Pre-lecture brain teaser

Remembering the edit distance example we saw in class last time, we formaluted the processing of the recursion as a table:

	ε	D	R	E	A	D
ε						
D						
E						
E						
D						

Is there an easier way to get the minimum alignment without having to calculate all the values in the cell?

ECE-374-B: Lecture 15 - Graph search

Instructor: Nickvash Kani

March 09, 2023

University of Illinois at Urbana-Champaign

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	ε	*D*	*R*	*E*	*A*	*D*
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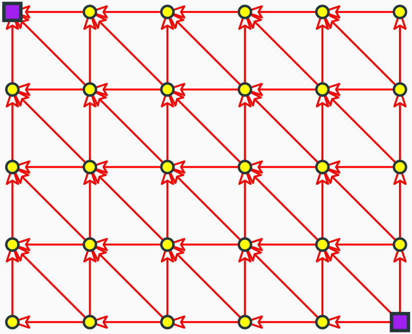
D						
E						
E						
D						



Look at the flow of the computation!

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D						
E						
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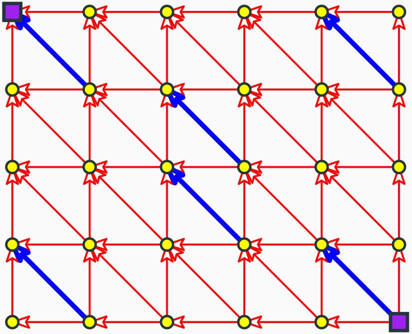

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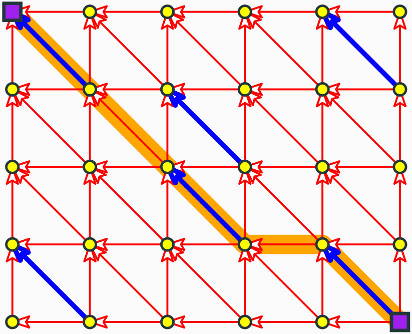
D						
E						
E						
D						



We can solve the problem by turning it into a graph!

Graph Basics

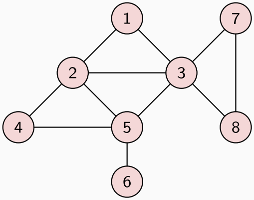
Why Graphs?

- Graphs help model networks which are ubiquitous: transportation networks (rail, roads, airways), social networks (interpersonal relationships), information networks (web page links), and many problems that don't even look like graph problems.
- Fundamental objects in Computer Science, Optimization, Combinatorics
- Many important and useful optimization problems are graph problems
- Graph theory: elegant, fun and deep mathematics

Graph

An undirected (simple) graph $G = (V, E)$ is a 2-tuple:

- V is a set of vertices (also referred to as nodes/points)
- E is a set of edges where each edge $e \in E$ is a set of the form $\{u, v\}$ with $u, v \in V$ and $u \neq v$.



Example

In figure, $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$.

Example: Modeling Problems as Search

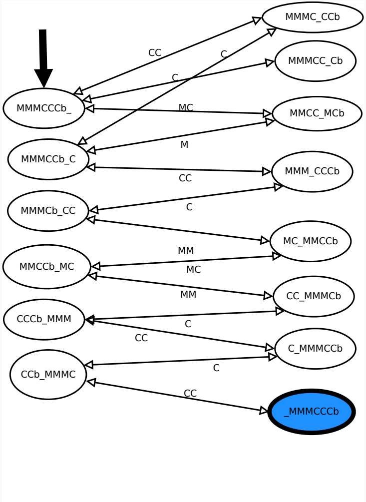
State Space Search
Many search problems can be modeled as search on a graph.
The trick is figuring out what the vertices and edges are.

Missionaries and Cannibals

- Three missionaries, three cannibals, one boat, one river
- Boat carries two people, must have at least one person
- Must all get across
- At no time can cannibals outnumber missionaries

How is this a graph search problem?
What are the vertices?
What are the edges?

Cannibals and Missionaries: Is the language empty?



Problems goes back to 800 CE

Versions with brothers and sisters.

Jealous Husbands.

Lions and buffalo

All bad names to a simple problem...

*Omitted states where cannibals outnumber missionaries

Problems on DFAs and NFAs sometimes are just problems on graphs

- M: DFA/NFA is $L(M)$ empty?
- M: DFA is $L(M) = \Sigma^*$?
- M: DFA, and a string w. Does M accepts w?
- N: NFA, and a string w. Does N accepts w?

Graph notation and representation

Notation and Convention

Notation

An edge in an undirected graphs is an unordered pair of nodes and hence it is a set. Conventionally we use uv for $\{u, v\}$ when it is clear from the context that the graph is undirected.

- u and v are the end points of an edge $\{u, v\}$
- Multi-graphs allow
    - loops which are edges with the same node appearing as both end points
    - multi-edges: different edges between same pairs of nodes
- In this class we will assume that a graph is a simple graph unless explicitly stated otherwise.

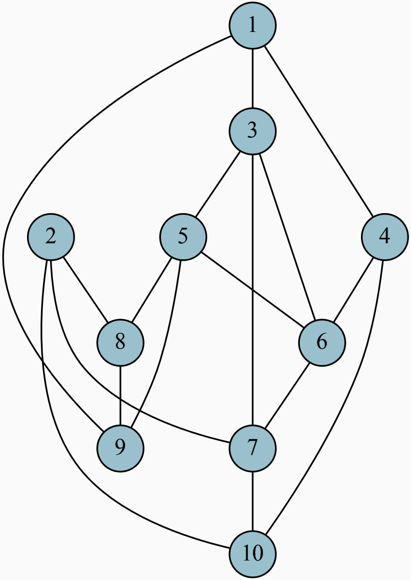
Graph Representation I

Adjacency Matrix

Represent $G = (V, E)$ with n vertices and m edges using a $n \times n$ adjacency matrix A where

- $A[i, j] = A[j, i] = 1$ if $\{i, j\} \in E$ and $A[i, j] = A[j, i] = 0$ if $\{i, j\} \notin E$.
- Advantage: can check if $\{i, j\} \in E$ in $O(1)$ time
- Disadvantage: needs $\Omega(n^2)$ space even when $m \ll n^2$

Graph adjacency matrix example [10 vertices]



	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	0	0	0	0	1	0
2	0	0	0	0	0	0	1	1	0	1
3	1	0	0	0	1	1	1	0	0	0
4	1	0	0	0	0	1	0	0	0	1
5	0	0	1	0	0	1	0	1	1	0
6	0	0	1	1	1	0	1	0	0	0
7	0	1	1	0	0	1	0	0	0	1
8	0	1	0	0	1	0	0	0	1	0
9	1	0	0	0	1	0	0	1	0	0
10	0	1	0	1	0	0	1	0	0	0

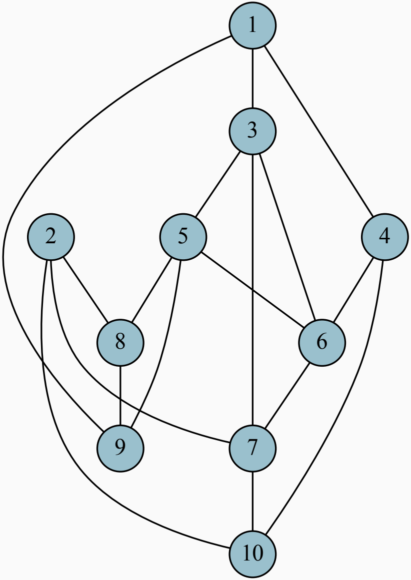
Adjacency Lists

Represent $G = (V, E)$ with n vertices and m edges using adjacency lists:

- For each $u \in V$, $\mathrm{Adj}(u) = \{v \mid \{u, v\} \in E\}$, that is neighbors of u. Sometimes $\mathrm{Adj}(u)$ is the list of edges incident to u.
- Advantage: space is $O(m + n)$
- Disadvantage: cannot "easily" determine in $O(1)$ time whether $\{i, j\} \in E$
    - By sorting each list, one can achieve $O(\log n)$ time
    - By hashing "appropriately", one can achieve $O(1)$ time

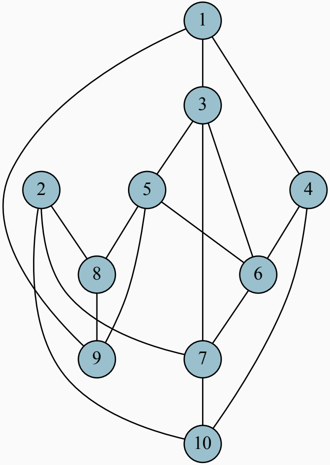
Note: In this class we will assume that by default, graphs are represented using plain vanilla (unsorted) adjacency lists.

Graph adjacency list example [10 vertices]



vertex	adjacency list
1	3, 4, 9
2	7, 8, 10
3	1, 5, 6, 7
4	1, 6, 10
5	3, 6, 8, 9
6	3, 4, 5, 7
7	2, 3, 6, 10
8	2, 5, 9
9	1, 5, 8
10	2, 4, 7

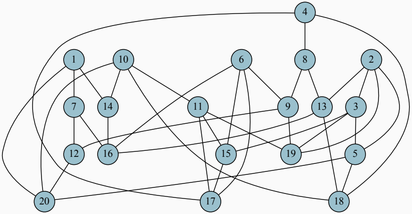
Graph adjacency matrix+list example [10 vertices]



vertex	adjacency list
1	3, 4, 9
2	7, 8, 10
3	1, 5, 6, 7
4	1, 6, 10
5	3, 6, 8, 9
6	3, 4, 5, 7
7	2, 3, 6, 10
8	2, 5, 9
9	1, 5, 8
10	2, 4, 7

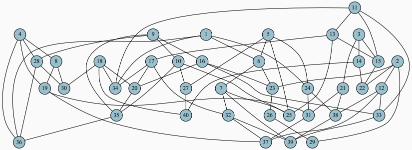
	1	2	3	4	5	6	7	8	9	10
1	0	0	1	1	0	0	0	0	1	0
2	0	0	0	0	0	0	1	1	0	1
3	1	0	0	0	1	1	1	0	0	0
4	1	0	0	0	0	1	0	0	0	1
5	0	0	1	0	0	1	0	1	1	0
6	0	0	1	1	1	0	1	0	0	0
7	0	1	1	0	0	1	0	0	0	1
8	0	1	0	0	1	0	0	0	1	0
9	1	0	0	0	1	0	0	1	0	0
10	0	1	0	1	0	0	1	0	0	0

Graph adjacency matrix example [20 vertices]



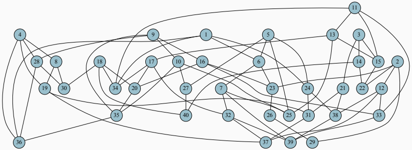
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1
2	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
3	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
4	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	0
5	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
6	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	0	0
7	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0
8	0	0	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
9	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	0	1	0
10	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0	1
11	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	1	0
12	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	1
13	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0
14	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
15	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0
16	0	0	0	0	0	1	1	0	0	0	0	0	1	1	0	0	0	0	0	0
17	0	0	0	1	0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0
18	0	0	0	1	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
19	0	1	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
20	1	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0

Graph adjacency matrix example [40 vertices]



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
5	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0
8	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0
10	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
12	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	0
13	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
15	0	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1
19	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
21	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
22	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
25	0	0	0	0	1	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
28	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
29	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
32	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
33	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
34	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
36	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
39	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0

Graph adjacency list example [40 vertices]

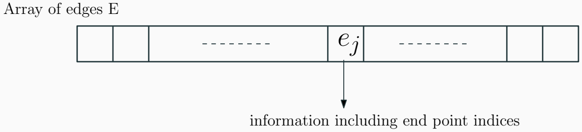


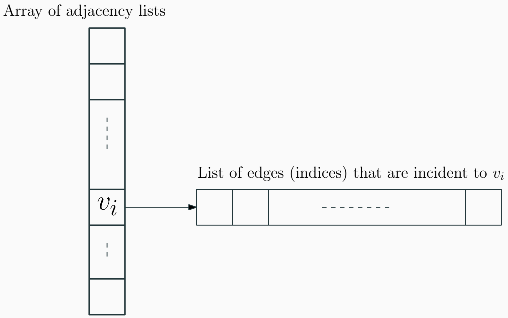
vertex	adjacency list
1	6, 24, 34, 36
2	12, 22, 23, 29
3	14, 15, 21
4	8, 19, 28, 36
5	6, 24, 25, 27
6	1, 5, 7, 23
7	6, 25, 32, 39
8	4, 19, 30
9	10, 16, 28, 35
10	9, 25, 27, 35
11	13, 15, 33, 34
12	2, 33, 37, 38
13	11, 15, 17, 25
14	3, 22, 40
15	3, 11, 13, 22
16	9, 20, 23, 33
17	13, 20, 32, 34
18	20, 30, 34, 40
19	4, 8, 31, 37
20	16, 17, 18, 35
21	3, 31, 38
22	2, 14, 15
23	2, 6, 16, 26
24	1, 5, 31, 38
25	5, 7, 10, 13
26	23, 29
27	5, 10, 40
28	4, 9, 30, 36
29	2, 26
30	8, 18, 28
31	19, 21, 24, 37
32	7, 17, 37, 39
33	11, 12, 16, 39
34	1, 11, 17, 18
35	9, 10, 20, 36
36	1, 4, 28, 35
37	12, 19, 31, 32
38	12, 21, 24, 39
39	7, 32, 33, 38
40	14, 18, 27

A Concrete Representation

- Assume vertices are numbered arbitrarily as $\{1, 2, \ldots, n\}$.
- Edges are numbered arbitrarily as $\{1, 2, \ldots, m\}$.
- Edges stored in an array/list of size m. $E[j]$ is j^{th} edge with info on end points which are integers in range 1 to n.
- Array *Adj* of size n for adjacency lists. $Adj[i]$ points to adjacency list of vertex i. $Adj[i]$ is a list of edge indices in range 1 to m.

A Concrete Representation





A Concrete Representation: Advantages

- Edges are explicitly represented/numbered. Scanning/processing all edges easy to do.
- Representation easily supports multigraphs including self-loops.
- Explicit numbering of vertices and edges allows use of arrays: $O(1)$-time operations are easy to understand.
- Can also implement via pointer based lists for certain dynamic graph settings.

Connectivity

Connectivity

Given a graph $G = (V, E)$:

- path: sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1} \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from v_1 to v_k. Note: a single vertex u is a path of length 0.

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- cycle: sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$ and $\{v_1, v_k\} \in E$. Single vertex not a cycle according to this definition.
  Caveat: Some times people use the term cycle to also allow vertices to be repeated; we will use the term tour.

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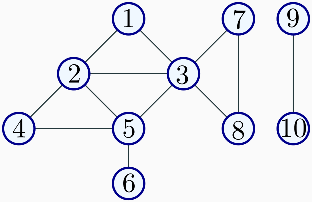
  Caveat: Some times people use the term cycle to also allow vertices to be repeated; we will use the term tour.
- A vertex u is connected to v if there is a path from u to v.
- The connected component of u, $\text{con}(u)$, is the set of all vertices connected to u. Is $u \in \text{con}(u)$?

Connectivity contd

Define a relation C on $V \times V$ as uCv if u is connected to v

- In undirected graphs, connectivity is a reflexive, symmetric, and transitive relation. Connected components are the equivalence classes.
- Graph is connected if there is only one connected component.



Algorithmic Problems

- Given graph G and nodes u and v, is u connected to v?
- Given G and node u, find all nodes that are connected to u.
- Find all connected components of G.

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Can be accomplished in $O(m + n)$ time using BFS or DFS.
BFS and DFS are refinements of a basic search procedure which is good to understand on its own.

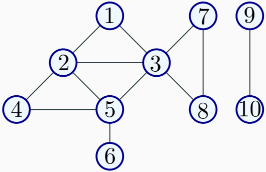
Computing connected components in undirected graphs using basic graph search

Basic Graph Search in Undirected Graphs

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore(G,u):
    Visited[1..n] ← FALSE
    // ToExplore, S: Lists
    Add u to ToExplore and to S
    Visited[u] ← TRUE
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge xy in Adj(x) do
            if (Visited[y] = FALSE)
                Visited[y] ← TRUE
                Add y to ToExplore
                Add y to S
    Output S
```

Example



Properties of Basic Search

Running Time:

Properties of Basic Search

Running Time:

BFS and DFS are special case of BasicSearch.

- Breadth First Search (BFS): use queue data structure to implementing the list *ToExplore*
- Depth First Search (DFS): use stack data structure to implement the list *ToExplore*

One can create a natural search tree T rooted at u during search.

```
Explore(G,u):
    array Visited[1..n]
    Initialize: Visited[i] ← FALSE for i = 1,...,n
    List: ToExplore, S
    Add u to ToExplore and to S, Visited[u] ← TRUE
    Make tree T with root as u
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge (x,y) in Adj(x) do
            if (Visited[y] = FALSE)
                Visited[y] ← TRUE
                Add y to ToExplore
                Add y to S
                Add y to T with x as its parent
    Output S
```

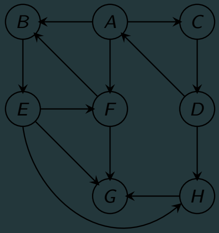
Finding all connected components

Modify Basic Search to find all connected components of a given graph G in $O(m+n)$ time.

Directed Graphs and Directed Connectivity

Definition
A directed graph $G = (V, E)$ consists of

- set of vertices/nodes V and
- a set of edges/arcs $E \subseteq V \times V$.



An edge is an ordered pair of vertices. (u, v) different from (v, u).

Examples of Directed Graphs

In many situations relationship between vertices is asymmetric:

- Road networks with one-way streets.
- Web-link graph: vertices are web-pages and there is an edge from page p to page p' if p has a link to p'. Web graphs used by Google with PageRank algorithm to rank pages.
- Dependency graphs in variety of applications: link from x to y if y depends on x. Make files for compiling programs.
- Program Analysis: functions/procedures are vertices and there is an edge from x to y if x calls y.

Directed Graph Representation

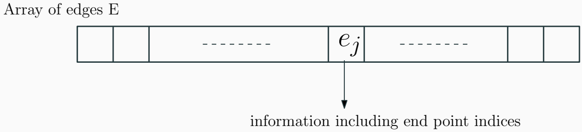
Graph $G = (V, E)$ with n vertices and m edges:

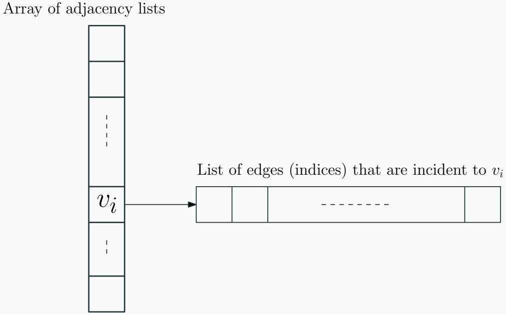
- Adjacency Matrix: $n \times n$ asymmetric matrix A. $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ if $(u, v) \notin E$. $A[u, v]$ is not same as $A[v, u]$.
- Adjacency Lists: for each node u, $Out(u)$ (also referred to as $Adj(u)$) and $In(u)$ store out-going edges and in-coming edges from u.

Default representation is adjacency lists.

A Concrete Representation for Directed Graphs

Concrete representation discussed previously for undirected graphs easily extends to directed graphs.





Directed Connectivity

Given a graph $G = (V, E)$:

- A (directed) path is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ and the path is from v_1 to v_k. By convention, a single node u is a path of length 0.

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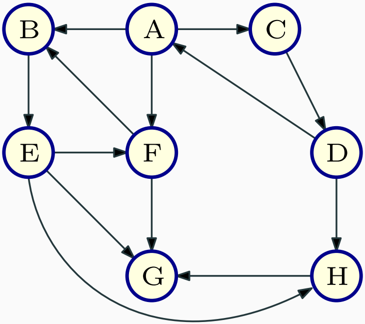
- A vertex u can <u>reach</u> v if there is a path from u to v. Alternatively v can be reached from u
- Let $\mathrm{rch}(u)$ be the set of all vertices reachable from u.

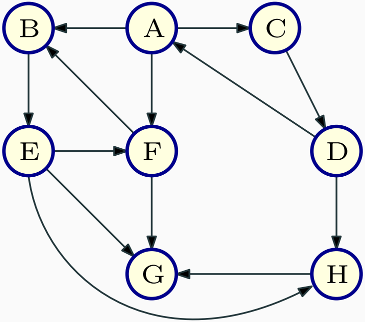
Connectivity contd

Asymmetricity: D can reach B but B cannot reach D



Connectivity contd

Asymmetricity: *D* can reach *B* but *B* cannot reach *D*



Questions:

- Is there a notion of connected components?
- How do we understand connectivity in directed graphs?

Strong connected components

Connectivity and Strong Connected Components

Definition
Given a directed graph G, u is strongly connected to v if u can reach v and v can reach u. In other words $v \in \mathsf{rch}(u)$ and $u \in \mathsf{rch}(v)$.

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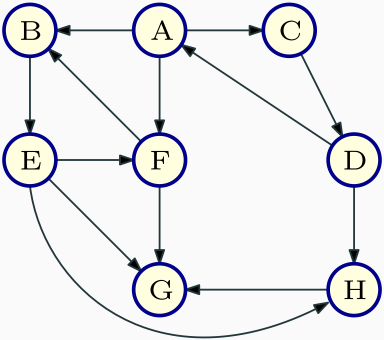
Define relation C where uCv if u is (strongly) connected to v.

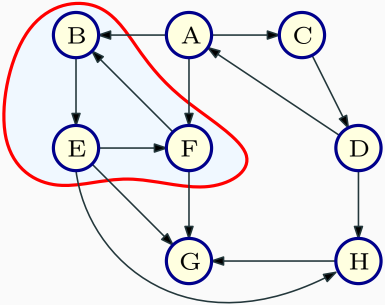
Proposition
C is an equivalence relation, that is reflexive, symmetric and transitive.

Equivalence classes of C: strong connected components of G.
They partition the vertices of G.
$SCC(u)$: strongly connected component containing u.

Strongly Connected Components: Example



Strongly Connected Components: Example



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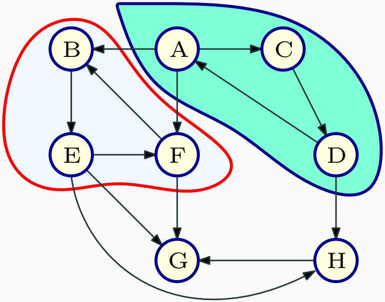

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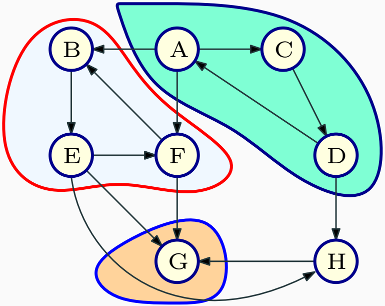
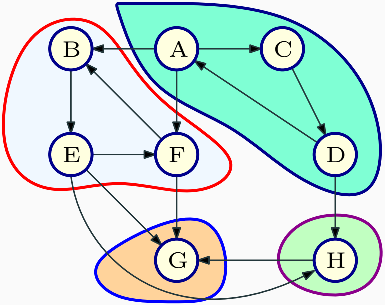


Strongly Connected Components: Example



Directed Graph Connectivity Problems

- Given G and nodes u and v, can u reach v?
- Given G and u, compute $\mathsf{rch}(u)$.
- Given G and u, compute all v that can reach u, that is all v such that $u \in \mathsf{rch}(v)$.
- Find the strongly connected component containing node u, that is $SCC(u)$.
- Is G strongly connected (a single strong component)?
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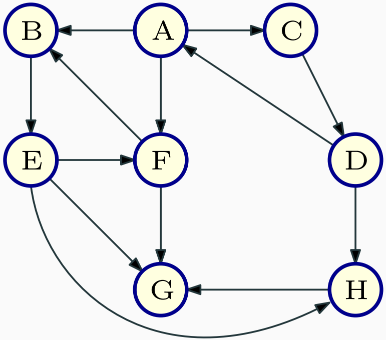
Graph exploration in directed graphs

Basic Graph Search in Directed Graphs

Given $G = (V, E)$ a directed graph and vertex $u \in V$. Let $n = |V|$.

```
Explore(G,u):
    array Visited[1..n]
    Initialize: Set Visited[i] ← FALSE for 1 ≤ i ≤ n
    List: ToExplore, S
    Add u to ToExplore and to S, Visited[u] ← TRUE
    Make tree T with root as u
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge (x,y) in Adj(x) do
            if (Visited[y] = FALSE)
                Visited[y] ← TRUE
                Add y to ToExplore
                Add y to S
                Add y to T with edge (x,y)
    Output S
```

Example



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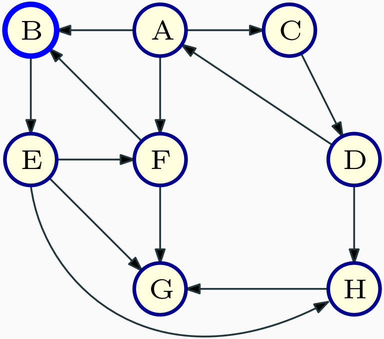

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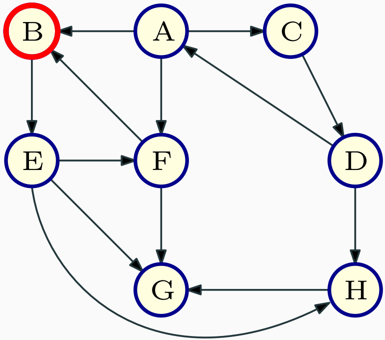

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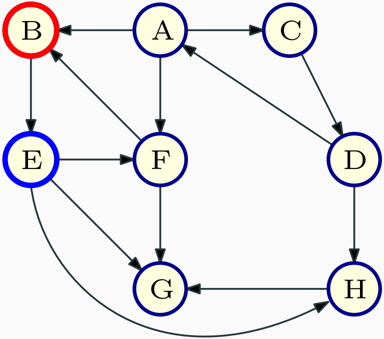

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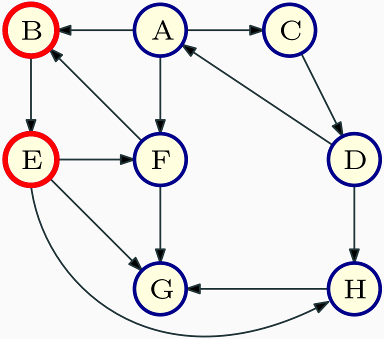

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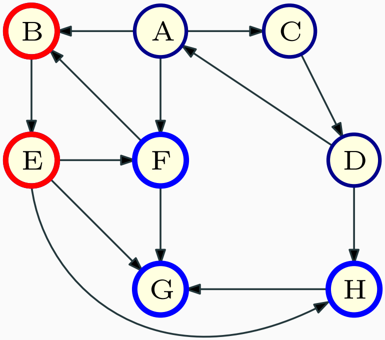

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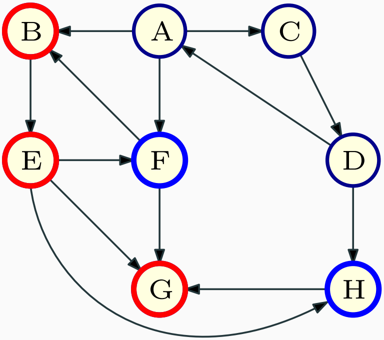

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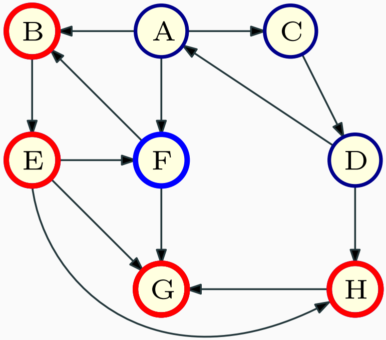

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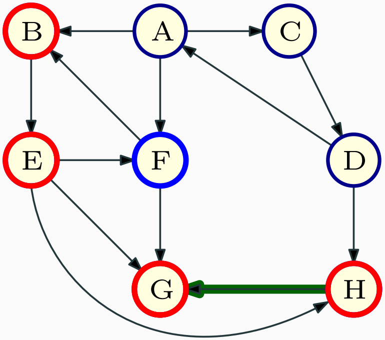

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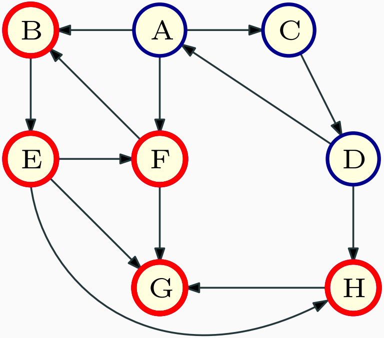

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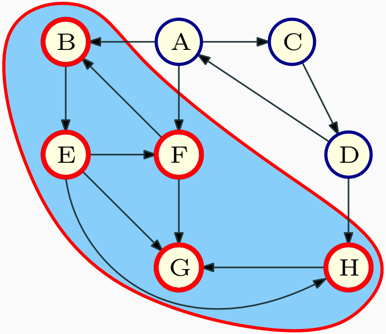

Properties of Basic Search

Proposition
Explore(G, u) *terminates with* $S = \text{rch}(u)$.

Proof Sketch.

- Once *Visited*$[i]$ is set to *TRUE* it never changes. Hence a node is added only once to *ToExplore*. Thus algorithm terminates in at most n iterations of while loop.
- By induction on iterations, can show $v \in S \Rightarrow v \in \text{rch}(u)$
- Since each node $v \in S$ was in *ToExplore* and was explored, no edges in G leave S. Hence no node in $V - S$ is in $\text{rch}(u)$. Caveat: In directed graphs edges can enter S.
- Thus $S = \text{rch}(u)$ at termination.

$\square$

Directed Graph Connectivity Problems

- Given G and nodes u and v, can u reach v?
- Given G and u, compute $\mathsf{rch}(u)$.
- Given G and u, compute all v that can reach u, that is all v such that $u \in \mathsf{rch}(v)$.
- Find the strongly connected component containing node u, that is $SCC(u)$.
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First five problems can be solved in $O(n + m)$ time by via Basic Search (or BFS/DFS). The last one can also be done in linear time but requires a rather clever DFS based algorithm (next lecture).

Algorithms via Basic Search

Algorithms via Basic Search - I

- Given G and nodes u and v, can u reach v?
- Given G and u, compute $\mathsf{rch}(u)$.

Algorithms via Basic Search - I

- Given G and nodes u and v, can u reach v?
- Given G and u, compute $\mathsf{rch}(u)$.

Use $Explore(G, u)$ to compute $\mathsf{rch}(u)$ in $O(n+m)$ time.

- Given G and u, compute all v that can reach u, that is all v such that $u \in \mathsf{rch}(v)$.

Algorithms via Basic Search - II

- Given G and u, compute all v that can reach u, that is all v such that $u \in \mathrm{rch}(v)$. Naive: $O(n(n+m))$

- Given G and u, compute all v that can reach u, that is all v such that $u \in \mathrm{rch}(v)$. Naive: $O(n(n+m))$

Definition (Reverse graph.)
Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

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Definition (Reverse graph.)
Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

Compute $\mathsf{rch}(u)$ in G^{rev}!

- **Running time:** $O(n+m)$ to obtain G^{rev} from G and $O(n+m)$ time to compute $\mathsf{rch}(u)$ via Basic Search. If both $Out(v)$ and $In(v)$ are available at each v then no need to explicitly compute G^{rev}. Can do $Explore(G, u)$ in G^{rev} implicitly.

Algorithms via Basic Search - III

$SCC(G, u) = \{v \mid u \text{ is strongly connected to } v\}$

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$SCC(G, u) = \mathrm{rch}(G, u) \cap \mathrm{rch}(G^{rev}, u)$

Algorithms via Basic Search - III

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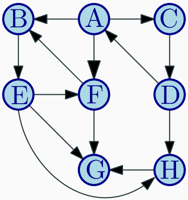
$SCC(G, u) = \mathbf{rch}(G, u) \cap \mathbf{rch}(G^{rev}, u)$

Hence, $SCC(G, u)$ can be computed with $Explore(G, u)$ and $Explore(G^{rev}, u)$. Total $O(n + m)$ time.

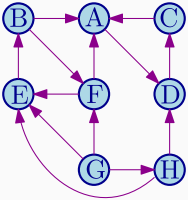
Why can $\mathbf{rch}(G, u) \cap \mathbf{rch}(G^{rev}, u)$ be done in $O(n)$ time?

SCC I

Graph G and its reverse graph G^{rev}



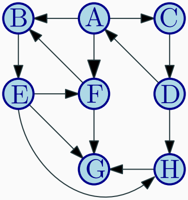
Graph G



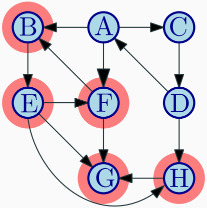
Reverse graph G^{rev}

SCC II

Graph G a vertex F and its reachable set rch(G, F)



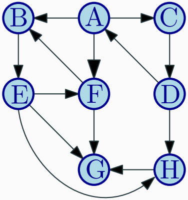
Graph G



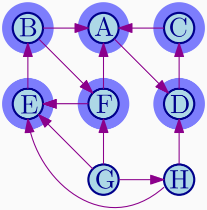
Reachable set of vertices from *F*

SCC III

Graph G a vertex F and the set of vertices that can reach it in G:rch(G^{rev}, F)

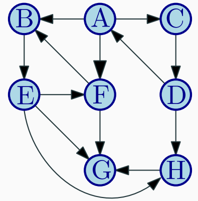


Graph G

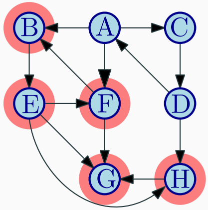


Graph G a vertex F and its strong connected component in G:

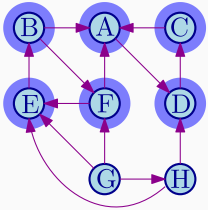
SCC(G, F)



Graph G



rch(G, F)



rch(G^{rev}, F)

Algorithms via Basic Search - IV

- Is G strongly connected?

Algorithms via Basic Search - IV

- Is G strongly connected?

Pick arbitrary vertex u. Check if $SCC(G, u) = V$.

Algorithms via Basic Search - V

- Find all strongly connected components of G.

Algorithms via Basic Search - V

- Find <u>all</u> strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Algorithms via Basic Search - V

- Find all strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

- Find <u>all</u> strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

Running time: $O(n(n+m))$.

- Find all strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

Running time: $O(n(n+m))$.

Question: Can we do it in $O(n+m)$ time?

Find out next time.....