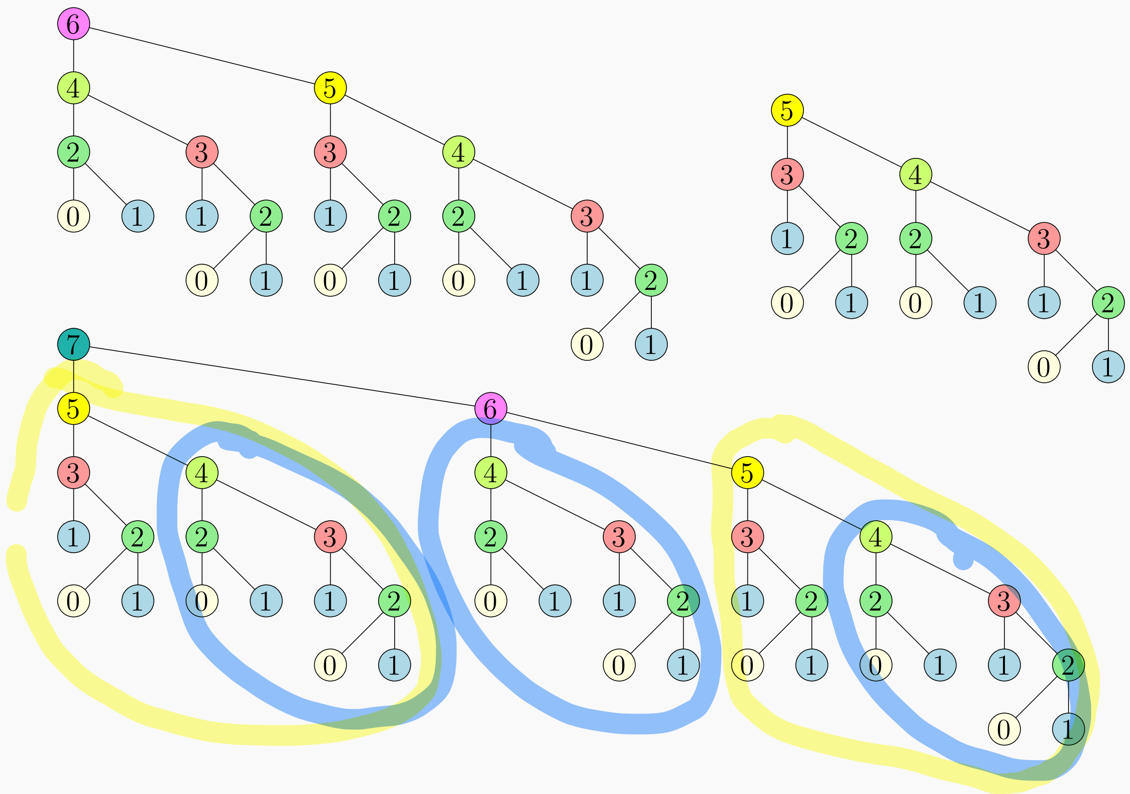

An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

F = [0, 1, 1, 2, ...]

An iterative algorithm for Fibonacci numbers

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    if (n = 0) then
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```

O(n) times

1 addition
O(1)

What is the running time of the algorithm?

$O(n) \cdot O(1) = O(n)$

An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
        return 0
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    F[0] = 0
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    return F[n]
```

What is the running time of the algorithm? $O(n)$ additions.

What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value.

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- Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

↳ when we store outputs of previous problem instances

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- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

Dynamic Programming: Finding a recursion that can be effectively/efficiently memorized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

Automatic/implicit memorization

Automatic Memorization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

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```
Fib(n):
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    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
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How do we keep track of previously computed values?

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        return stored value of Fib(n)
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        return Fib(n − 1) + Fib(n − 2)
```

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure)

Automatic implicit memorization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib(n):
        if (n = 0)
            return 0
        if (n = 1)
            return 1
        if (n is already in D)
            return value stored with n in D
        val ⇐ Fib(n − 1) + Fib(n − 2)
        Store (n, val) in D
        return val
```

Use hash-table or a map to remember which values were already computed.

Explicit memorization (not automatic)

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.

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- Resulting code:

```
Fib(n):
        if (n = 0)
            return 0
        if (n = 1)
            return 1
        if (M[n] ≠ −1) // M[n]: stored value of Fib(n)
            return M[n]
        M[n] ⇐ Fib(n − 1) + Fib(n − 2)
        return M[n]
```

Explicit memorization (not automatic)

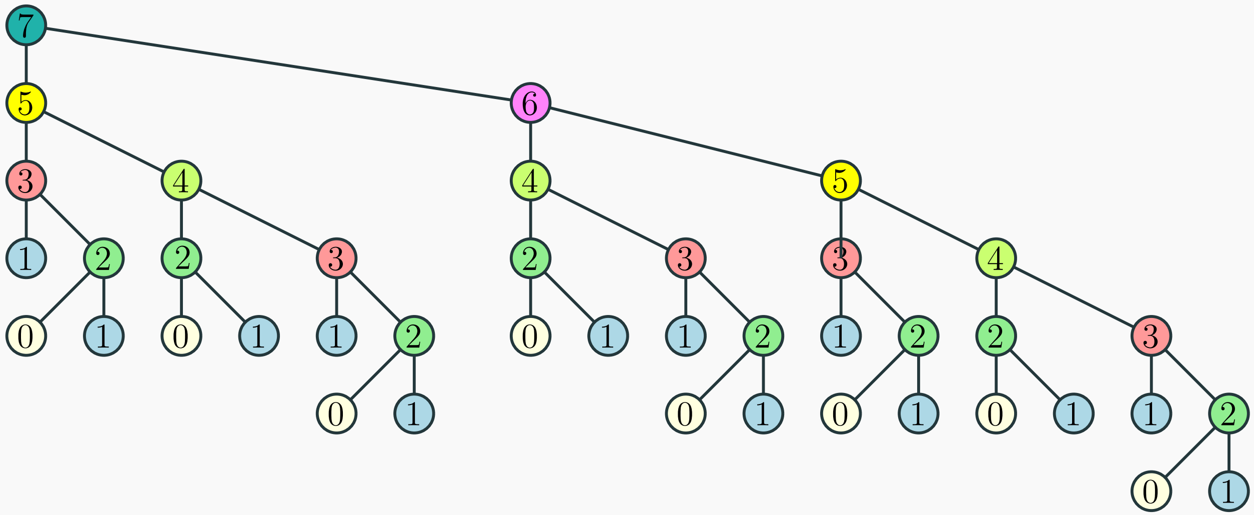
- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.
- Resulting code:

Fib(n):

> **if** ($n = 0$)
>
>   **return** 0
>
> **if** ($n = 1$)
>
>   **return** 1
>
> **if** ($M[n] \neq -1$) // $M[n]$: stored value of **Fib**(n)
>
>   **return** $M[n]$
>
> $M[n] \Leftarrow$ **Fib**$(n-1)$ + **Fib**$(n-2)$
>
> **return** $M[n]$

- Need to know upfront the number of sub-problems to allocate memory.

Recursion tree for the memorized Fib...



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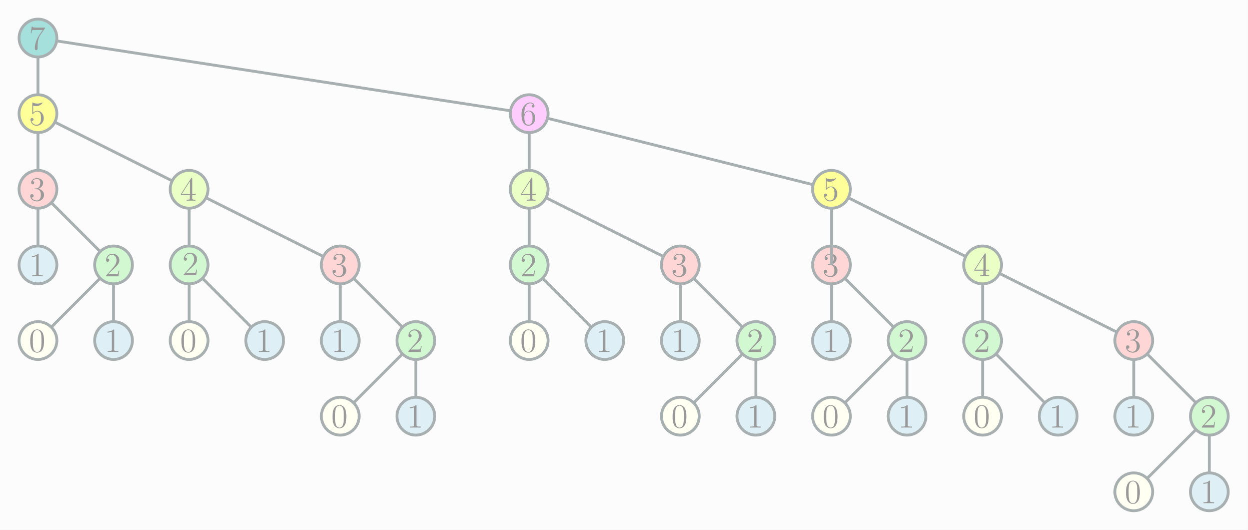

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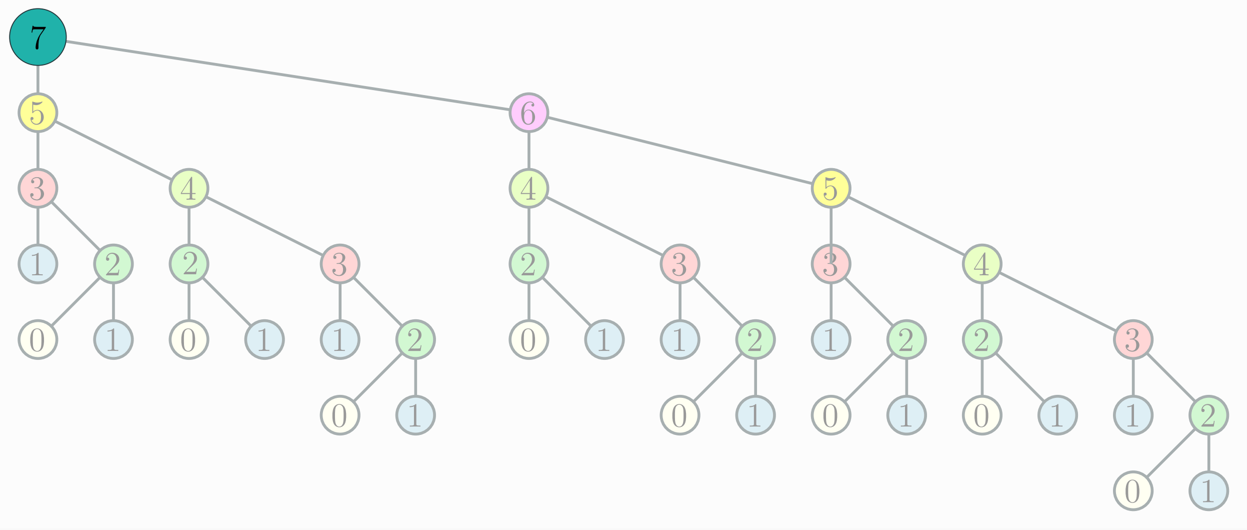

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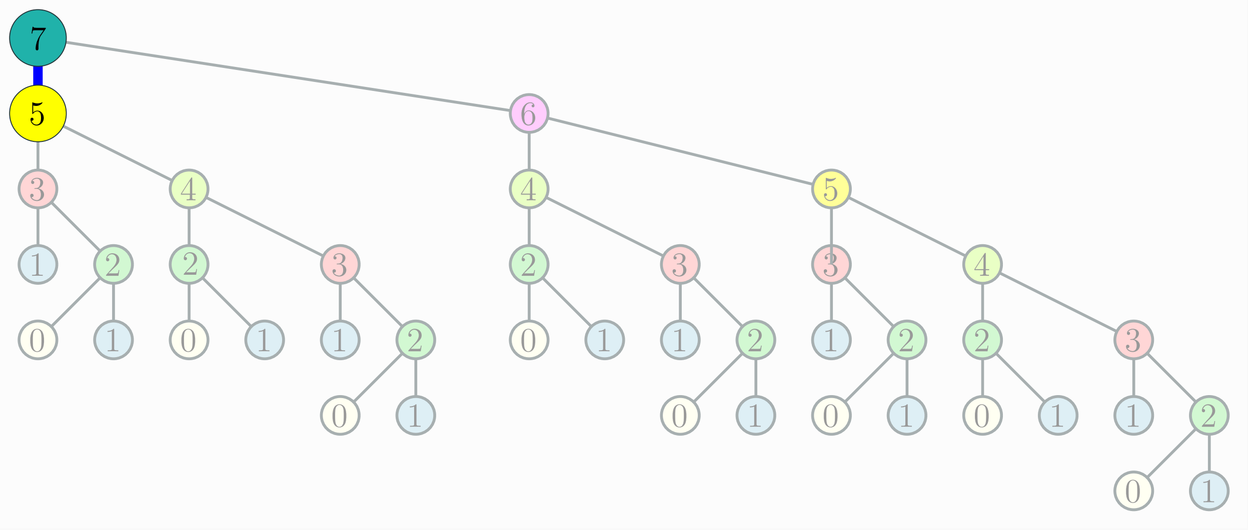

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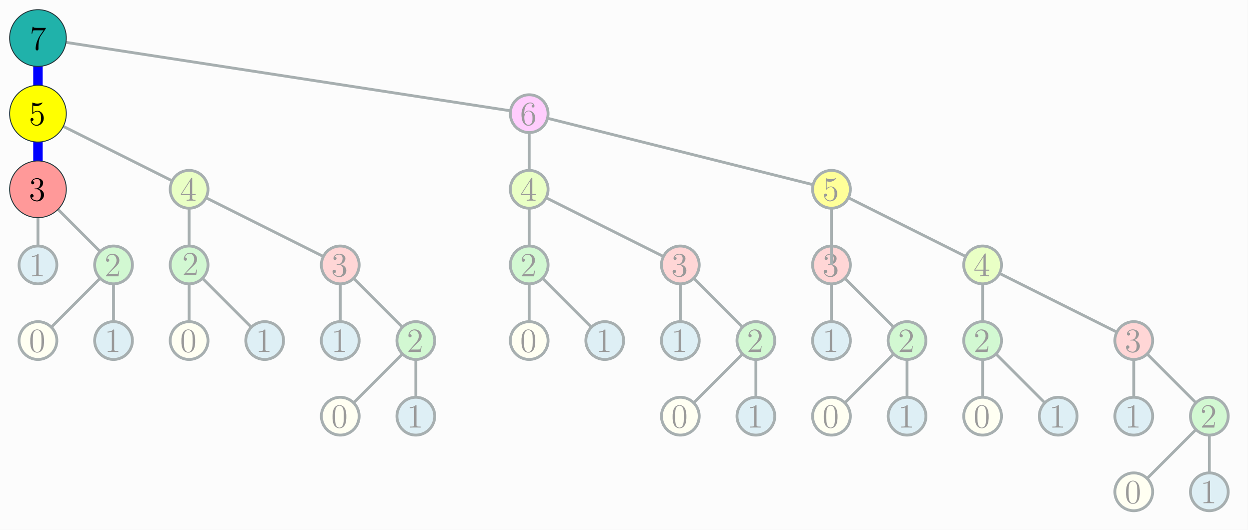

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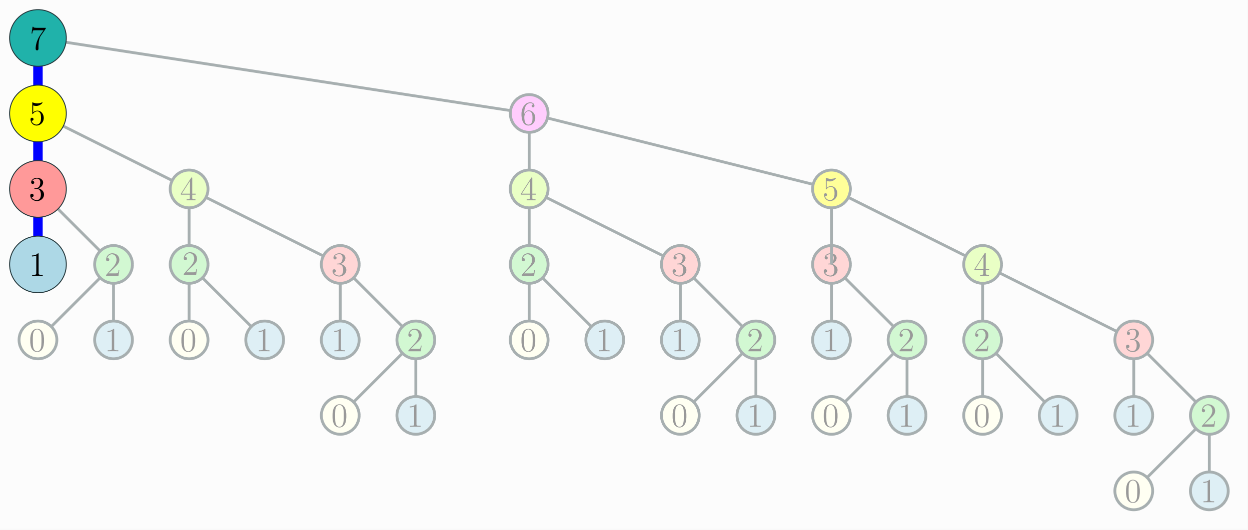

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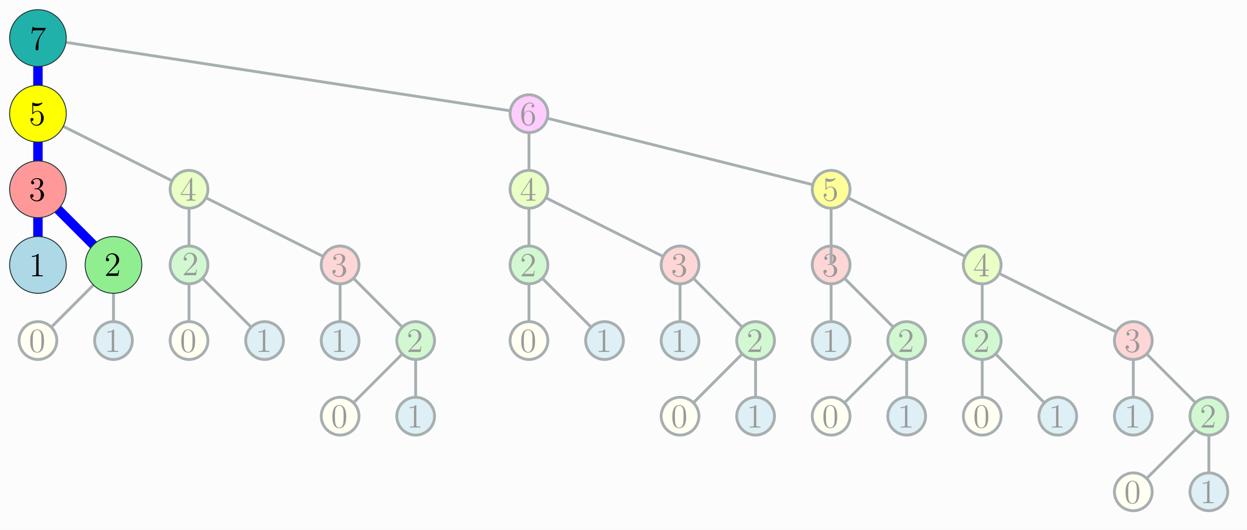

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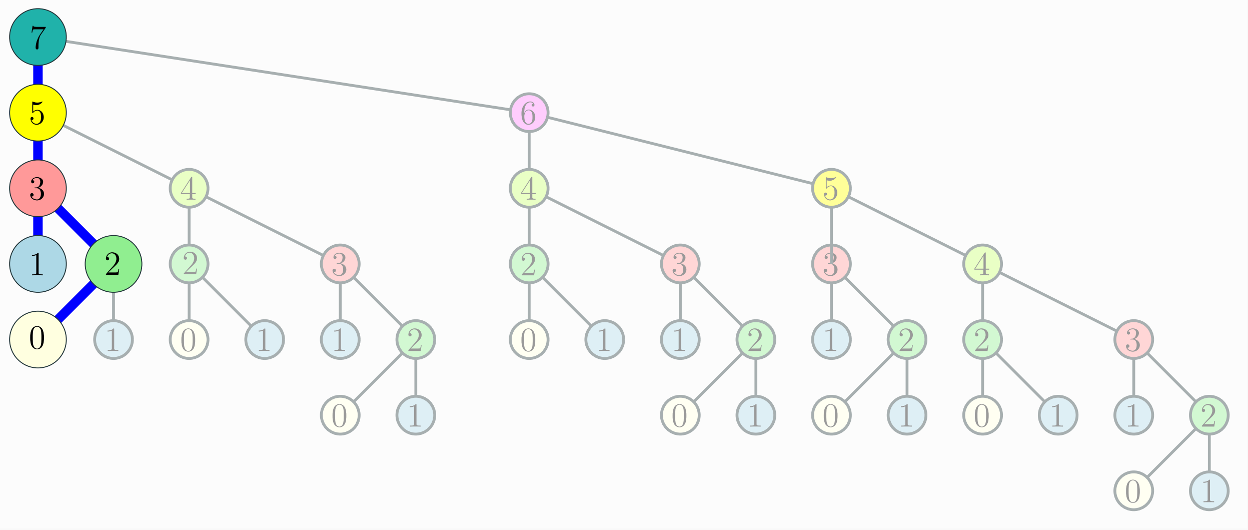

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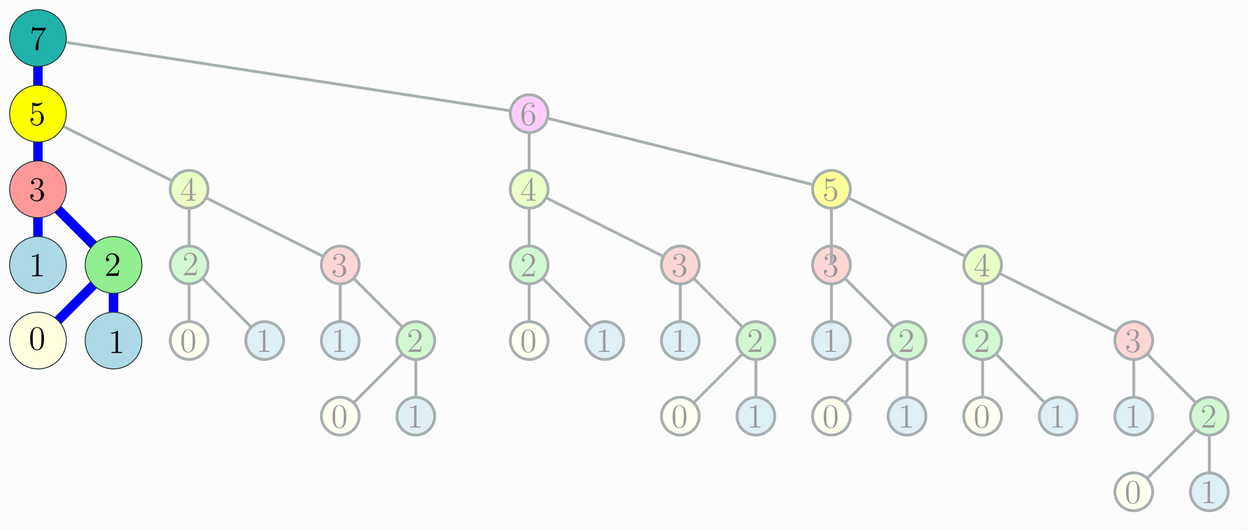

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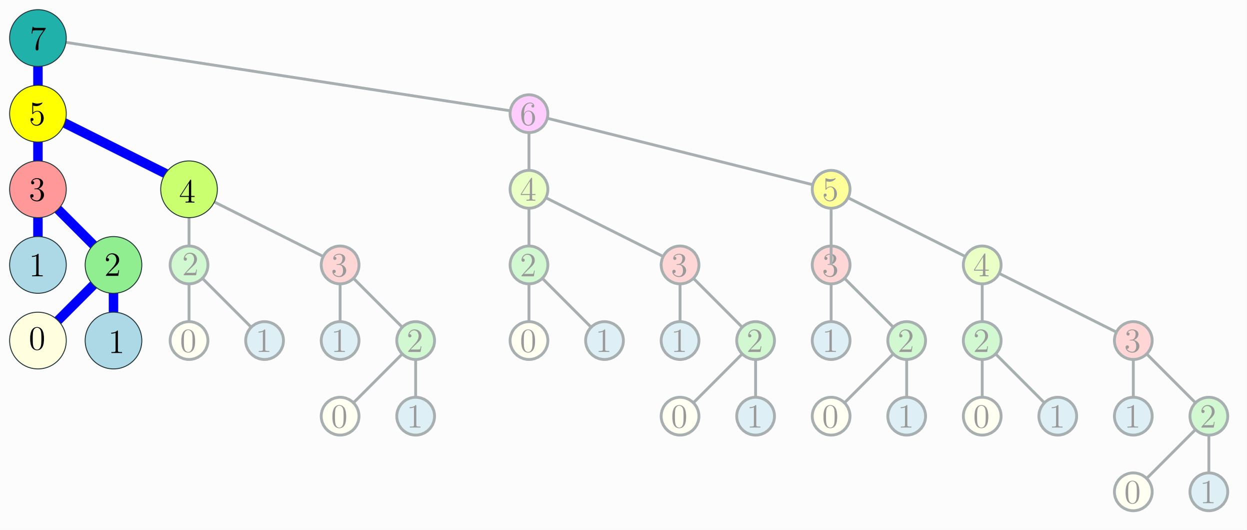

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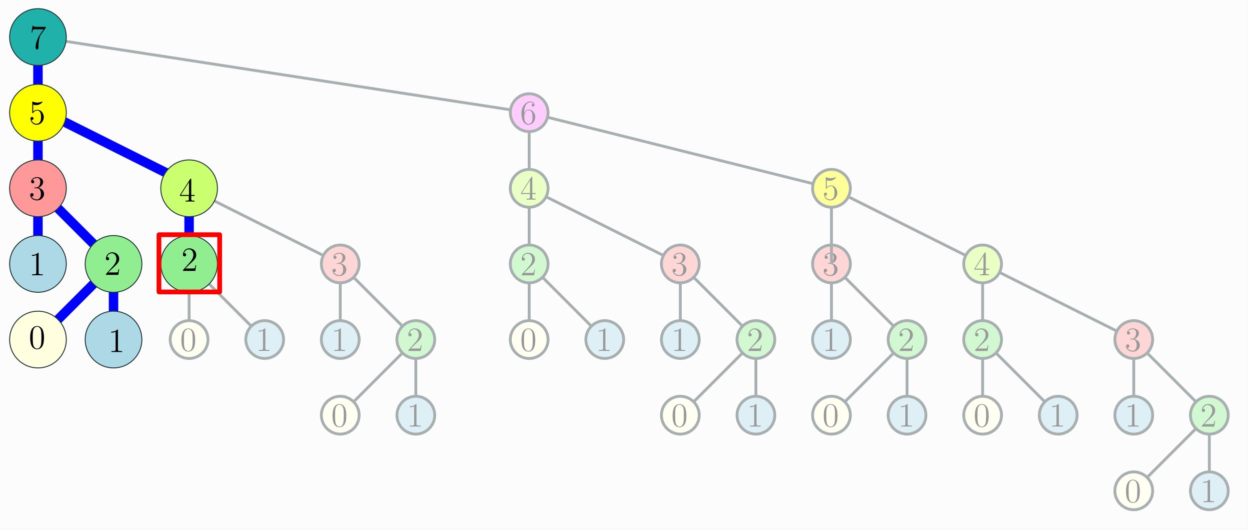

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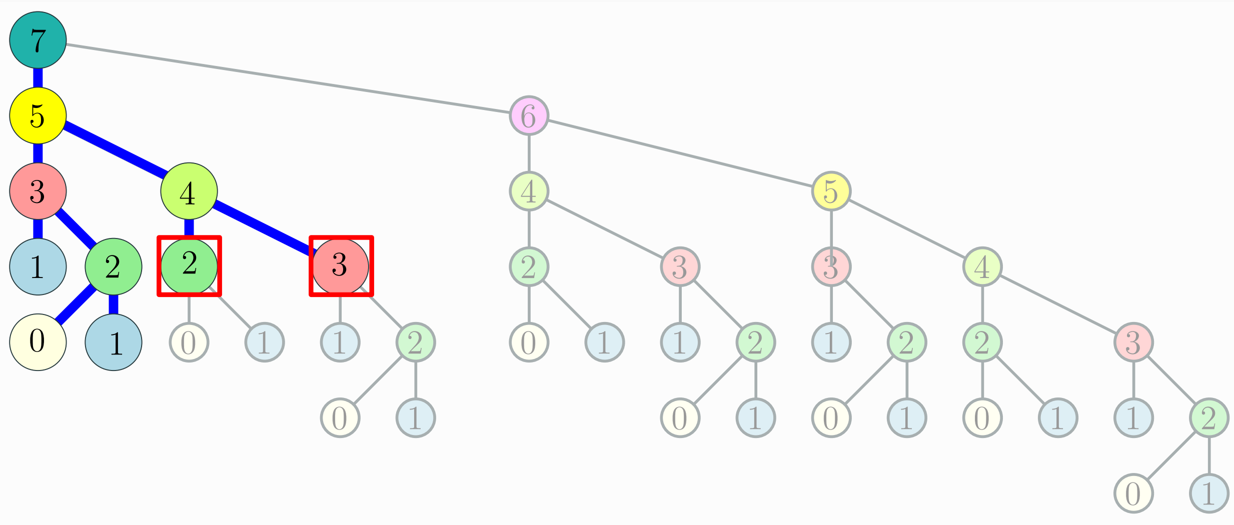

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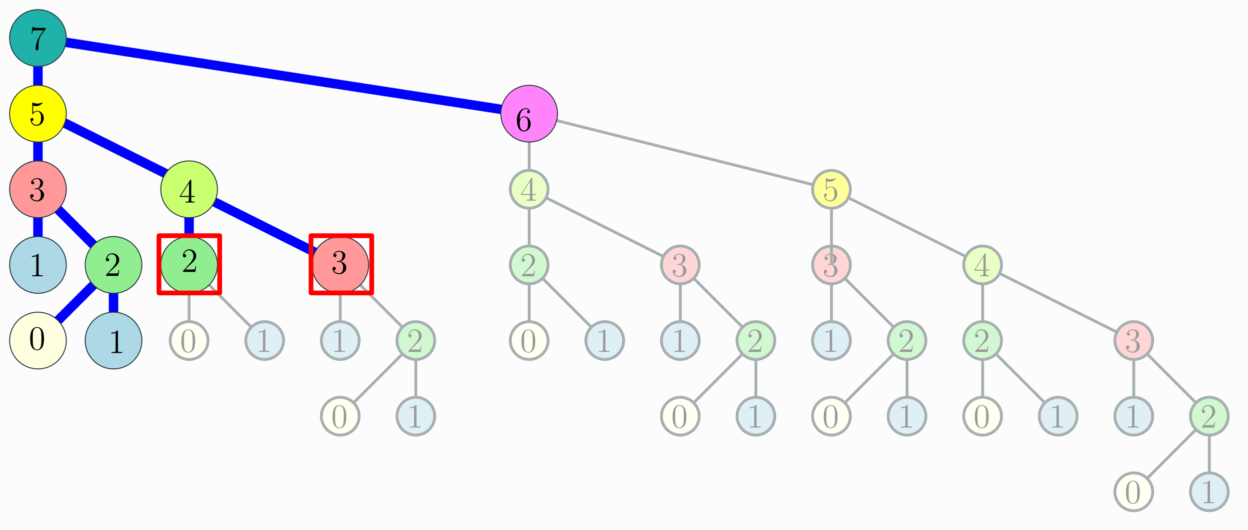

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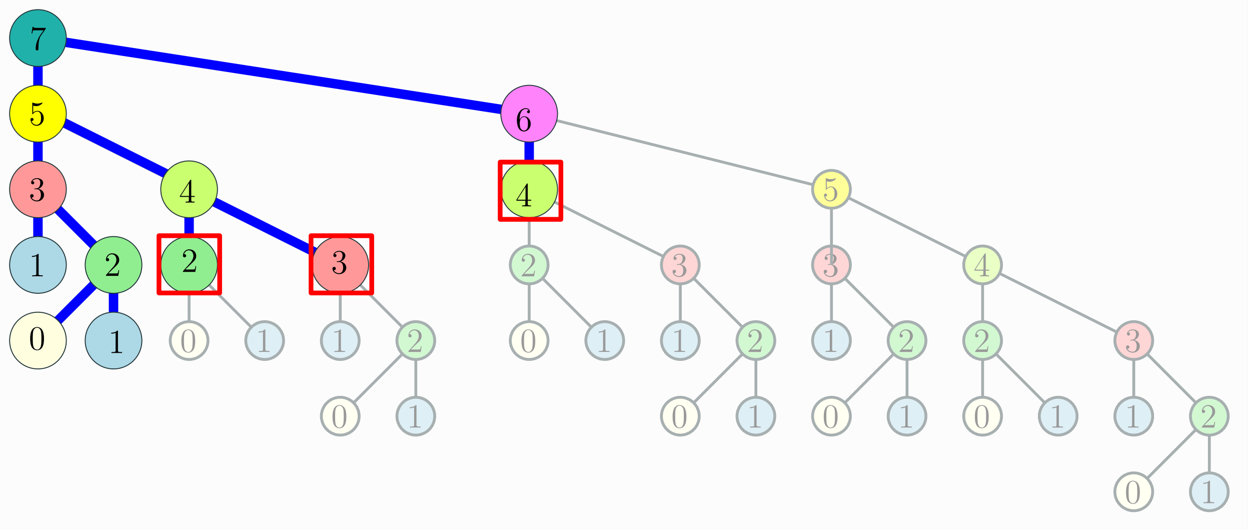

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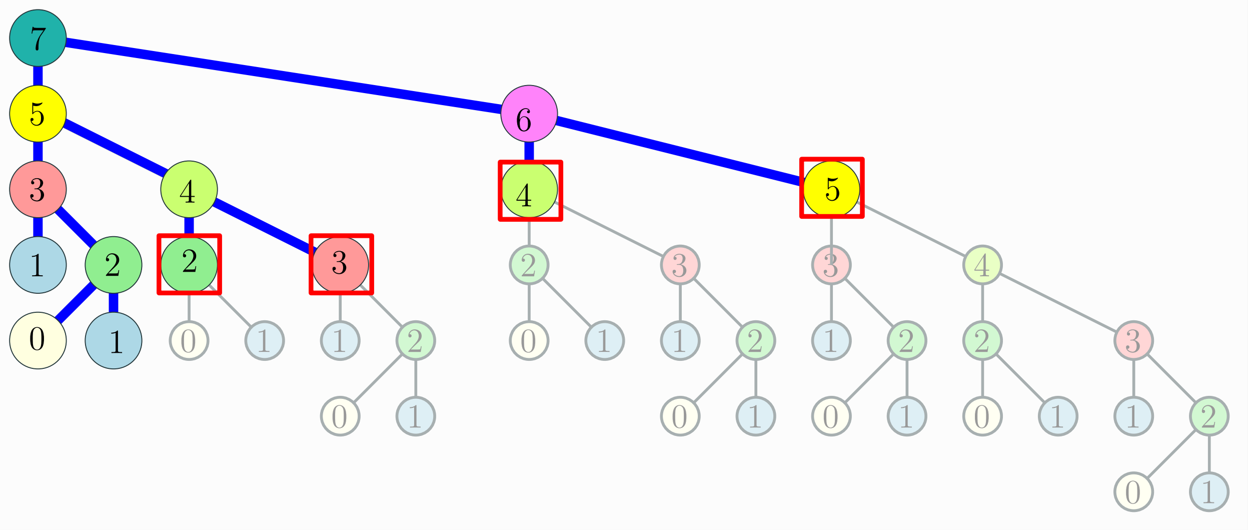

Automatic (Implicit) Memorization

- Recursive version:

```
f(x_1, x_2, ..., x_d):
        CODE
```

- Recursive version with memoization:

```
g(x_1, x_2, ..., x_d):
        if f already computed for (x_1, x_2, ..., x_d) then
            return value already computed
        NEW_CODE
```

- NEW_CODE:
  - Replaces any "**return** α" with
  - Remember "$f(x_1, \ldots, x_d) = \alpha$"; **return** α.

Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
    - analyze problem ahead of time

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  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.
- Implicit (automatic) memoization:
  - problem structure or algorithm is not well understood.
  - Need to pay overhead of data-structure.
  - Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.

Explicit/implicit memorization for Fibonacci

```
Init: M[i] = −1, i = 0,...,n.

Fib(k):
    if (k = 0)
        return 0
    if (k = 1)
        return 1
    if (M[k] ≠ −1)
        return M[n]
    M[k] ⇐ Fib(k − 1) + Fib(k − 2)
    return M[k]
```

Explicit memorization

```
Init: Init dictionary D

Fib(n):
    if (n = 0)
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    if (n is already in D)
        return value stored with n in D
        val ⇐ Fib(n − 1) + Fib(n − 2)
    Store (n, val) in D
    return val
```

Implicit memorization

Dynamic programming

Removing the recursion by filling the table in the right order

```
Fib(n):
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```

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

Dynamic programming: Saving space!

Saving space. Do we need an array of n numbers? Not really.

```
FibIter(n):
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    F[0] = 0
    F[1] = 1
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        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n do
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp

    return prev1
```

Recurrence: $F(n) = F(n-1) + F(n-2)$

Memoized the output

Dynamic programming – quick review

Dynamic Programming is smart recursion

Dynamic programming – quick review

Dynamic Programming is smart recursion

$+$ explicit memorization

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memorization

+ filling the table in right order

+ removing recursion.

Analyzing memorized recursive function

Suppose we have a recursive program *foo*(x) that takes an input x.

$|x| = n$

- On input of size n the number of distinct sub-problems that *foo*(x) generates is at most $A(n)$
- *foo*(x) spends at most $B(n)$ time not counting the time for its recursive calls.

Fib: $A(n) = n$

$B(n) = O(1)$

(1) addition

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Suppose we memorize the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

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Suppose we memorize the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of memorized version of $foo(x)$ if $|x| = n$? $A(n) \cdot B(n)$

Analyzing memorized recursive function

Suppose we have a recursive program $foo(x)$ that takes an input x.

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Suppose we memorize the recursion.
Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of memorized version of $foo(x)$ if $|x| = n$? $O(A(n)B(n))$.

Fibonacci numbers are big – corrected running time analysis

Back to Fibonacci Numbers

T Is the iterative algorithm a polynomial time algorithm? Does it take $O(n)$ time?

- input is n and hence input size is $\Theta(\log n)$
- output is $F(n)$ and output size is $\Theta(n)$. Why?
- Hence output size is exponential in input size so no polynomial time algorithm possible!
- Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?

Longest Increasing Sub-sequence Revisited

Sequences

Definition
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

Definition
$a_{i_1}, \ldots, a_{i_k}$ is a sub-sequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Sequences - Example...

Example

$n = 7$

- Sequence: $6, 3, 5, 2, 7, 8, 1$
- Subsequence of above sequence: $5, 2, 1$
- Increasing sequence: $3, 5, 9, 17, 54$
- Decreasing sequence: $34, 21, 7, 5, 1$
- Increasing subsequence of the first sequence: $2, 7, 8$.
- *Longest* Increasing subsequence of the first sequence: $3, 5, 7, 8$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

Goal Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

Longest Increasing Subsequence Problem

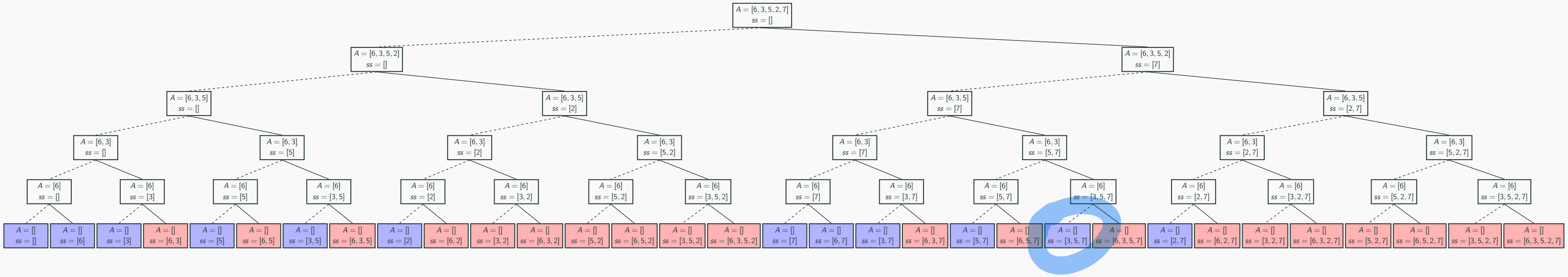
Input A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

Goal Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

Naive Recursion Enumeration - State Tree



- This is just for [6,3,5,2,7]! (Tikz won't print larger trees)
- How many leafs are there for the full [6,3,5,2,7, 8, 1] sequence
- What is the running time?

Naive Recursion Enumeration - Code

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Backtracking Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[0..n-1]$):

Backtracking Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[0..n-1]$):

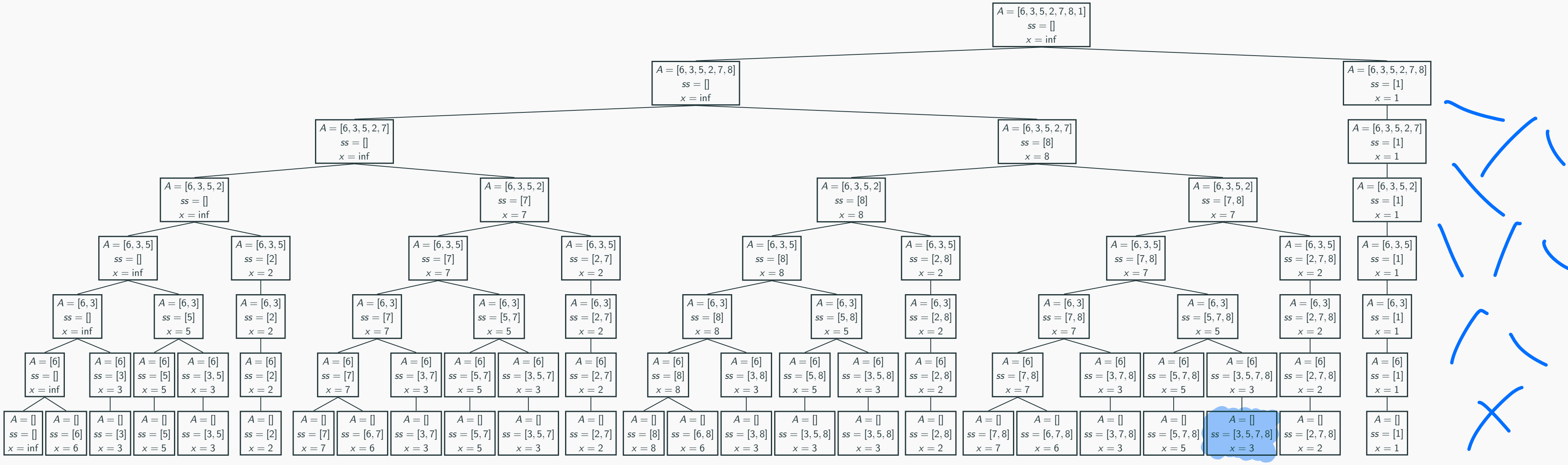
- Case 1: Does not contain $A[n-1]$ in which case LIS($A[0..n-1]$) = LIS($A[0..(n-1)]$)
- Case 2: contains $A[n-1]$ in which case LIS($A[0..n-1]$) is not so clear.

Observation

For second case we want to find a subsequence in $A[1..(n-2)]$ that is restricted to numbers less than $A[n-1]$. This suggests that a more general problem is **LIS_smaller**($A[0..n-1], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Example

Sequence: $A[0..6] = 6, 3, 5, 2, 7, 8, 1$



Recursive Approach

$LIS(A[1..n])$: the length of longest increasing subsequence in A

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..i], x):
    if i = 0 then return 0
    m = LIS_smaller(A[1..i − 1], x)
    if A[i] < x then
        m = max(m, 1 + LIS_smaller(A[1..i − 1], A[i]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

Recursive Approach

LIS_smaller$(A[1..i], x)$:
  if $i = 0$ **then return** 0
  $m =$ **LIS_smaller**$(A[1..i-1], x)$
  if $A[i] < x$ **then**
    $m = max(m, 1 +$ **LIS_smaller**$(A[1..i-1], A[i]))$
  `Output` m

LIS$(A[1..n])$:
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- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate?

A[1 ... n-1]
A[1 ... n-2] (n)
⋮

x must be one of the values in A
(n values)

Recursive Approach

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- How many distinct sub-problems will LIS_smaller($A[1..n], \infty$) generate? $O(n^2)$
- What is the running time if we memorize recursion? $O(n^2)$

$A(n) = O(n^2)$

$B(n) = O(1)$

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Naming sub-problems and recursive equation

After seeing that number of sub-problems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

A[1...i] x = A[j]

$LIS(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

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$LIS(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

; A[0..0]

Base case: $LIS(0, j) = 0$ for $1 \leq j \leq n+1$

Recursive relation:

- $LIS(i, j) = LIS(i-1, j)$ if $A[i] \geq A[j]$ ← if we don't include A[i] in LIS
- $LIS(i, j) = \max\{LIS(i-1, j), 1 + LIS(i-1, i)\}$ if $A[i] < A[j]$ ← if we can include A[i] in LIS

Output: $LIS(n, n+1)$.

How to order bottom up computation?

		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
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Iterative algorithm

The dynamic program for longest increasing subsequence

```
LIS-Iterative(A[1..n]):
    A[n + 1] = ∞
    int LIS[0..n − 1, 0..n]
    for j = 0 . . . n) if A[i] ≤ A[j] then LIS[0][j] = 1

    for i = 1 . . . n − 1 do
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    Return LIS[n, n + 1]
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Running time: $O(n^2)$

Space: $O(n^2)$

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Space: $O(n^2)$ Can be done in linear space. How?

Two comments

Question: Can we compute an optimum solution and not just its value?

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Yes! See notes.

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We know the LIS length (4) but how do we find the LIS itself?

$$LIS = [3, 5, 7, 8]$$

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Question: Is there a faster algorithm for LIS?

Two comments

Question: Can we compute an optimum solution and not just its value?
Yes!

Question: Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $O(n \log n)$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.

How to come up with dynamic programming algorithm: summary

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- Optimize the resulting algorithm further

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- ...need to find the right way or order the sub-problems evaluation. Th is leads to an a dynamic programming algorithm.
- Optimize the resulting algorithm further
- ...

Dynamic Programming

- Find a “smart” recursion for the problem in which the number of distinct sub-problems is small; polynomial in the original problem size.
- Estimate the number of sub-problems, the time to evaluate each sub-problem and the space needed to store the value.
- This gives an upper bound on the total running time if we use automatic/explicit memorization.
- Come up with an explicit memorization algorithm for the problem.
- Eliminate recursion and find an iterative algorithm.
- ...need to find the right way or order the sub-problems evaluation. Th is leads to an a dynamic programming algorithm.
- Optimize the resulting algorithm further
- ...
- Get rich!