Pre-lecture brain teaser

Last time we looked at the BasicSearch algorithm:

```
Explore(G,u):
    Visited[1..n] ← FALSE
    Add u to S
    Visited[u] ← TRUE
    ExploreStep(G,u,Visited, S)
    Output S

ExploreStep(G,x,Visited, S):
    for each edge xy in Adj(x) do
        if (Visited[y] = FALSE)
            Visited[y] ← TRUE
            ExploreStep(G,x,Visited, S):
    return
```

We said that if ToExplore was a:

- Stack, the algorithm is equivalent to DFS
- Queue, the algorithm is equivalent to BFS

What if the algorithm was written recursively (instead of the while loop, you recursively call explore). What would the algorithm be equivalent to?

ECE-374-B: Lecture 16 - Directed Graphs (DFS, DAGs, Topological Sort)

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March 21, 2023

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To Explore

```
Explore(G,u):
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ExploreStep(G,x,Visited, S):
    for each edge xy in Adj(x) do
        if (Visited[y] = FALSE)
            Visited[y] ← TRUE
            ExploreStep(G,x y,Visited, S):
    return
```

We said that if ToExplore was a:

- Stack, the algorithm is equivalent to DFS
- Queue, the algorithm is equivalent to BFS

x

y

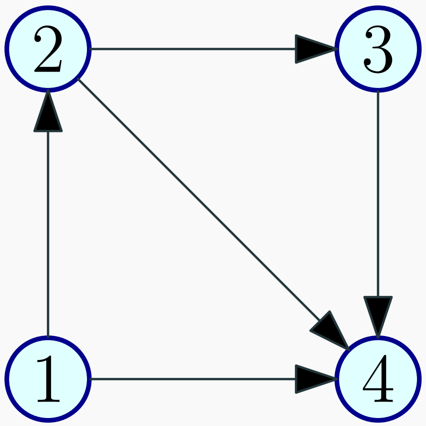
z

What if the algorithm was written recursively (instead of the while loop, you recursively call explore). What would the algorithm be equivalent to?

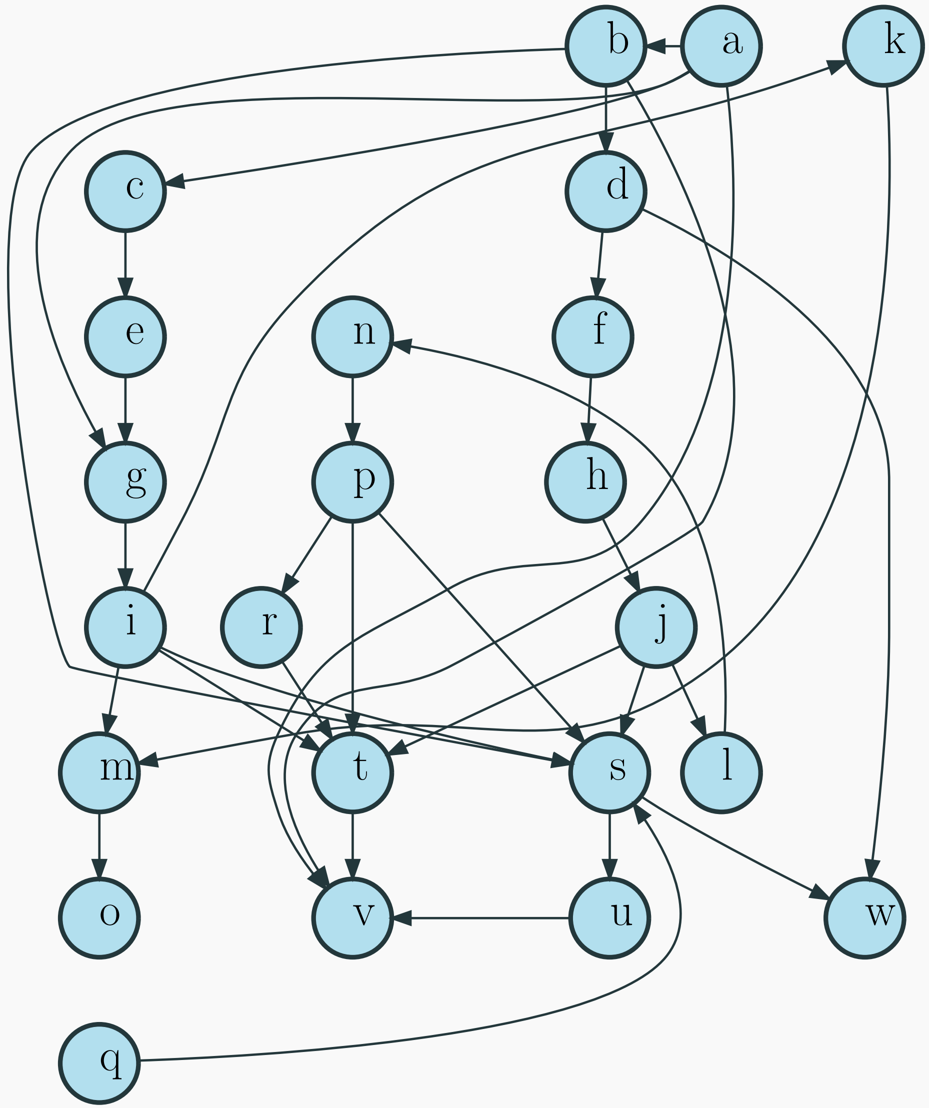
Directed Acyclic Graphs - definition and basic properties

Directed Acyclic Graphs

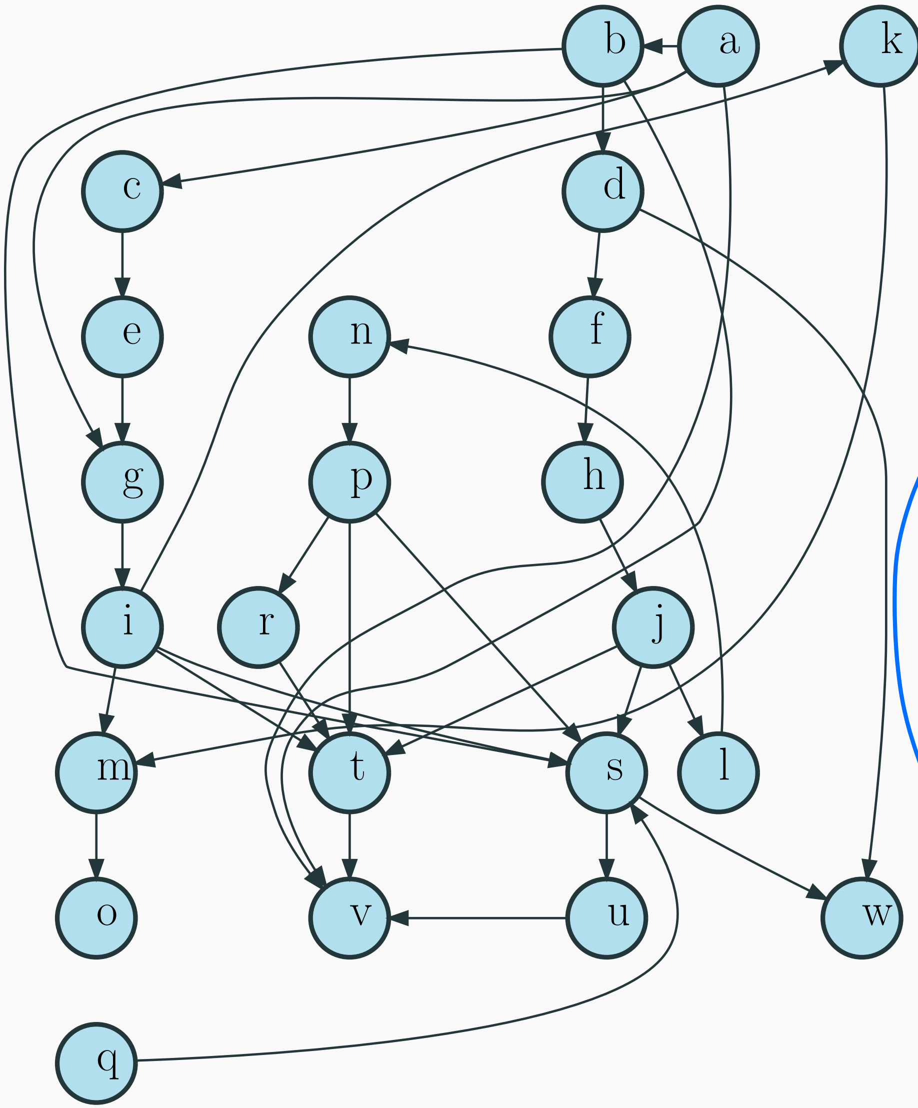
Definition
A directed graph G is a directed acyclic graph (DAG) if there is no directed cycle in G.

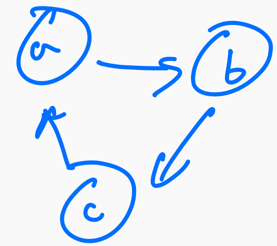


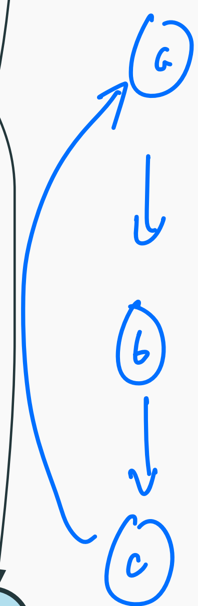
Is this a DAG?

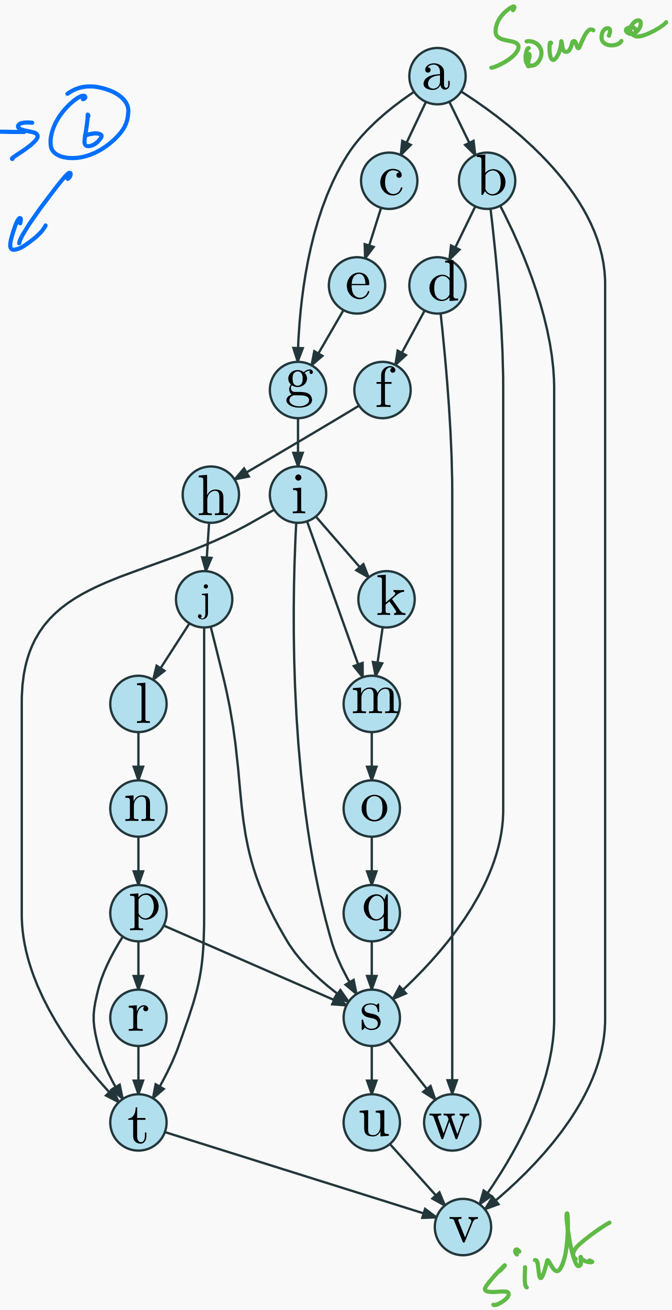


Is this a DAG?









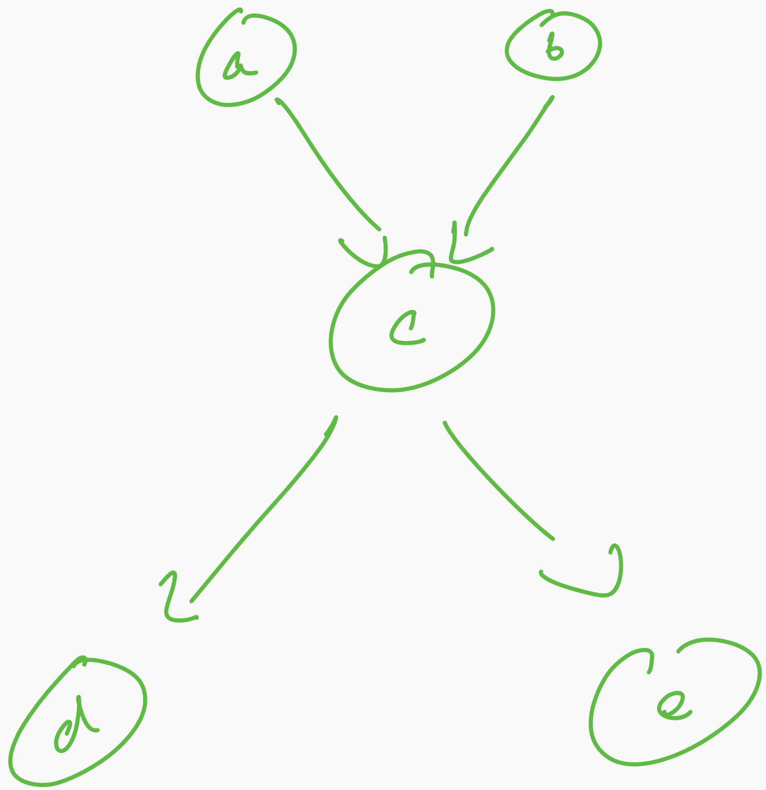
Sources and Sinks

Definition

- A vertex u is a source if it has no in-coming edges.
- A vertex u is a sink if it has no out-going edges.

Simple DAG Properties

Proposition
Every DAG G has at least one source and at least one sink.



Simple DAG Properties

Proposition
Every DAG G has at least one source and at least one sink.

Proof.
Let $P = v_1, v_2, \ldots, v_k$ be a longest path in G. Claim that v_1 is a source and v_k is a sink. Suppose not. Then v_1 has an incoming edge which either creates a cycle or a longer path both of which are contradictions. Similarly if v_k has an outgoing edge. $\square$

Topological ordering

Total recall: Order on a set

Order or strict total order on a set X is a binary relation $\prec$ on X, such that

- Transitivity: $\forall x.y, z \in X \qquad x \prec y$ and $y \prec z \implies x \prec z$.
- For any $x, y \in X$, exactly one of the following holds: $x \prec y$, $y \prec x$ or $x = y$.

increasing order $<$ = $\prec$

- transitive
  if $a < b$ & $b < c$ then $a < c$
- for any a, b
  either $a < b$ or $b < a$ or $a = b$

Convention about writing edges

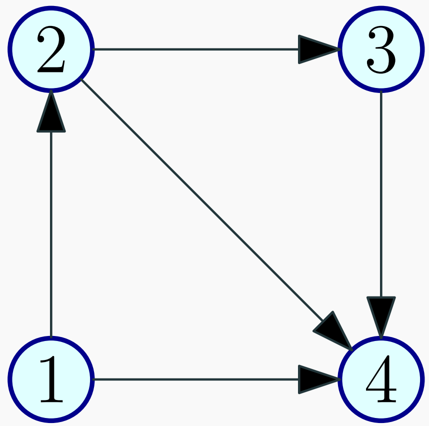
- Undirected graph edges:

$$uv = \{u, v\} = vu \in E$$

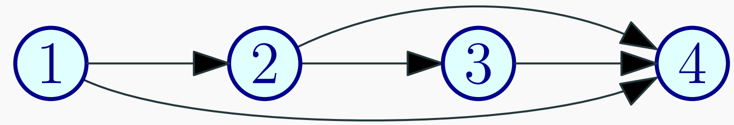
- Directed graph edges:

$$u \to v \qquad \equiv \qquad (u, v) \qquad \equiv \qquad (u \to v)$$

Topological Ordering/Sorting



Graph G



Topological Ordering of G

Definition
A topological ordering/topological sorting of $G = (V, E)$ is an ordering $\prec$ on V such that if $(u \to v) \in E$ then $u \prec v$.

Informal equivalent definition: One can order the vertices of the graph along a line (say the x-axis) such that all edges are from left to right.

Topological ordering in linear time

for a DAG

Exercise: show algorithm can be implemented in $O(m+n)$ time.

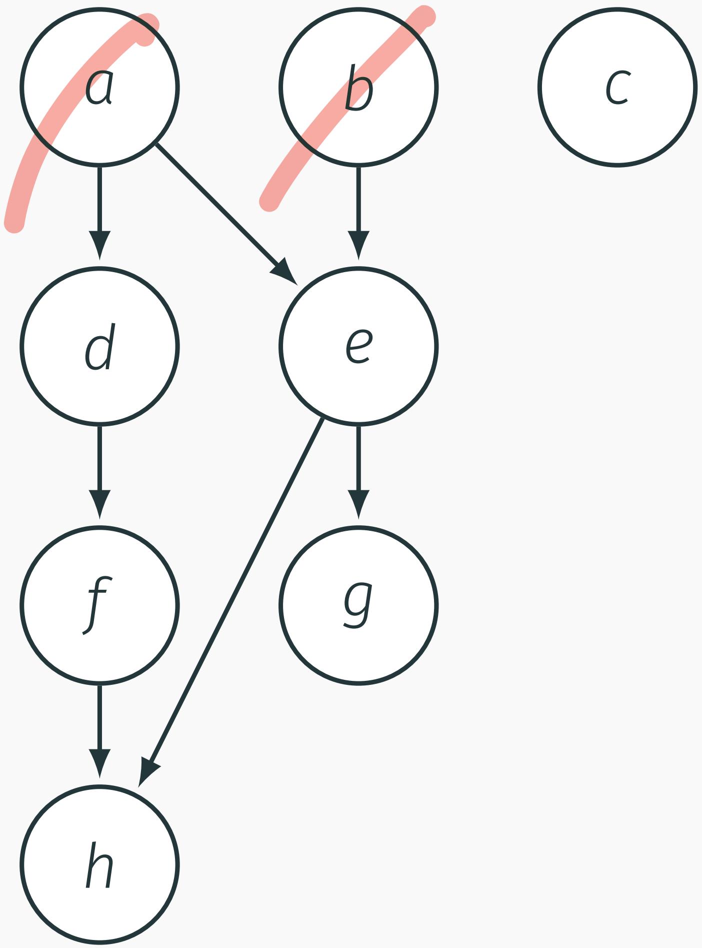
Topological ordering in linear time

Exercise: show algorithm can be implemented in $O(m+n)$ time.

Simple Algorithm:

1. Count the in-degree of each vertex
2. For each vertex that is source ($deg_{in}(v) = 0$):
   2.1 Add v to the topological sort
   2.2 Lower degree of vertices v is connected to. [1]

Topological Sort: Example



Adjacency List:

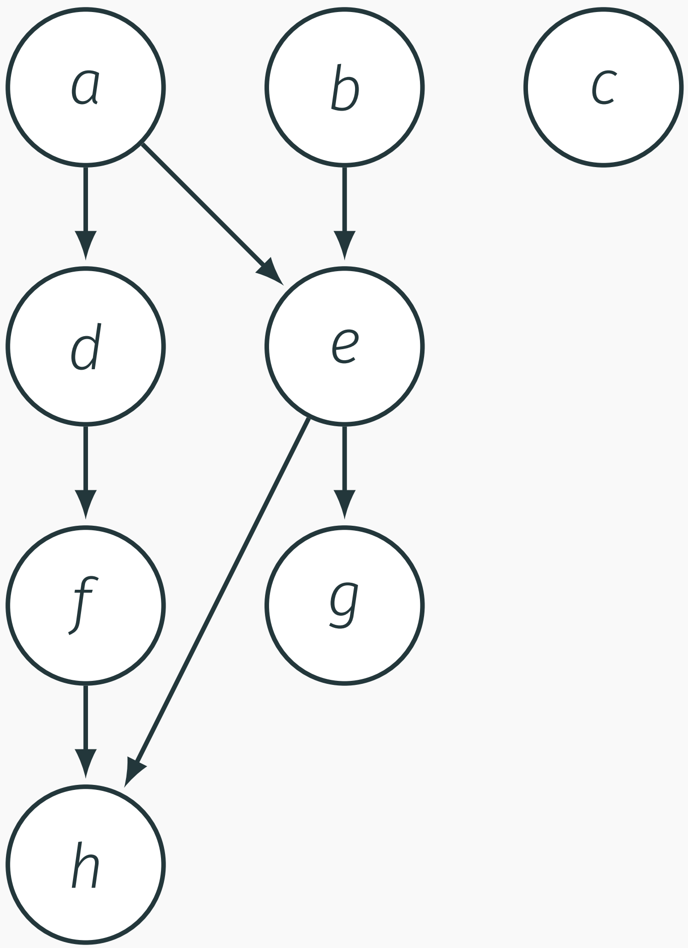
Node	Neighbors	
a	d	e
b	e	
c		
d	f	
e	h	g
f	h	
g		
h		

Generate $deg_{in}(v)$:

Degree	Vertices
0	a, b, c
1	d, f, g, e
2	e, h

a b

Topological Sort: Example



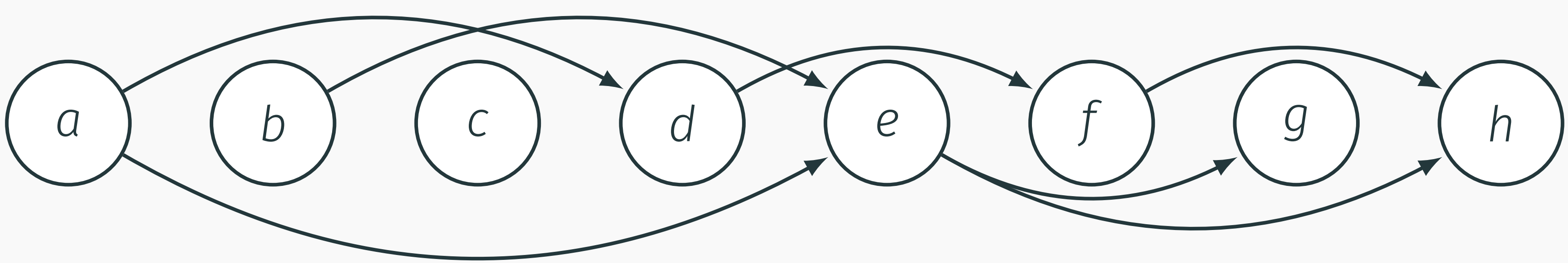
Adjacency List:

Node	Neighbors	
a	d	e
b	e	
c		
d	f	
e	h	g
f	h	
g		
h		

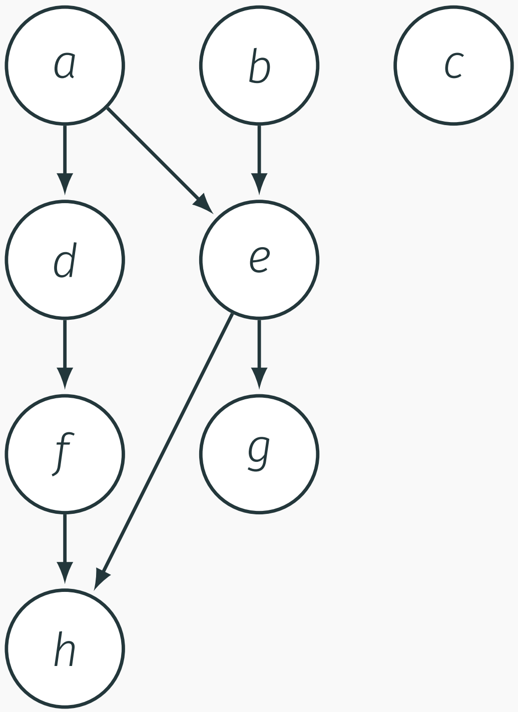
Generate $deg_{in}(v)$:

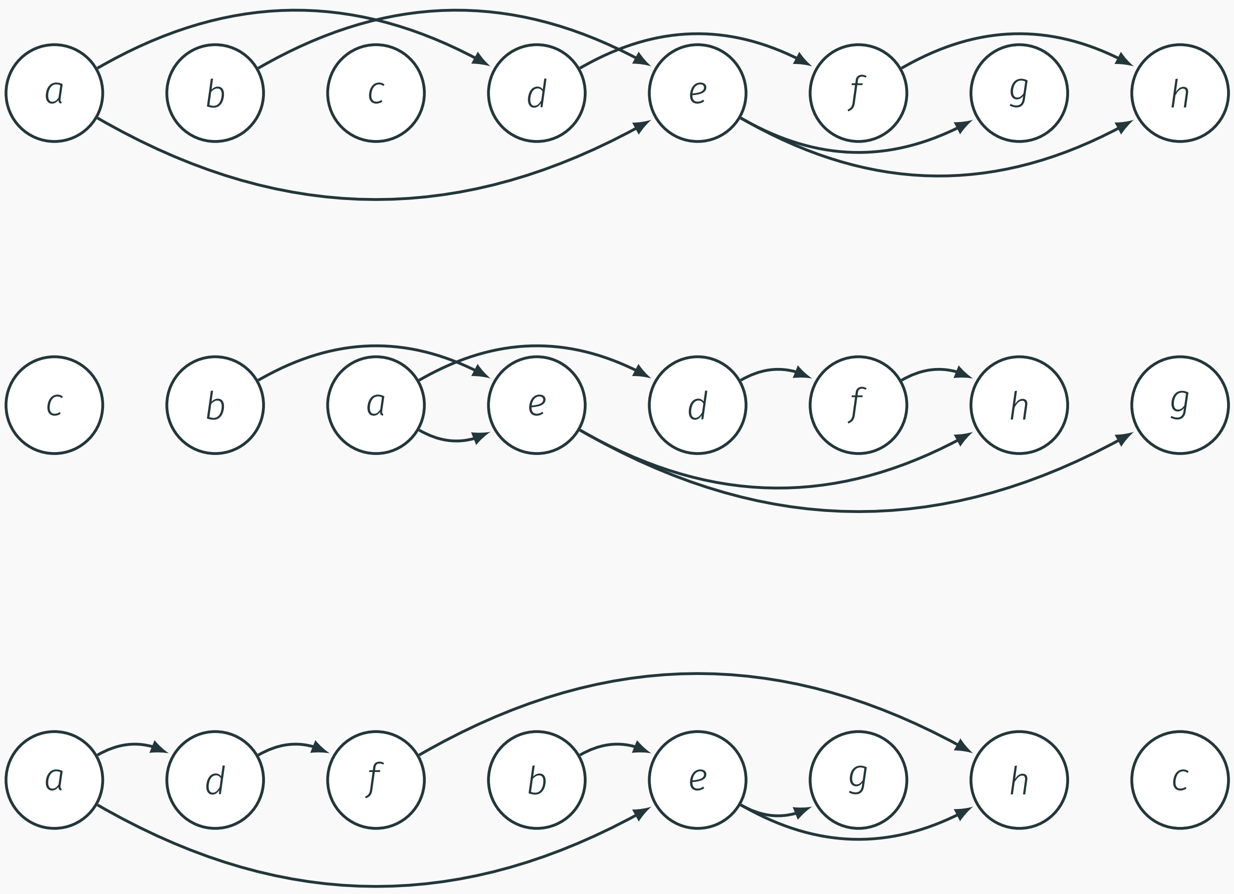
Degree	Vertices
0	a, b, c
1	d, f, g
2	e, h

Topological Ordering:



Multiple possible topological orderings





DAGs and Topological Sort

- **Note:** A DAG G may have many different topological sorts.

- Exercise: What is a DAG with the most number of distinct topological sorts for a given number n of vertices?

  - No edges
  - All disconnected

- Exercise: What is a DAG with the least number of distinct topological sorts for a given number n of vertices?

$a \to b \to c$ a path

Direct Topological ordering - code

```
TopSort(G):
    Sorted ← NULL
    deg_in[1..n] ← −1
    Tdeg_in[1..n] ← NULL
    Generate in-degree for each vertex
    for each edge xy in G do
        deg_in[y]++
    for each vertex v in G do
        Tdeg_in[deg_in[v]].append(v)
    Next we recursively add vertices
     with in-degree = 0 to the sort list
    while (Tdeg_in[0] is non-empty) do
        Remove node x from Tdeg_in[0]
        Sorted.append(x)
        for each edge xy in Adj(x) do
            deg_in[y]−−
            move y to Tdeg_in[deg_in[y]]
    Output Sorted
```

Lemma
A directed graph G can be topologically ordered $\implies$ G is a DAG.

Proof.
Proof by contradiction. Suppose G is not a DAG and has a topological ordering $\prec$. G has a cycle

$$C = u_1 \to u_2 \to \cdots \to u_k \to u_1.$$

Then $u_1 \prec u_2 \prec \ldots \prec u_k \prec u_1$

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Proof.
Proof by contradiction. Suppose G is not a DAG and has a topological ordering $\prec$. G has a cycle

$$C = u_1 \to u_2 \to \cdots \to u_k \to u_1.$$

Then $u_1 \prec u_2 \prec \ldots \prec u_k \prec u_1$

$\implies u_1 \prec u_1$.

A contradiction (to $\prec$ being an order). Not possible to topologically order the vertices. $\square$

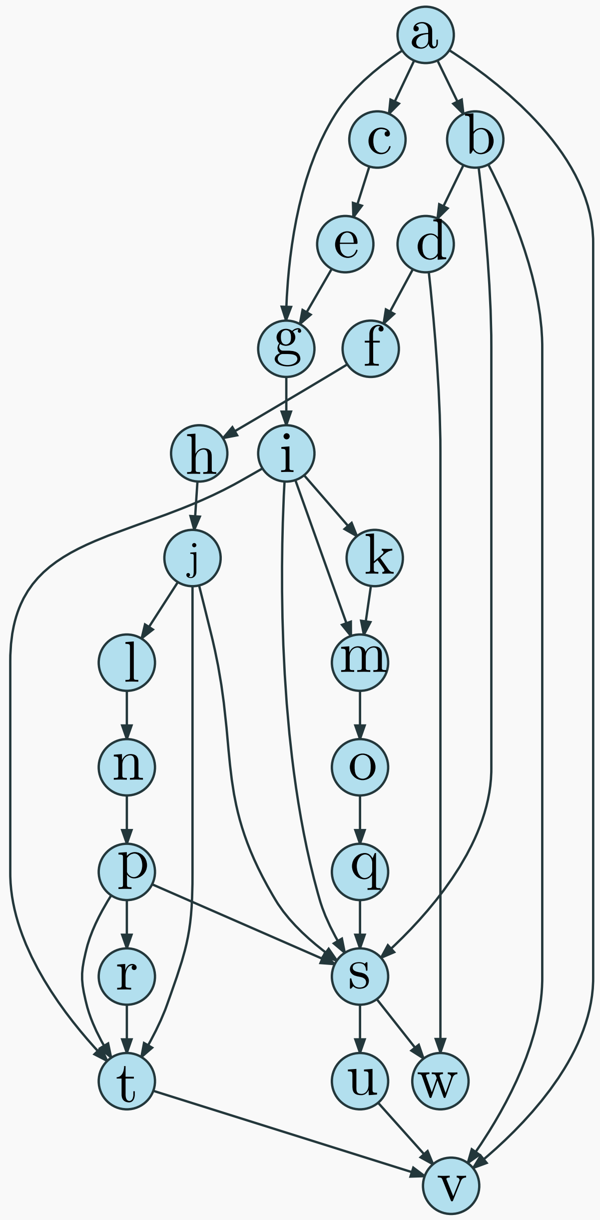
An explicit definition of what topological ordering of a graph is

For a graph $G = (V, E)$ a topological ordering of a graph is a numbering $\pi : V \to \{1, 2, \ldots, n\}$, such that

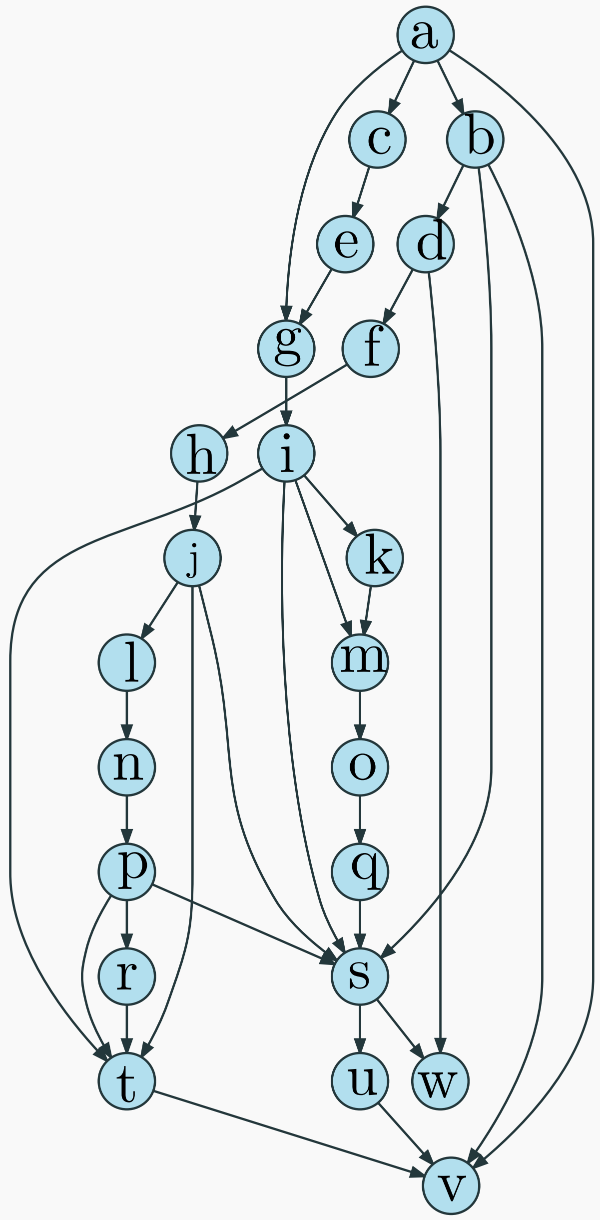
$$\forall (u \to v) \in E(G) \implies \pi(u) < \pi(v).$$

(That is, π is one-to-one, and $n = |V|$)

Example...



Example...



Assuming:

$V = \{a, \ldots w\}$
$\pi = \{1, \ldots 23\}$

Depth First Search (DFS)

Depth First Search (DFS) in Undirected Graphs

Depth First Search

- DFS special case of Basic Search.
- DFS is useful in understanding graph structure.
- DFS used to obtain linear time ($O(m+n)$) algorithms for
  - Finding cut-edges and cut-vertices of undirected graphs
  - Finding strong connected components of directed graphs
- ...many other applications as well.

DFS in Undirected Graphs

Recursive version. Easier to understand some properties.

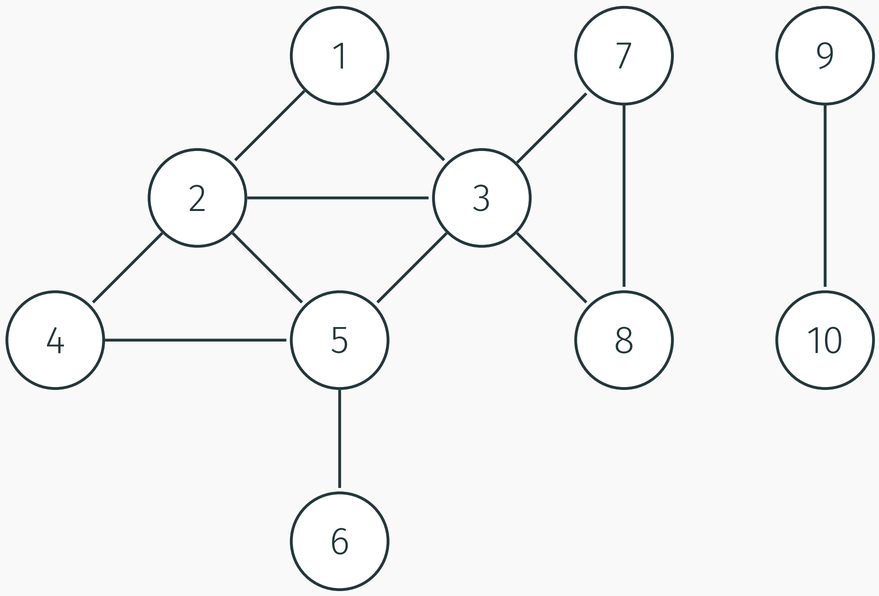
```
DFS(G)
    for all u ∈ V(G) do
        Mark u as unvisited
        Set pred(u) to null
    T is set to ∅
    while ∃ unvisited u do
        DFS(u)
    Output T
```

```
DFS(u)
    Mark u as visited
    for each uv in Out(u) do
        if v is not visited then
            add edge uv to T
            set pred(v) to u
            DFS(v)
```

Implemented using a global array *Visited* for all recursive calls.

T is the search tree/forest.

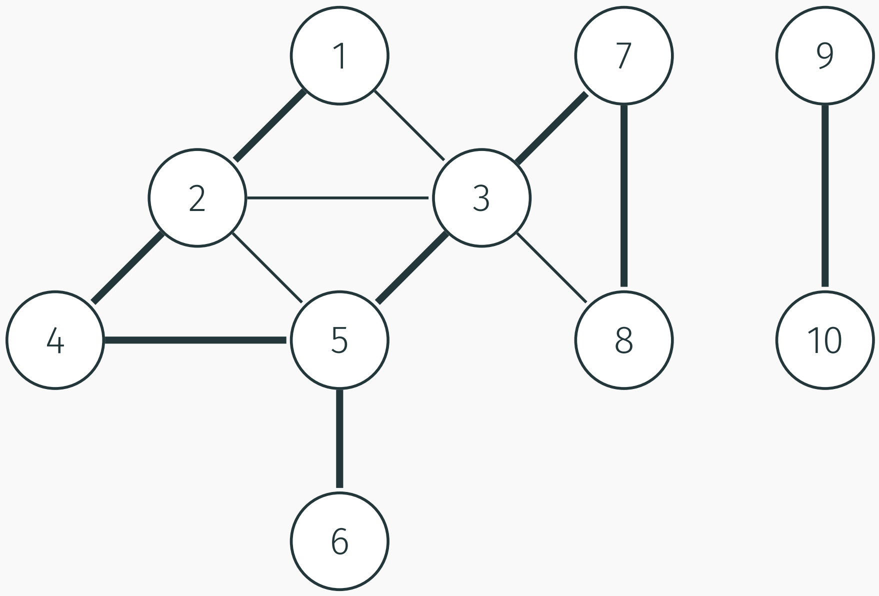
Example



Edges classified into two types: $uv \in E$ is a

- tree edge: belongs to T
- non-tree edge: does not belong to T

Example



Edges classified into two types: $uv \in E$ is a

- tree edge: belongs to T
- non-tree edge: does not belong to T

DFS with pre-post numbering

DFS with Visit Times

Keep track of when nodes are visited.

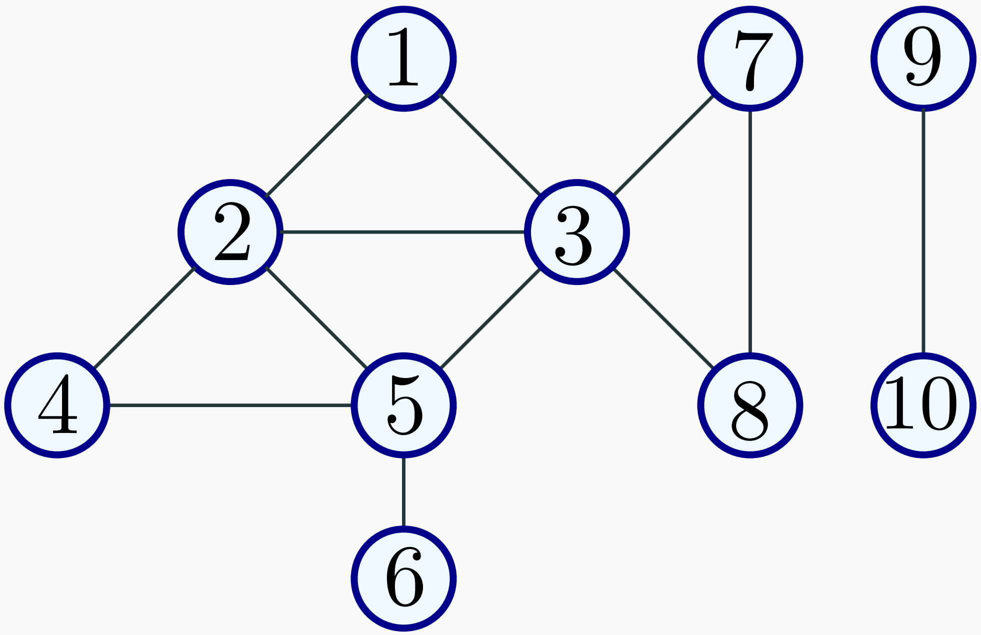
```
DFS(G)
    for all u ∈ V(G) do
        Mark u as unvisited
    T is set to ∅
    time = 0
    while ∃ unvisited u do
        DFS(u)
    Output T
```

```
DFS(u)
    Mark u as visited
    pre(u) = ++time
    for each uv in Out(u) do
        if v is not marked then
            add edge uv to T
            DFS(v)
    post(u) = ++time
```

Animation

$time = 0$

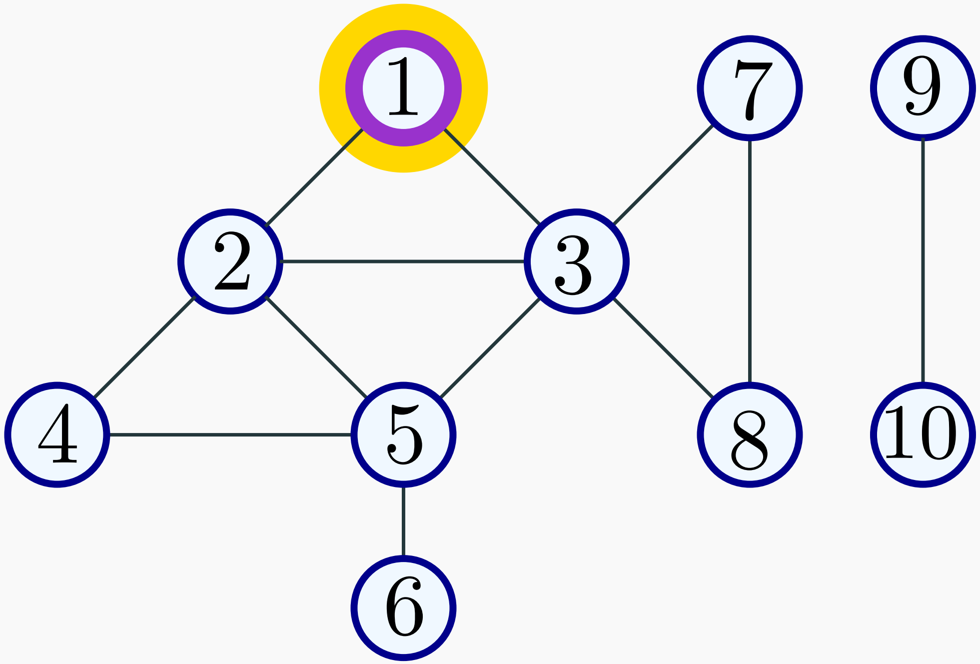
vertex	$[pre, post]$



Animation

$time = 1$

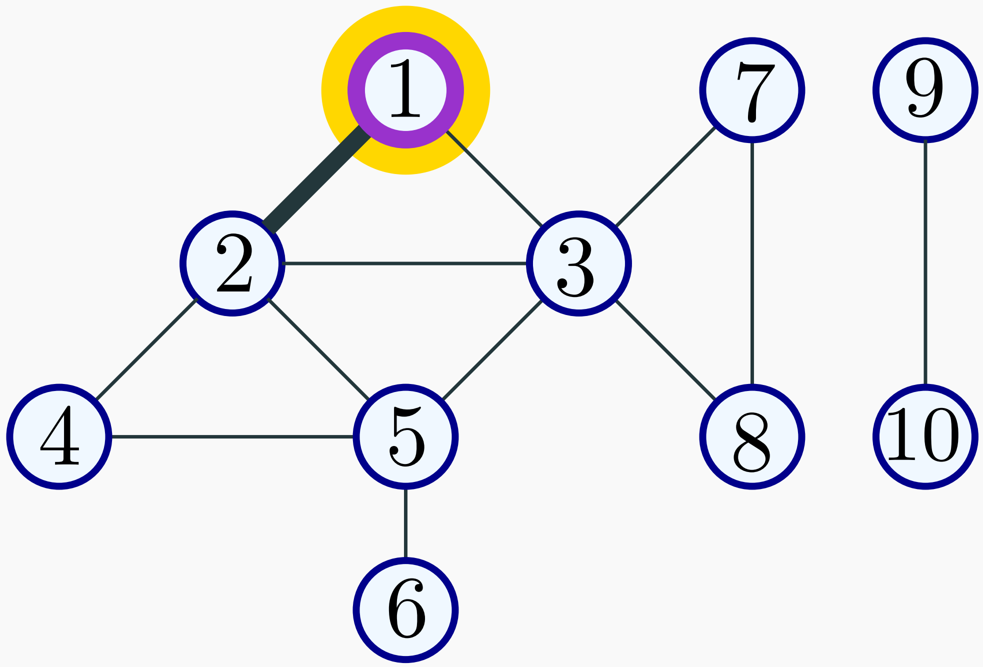
vertex	[pre, $post$]
1	[1,]



Animation

$time = 1$

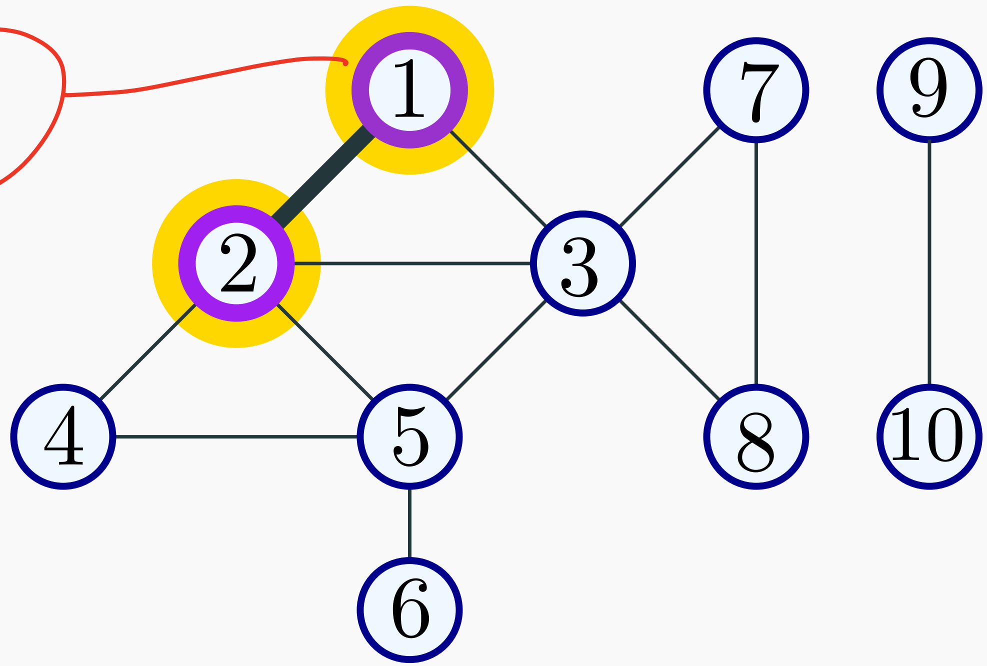
vertex	$[pre, post]$
1	[1,]



Animation

$$time = 2$$

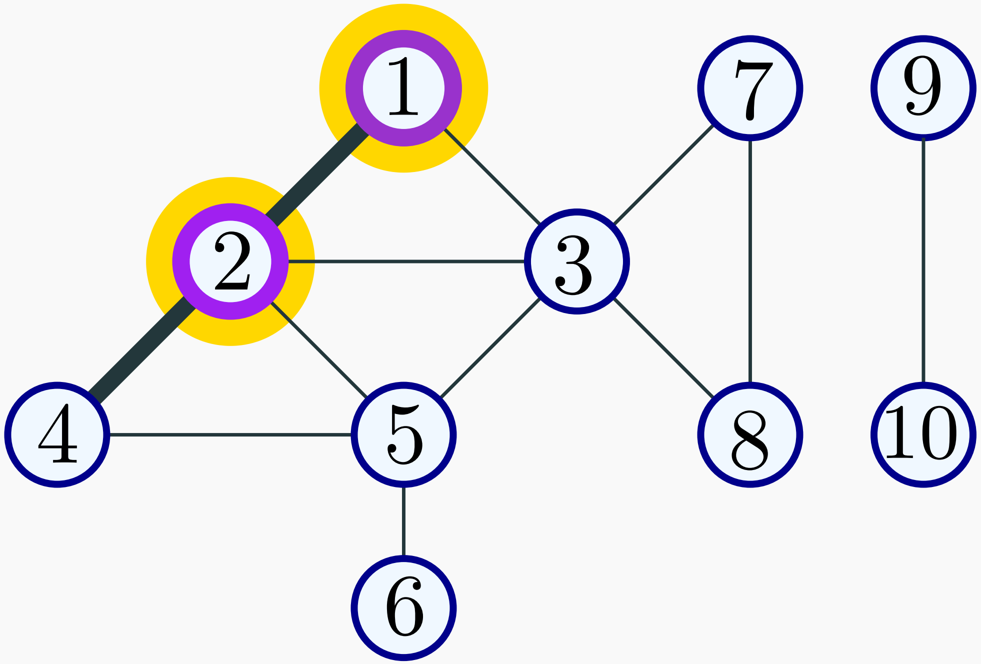
vertex	$[pre, post]$
1	[1,]
2	[2,]



Animation

$$time = 2$$

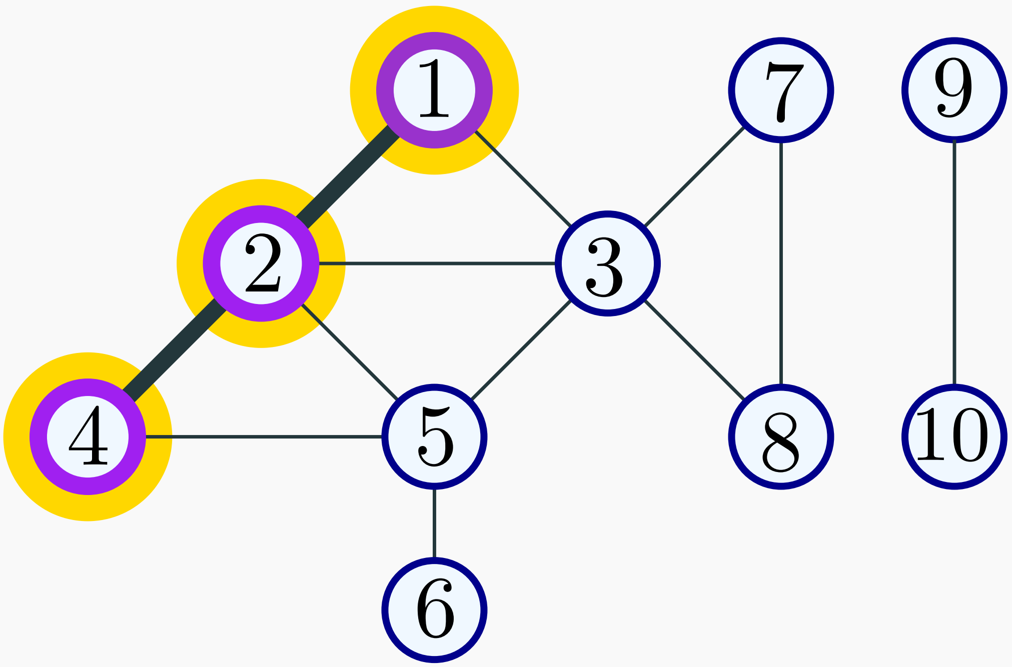
vertex	[*pre*, *post*]
1	[1,]
2	[2,]



Animation

$$time = 3$$

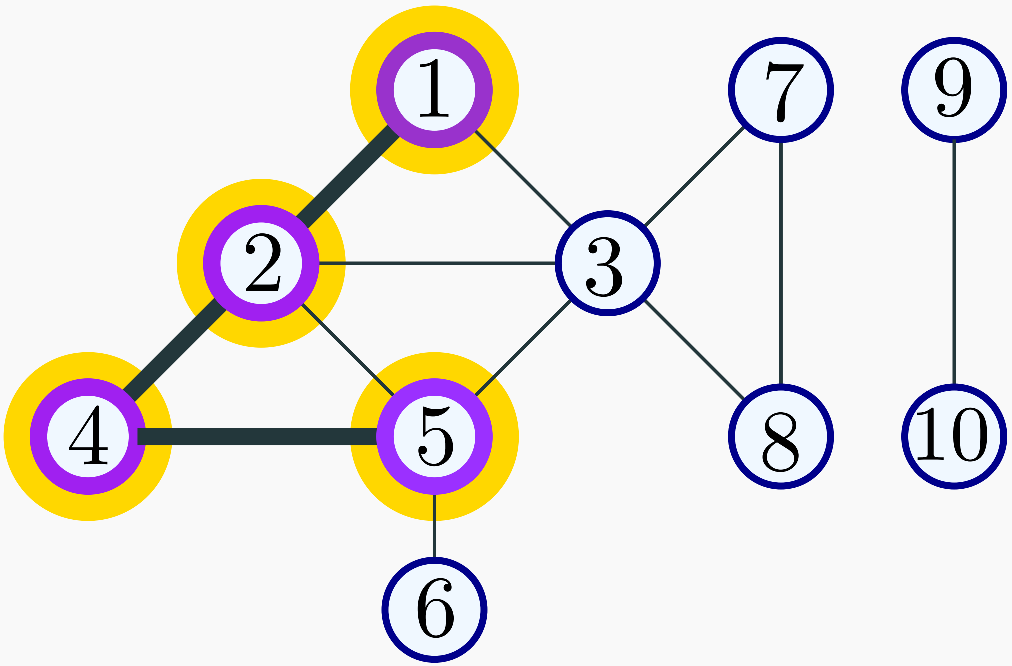
vertex	$[pre, post]$
1	[1,]
2	[2,]
4	[3,]



Animation

$time = 4$

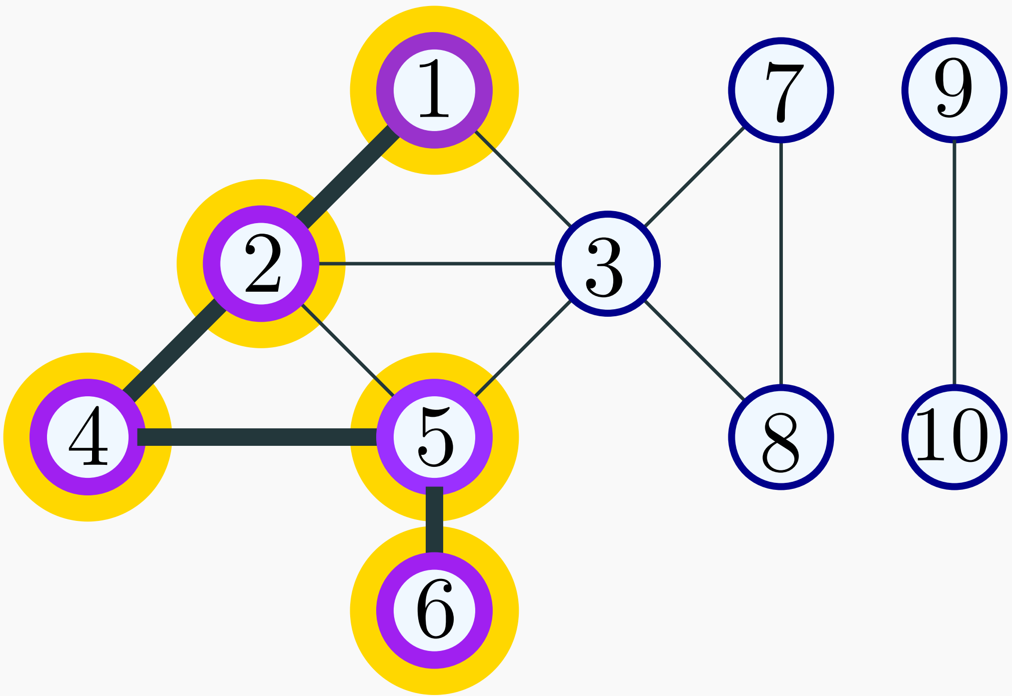
vertex	$[pre, post]$
1	[1,]
2	[2,]
4	[3,]
5	[4,]



Animation

$time = 5$

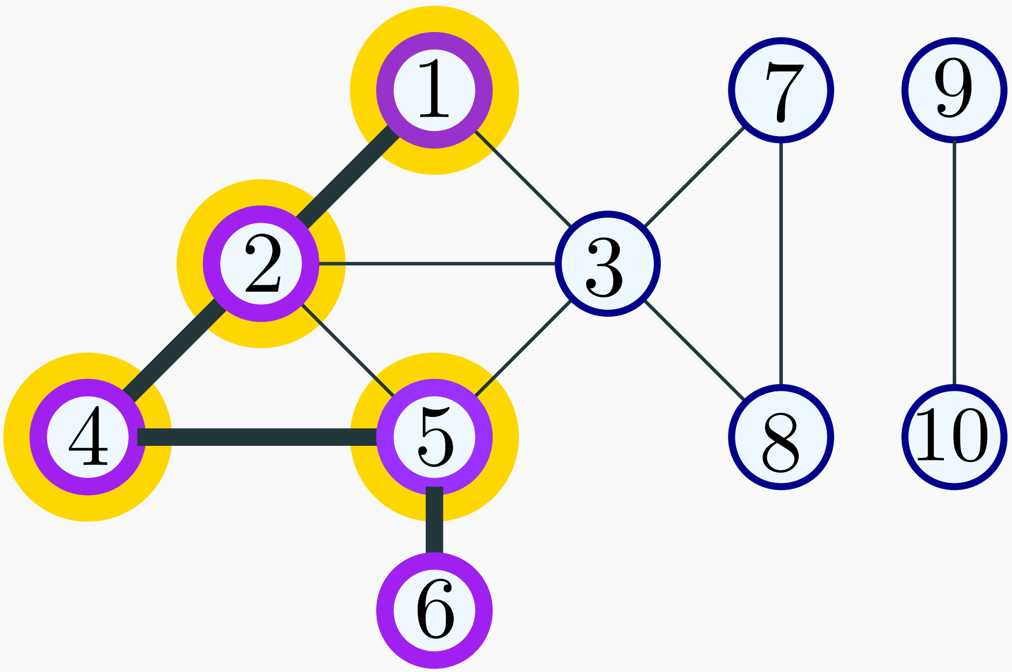
vertex	$[pre, post]$
1	[1,]
2	[2,]
4	[3,]
5	[4,]
6	[5,]



Animation

$time = 6$

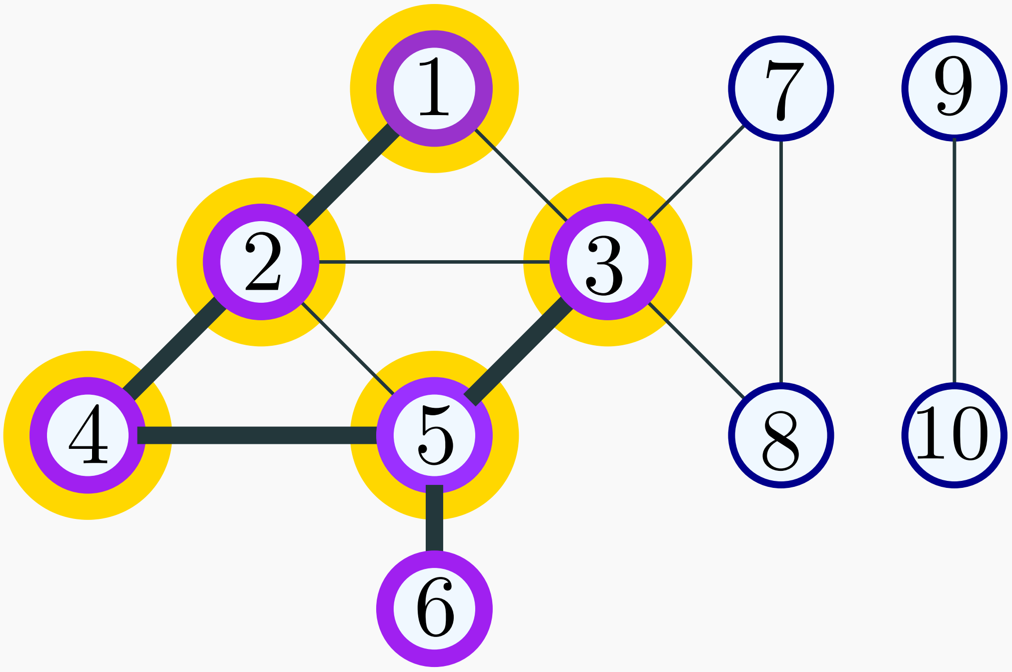
vertex	$[pre, post]$
1	[1,]
2	[2,]
4	[3,]
5	[4,]
6	[5, 6]



Animation

$$time = 7$$

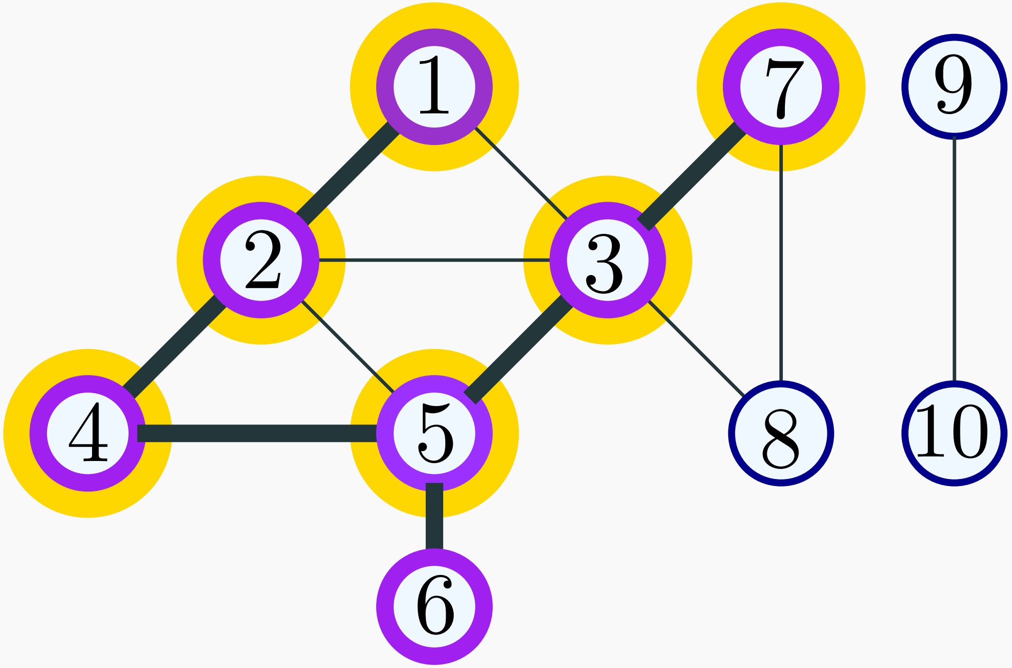
vertex	[*pre*, *post*]
1	[1,]
2	[2,]
4	[3,]
5	[4,]
6	[5, 6]
3	[7,]



Animation

$time = 8$

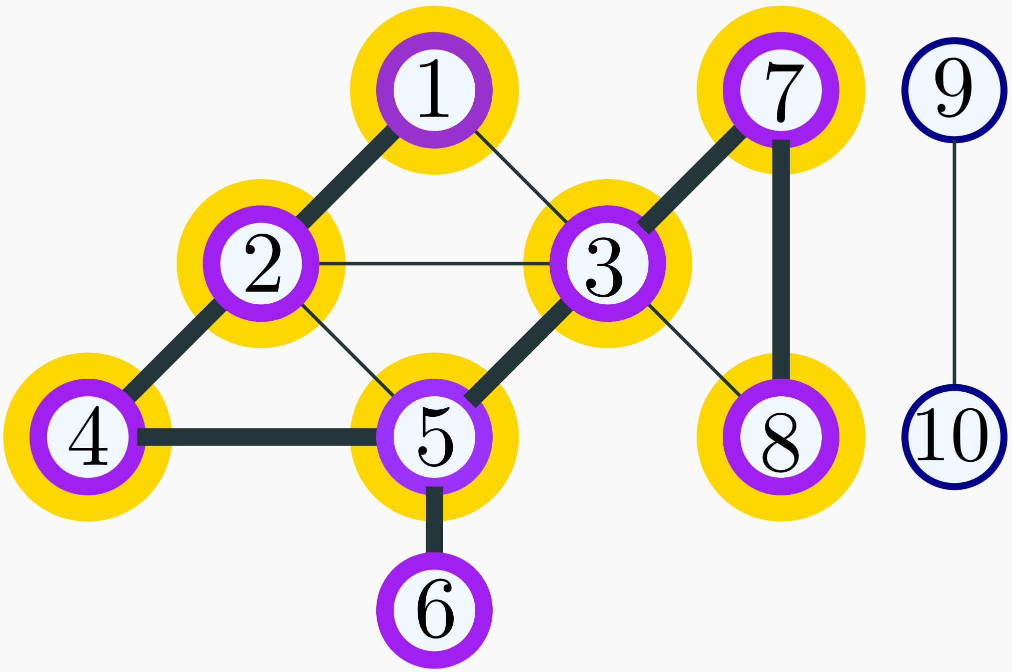
vertex	$[pre, post]$
1	[1,]
2	[2,]
4	[3,]
5	[4,]
6	[5, 6]
3	[7,]
7	[8,]



Animation

$time = 9$

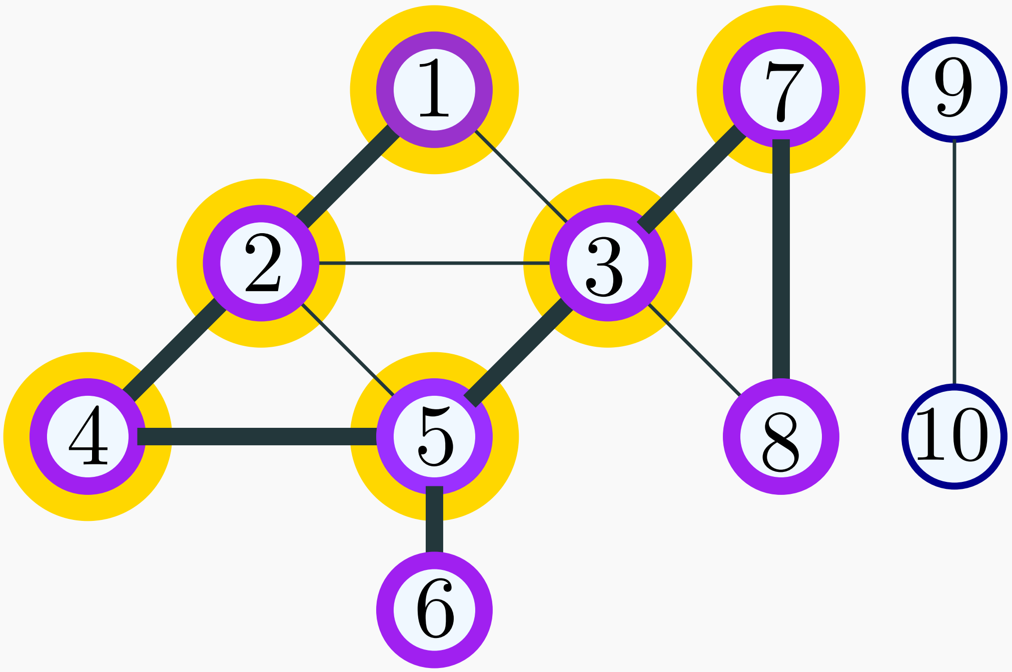
vertex	$[pre, post]$
1	$[1,]$
2	$[2,]$
4	$[3,]$
5	$[4,]$
6	$[5, 6]$
3	$[7,]$
7	$[8,]$
8	$[9,]$



Animation

$time = 10$

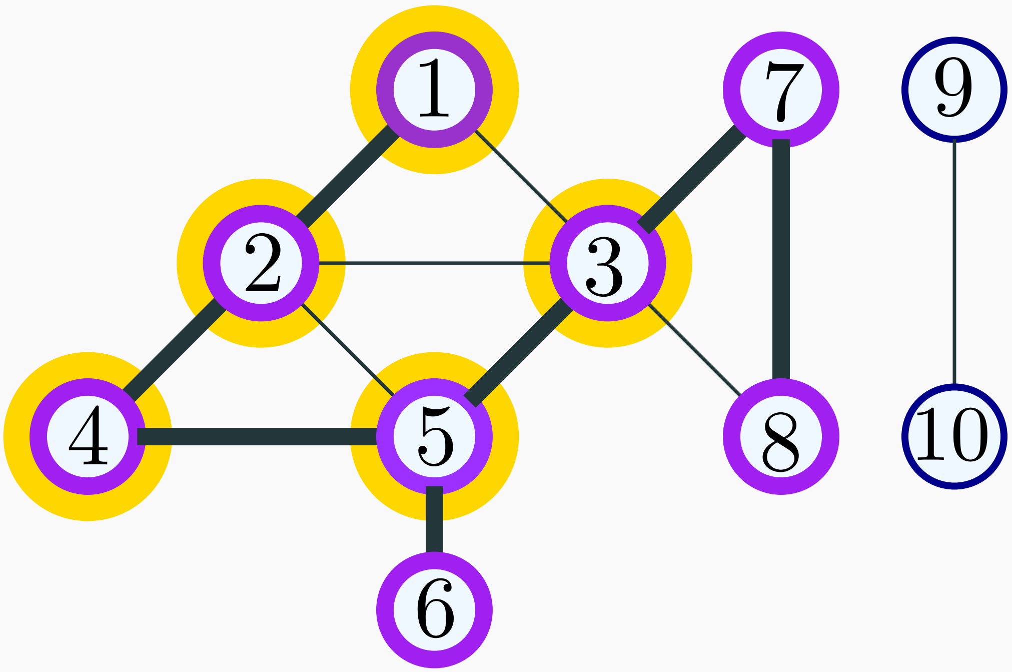
vertex	$[pre, post]$
1	$[1,]$
2	$[2,]$
4	$[3,]$
5	$[4,]$
6	$[5, 6]$
3	$[7,]$
7	$[8,]$
8	$[9, 10]$



Animation

$$time = 11$$

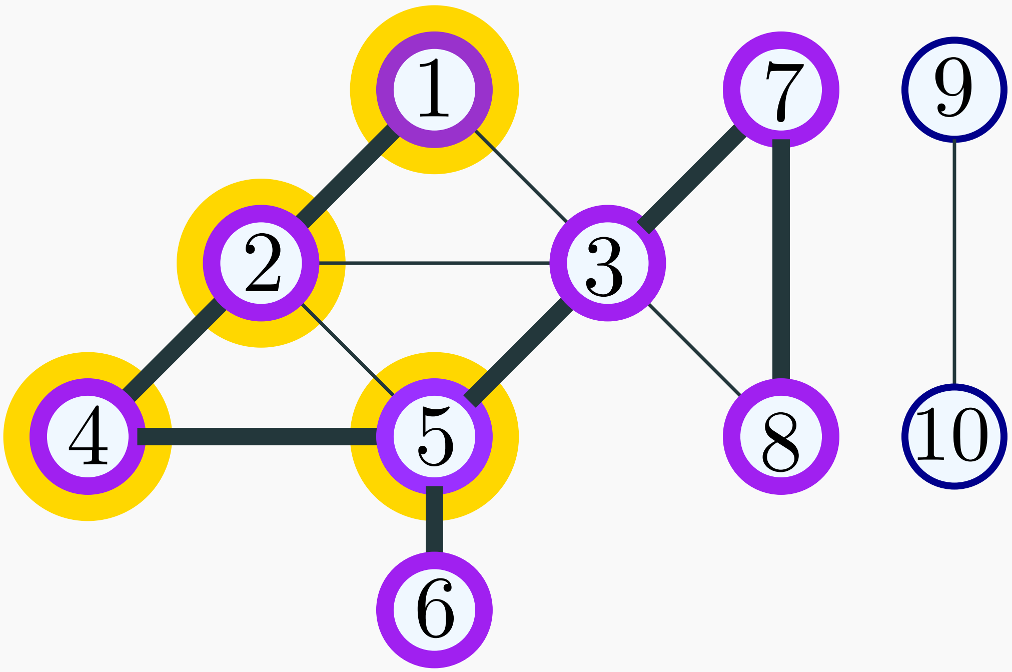
vertex	$[pre, post]$
1	$[1,]$
2	$[2,]$
4	$[3,]$
5	$[4,]$
6	$[5, 6]$
3	$[7,]$
7	$[8, 11]$
8	$[9, 10]$



Animation

$time = 12$

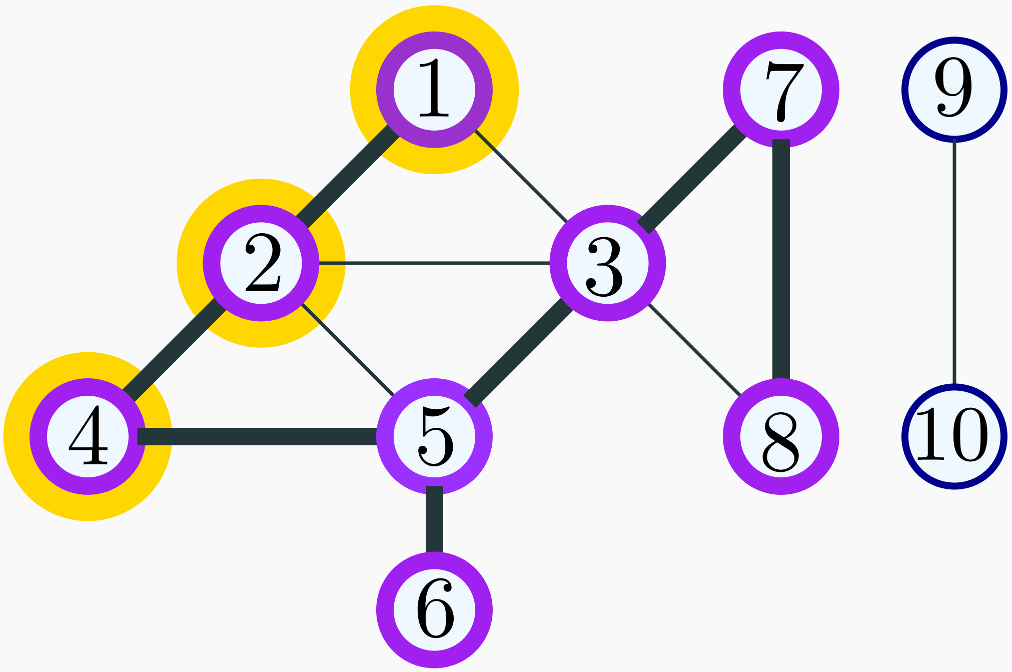
vertex	$[pre, post]$
1	$[1,]$
2	$[2,]$
4	$[3,]$
5	$[4,]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$



Animation

$$time = 13$$

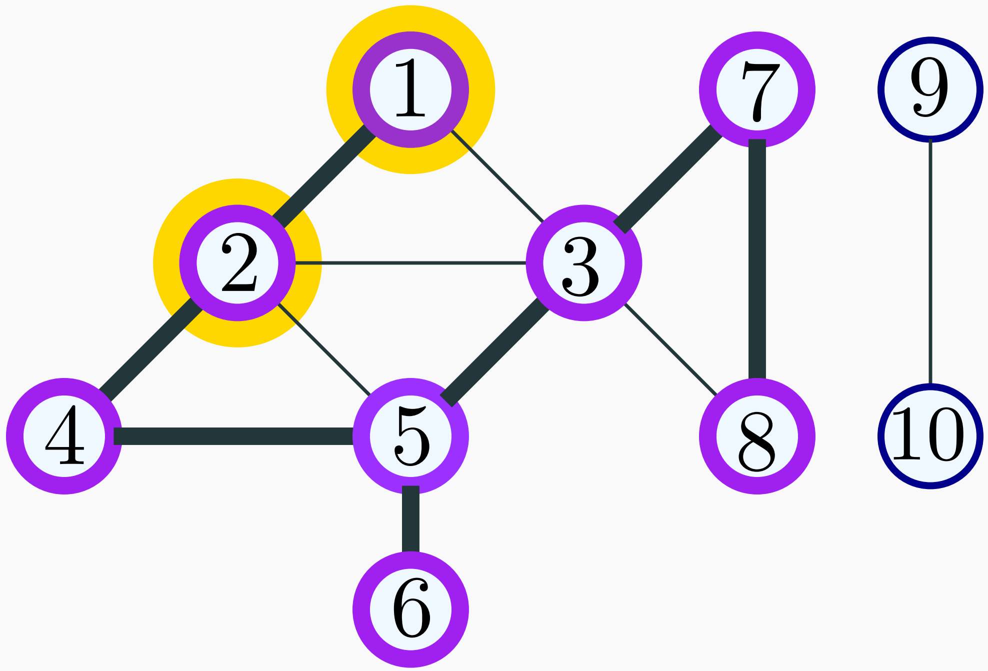
vertex	$[pre, post]$
1	[1,]
2	[2,]
4	[3,]
5	[4, 13]
6	[5, 6]
3	[7, 12]
7	[8, 11]
8	[9, 10]



Animation

$time = 14$

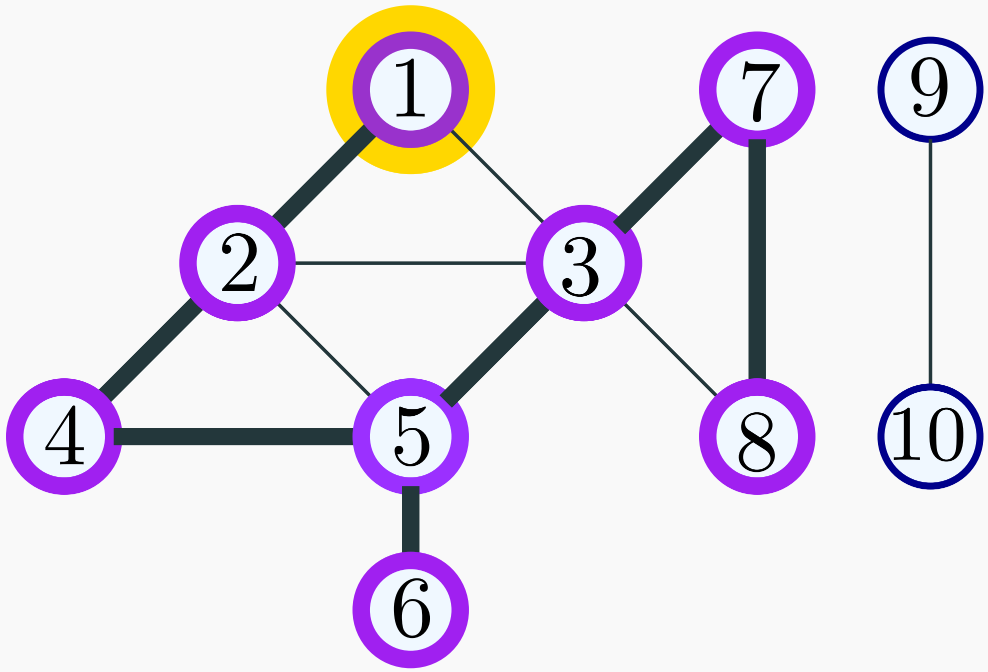
vertex	$[pre, post]$
1	$[1,]$
2	$[2,]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$



Animation

$time = 15$

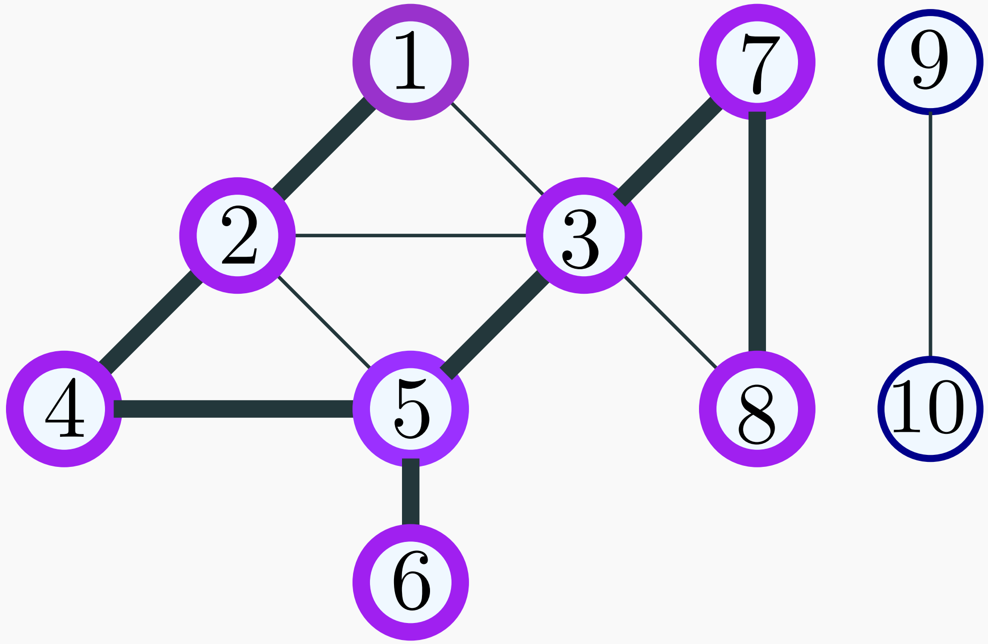
vertex	$[pre, post]$
1	$[1,]$
2	$[2, 15]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$



Animation

$time = 16$

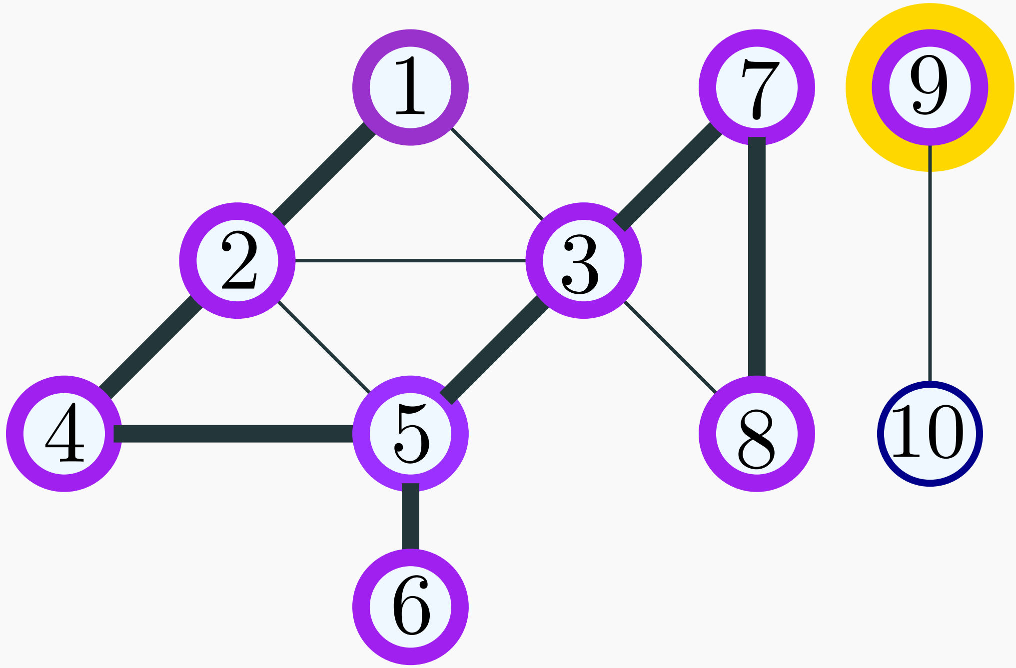
vertex	$[pre, post]$
1	$[1, 16]$
2	$[2, 15]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$



Animation

$$time = 17$$

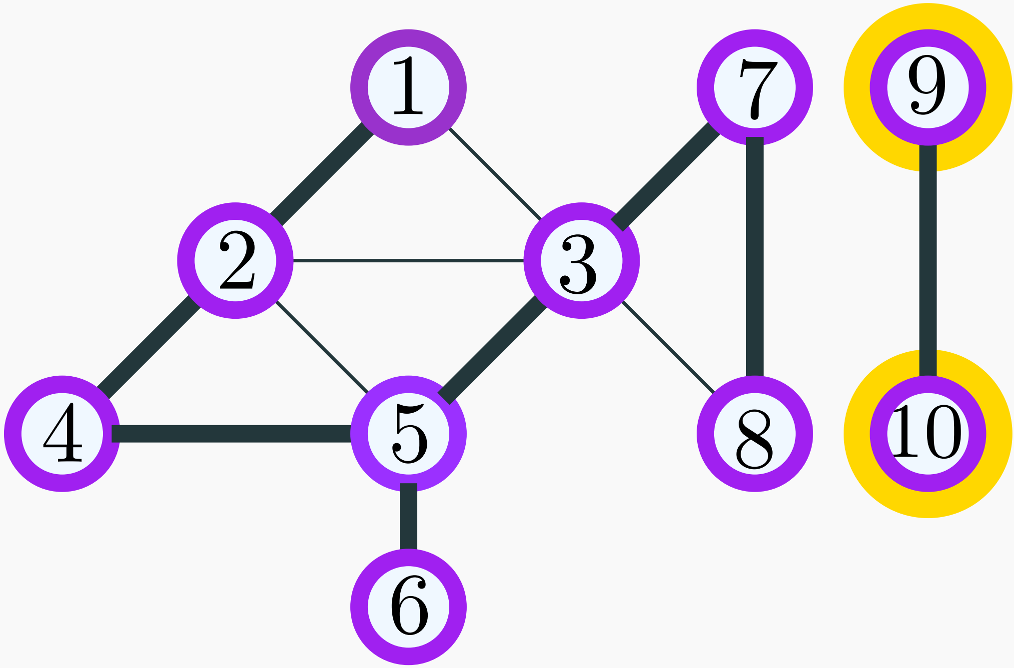
vertex	$[pre, post]$
1	$[1, 16]$
2	$[2, 15]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$
9	$[17,]$



Animation

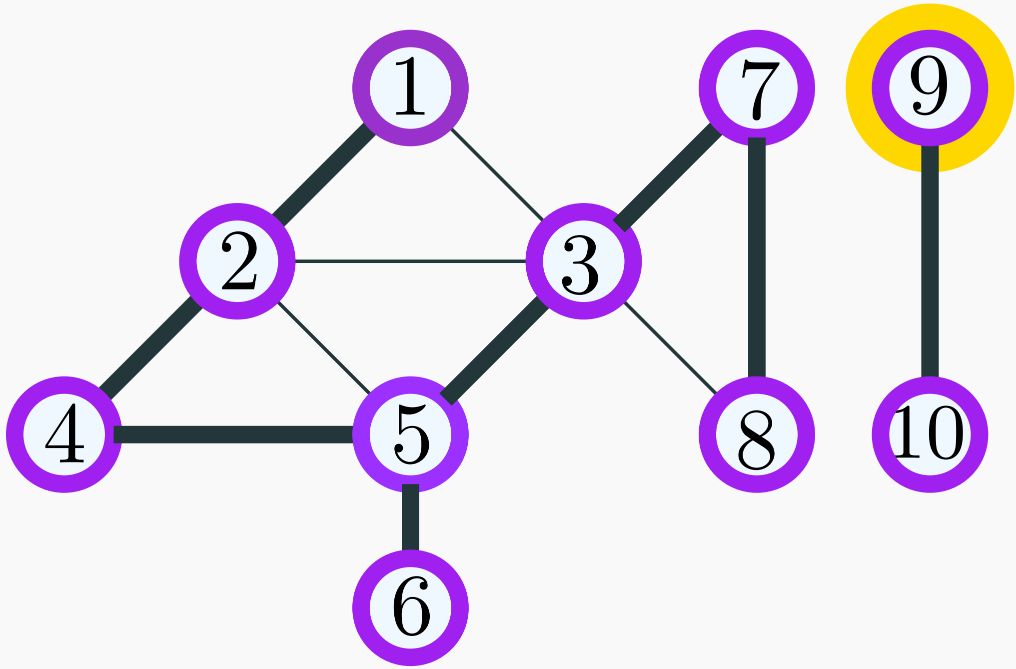
$time = 18$

vertex	$[pre, post]$
1	$[1, 16]$
2	$[2, 15]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$
9	$[17,]$
10	$[18,]$



$time = 19$

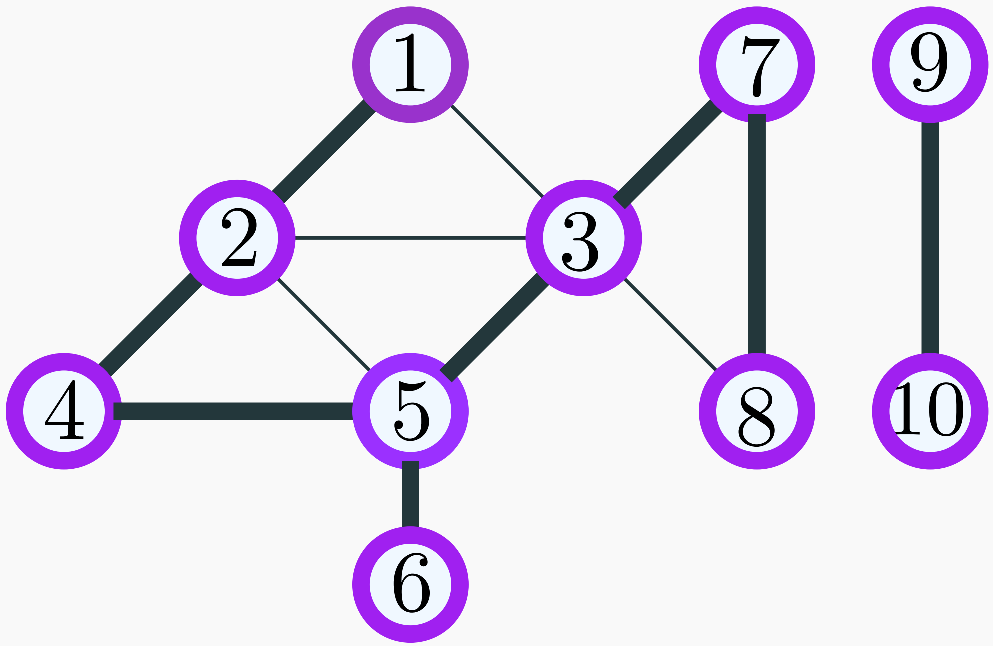
vertex	$[pre, post]$
1	$[1, 16]$
2	$[2, 15]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$
9	$[17,]$
10	$[18, 19]$



Animation

$$time = 20$$

vertex	$[pre, post]$
1	$[1, 16]$
2	$[2, 15]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$
9	$[17, 20]$
10	$[18, 19]$



Animation

vertex	[*pre*, *post*]
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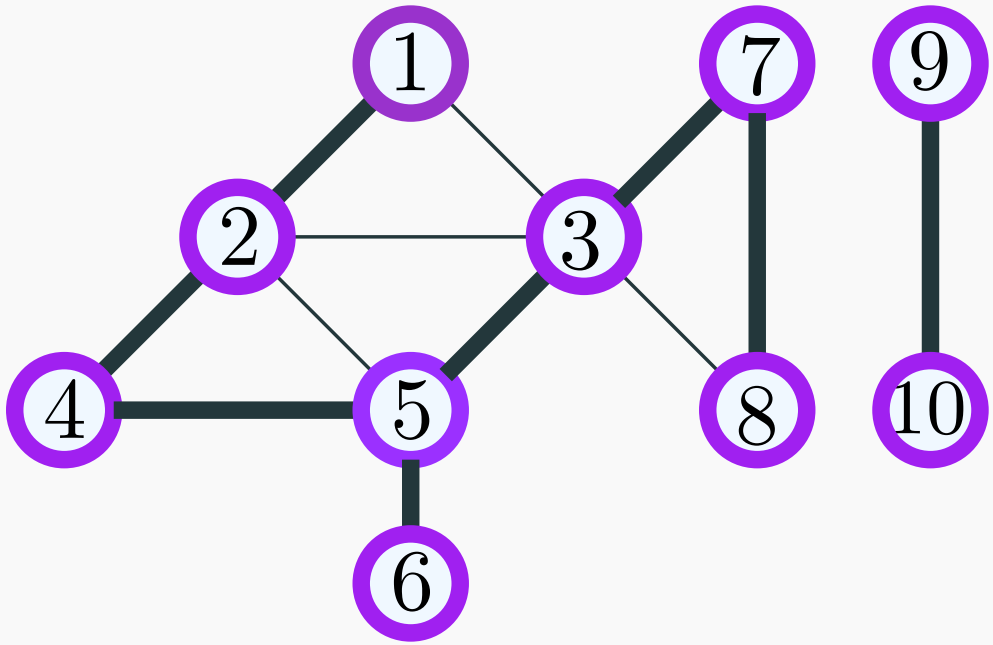
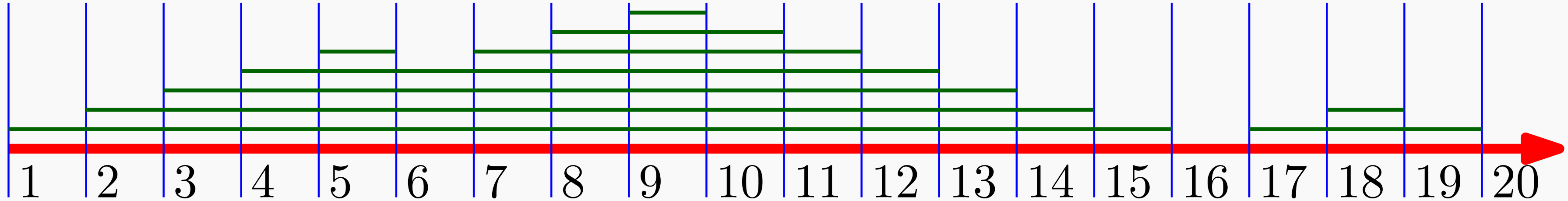
3	[7, 12]
7	[8, 11]
8	[9, 10]
9	[17, 20]
10	[18, 19]





pre and post numbers

Node u is active in time interval $[\mathrm{pre}(u), \mathrm{post}(u)]$

Proposition
For any two nodes u and v, the two intervals $[\mathrm{pre}(u), \mathrm{post}(u)]$ *and* $[\mathrm{pre}(v), \mathrm{post}(v)]$ *are disjoint or one is contained in the other.*

pre and post numbers useful in several applications of DFS

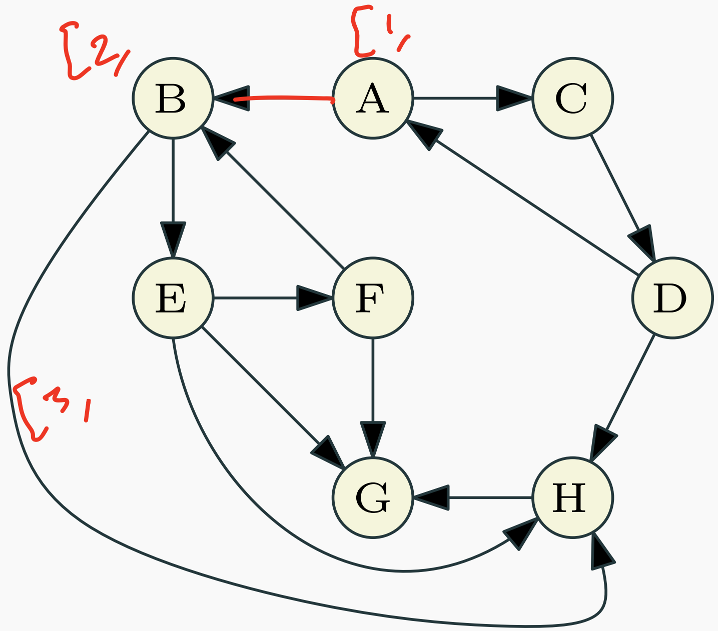
DFS in Directed Graphs

DFS in Directed Graphs

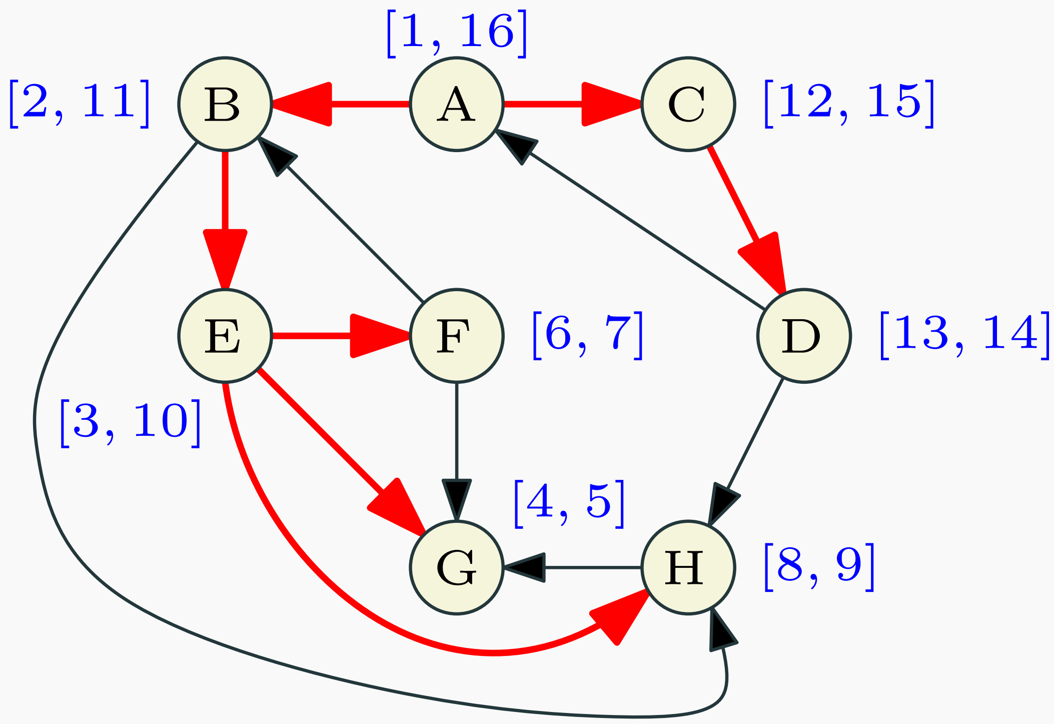
```
DFS(G)
    Mark all nodes u as unvisited
    T is set to ∅
    time = 0
    while there is an unvisited node u do
        DFS(u)
    Output T
```

```
DFS(u)
    Mark u as visited
    pre(u) = ++time
    for each edge (u,v) in Out(u) do
        if v is not visited
            add edge (u,v) to T
            DFS(v)
    post(u) = ++time
```

Example of DFS in directed graph



Example of DFS in directed graph



DFS Properties

Generalizing ideas from undirected graphs:

- DFS(G) takes $O(m + n)$ time.

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DFS Properties

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- If u is the first vertex considered by $DFS(G)$ then $DFS(u)$ outputs a directed out-tree T rooted at u and a vertex v is in T if and only if $v \in \mathrm{rch}(u)$

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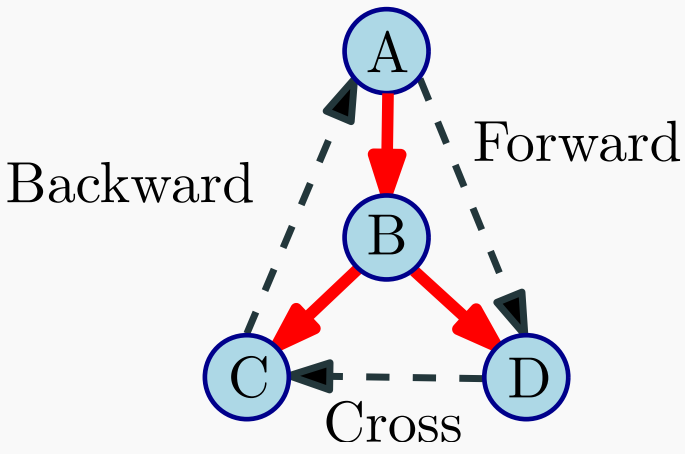
DFS tree and related edges

Edges of G can be classified with respect to the DFS tree T as:

- Tree edges that belong to T
- A forward edge is a non-tree edges (x, y) such that y is a descendant of x .
- A backward edge is a non-tree edge (x, y) such that y is an ancestor of x.
- A cross edge is a non-tree edges (x, y) such that they don't have a ancestor/descendant relationship between them.



DFS tree and related edges

Edges of G can be classified with respect to the DFS tree T as:

- Tree edges that belong to T
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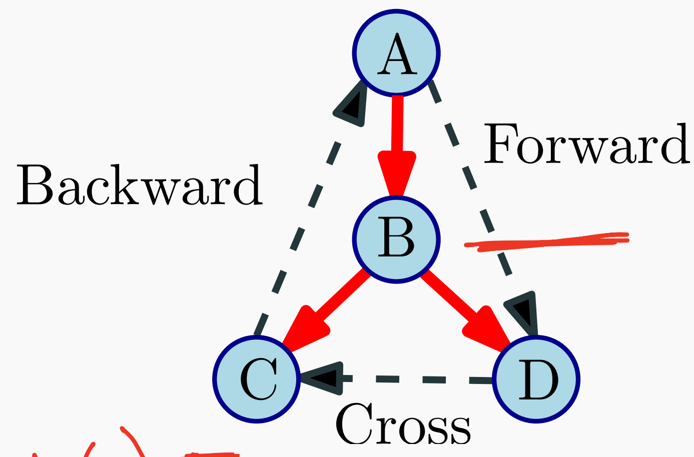
- A backward edge is a non-tree edge (x, y) such that . $\mathrm{pre}(y) < \mathrm{pre}(x) < \mathrm{post}(x) < \mathrm{post}(y)$

  C A
- A cross edge is a non-tree edges (x, y) such that the intervals are disjoint

$\mathrm{pre}(x) < \mathrm{post}(x) < \mathrm{pre}(y) < \mathrm{post}(y)$

or $\mathrm{pre}(y) < \mathrm{post}(y) < \mathrm{pre}(x) < \mathrm{post}(x)$



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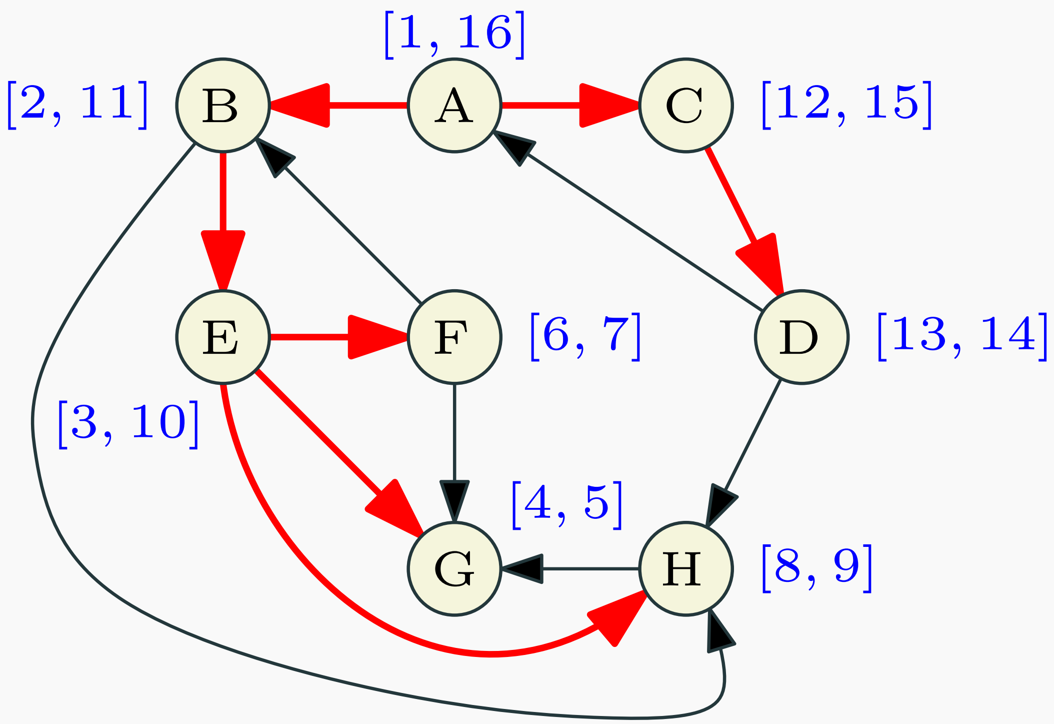
Types of Edges



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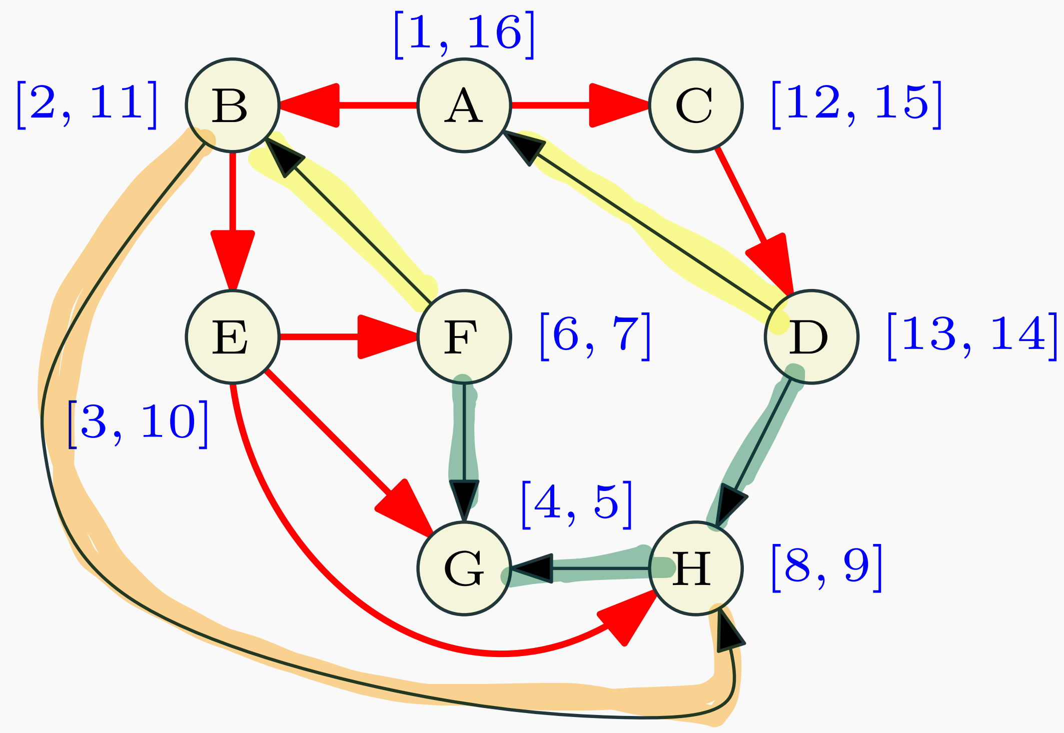

- Back edges:
- Forward edges:
- Cross edges:

DFS and cycle detection: Topological sorting using DFS

Cycles in graphs

Given an undirected graph how do we check whether it has a cycle and output one if it has one? If any edge isn't in T then there has to be a cycle

Given an undirected graph how do we check whether it has a cycle and output one if it has one?

Question: Given an directed graph how do we check whether it has a cycle and output one if it has one?

If we have a back edge (u,v)

Cycle = tree edges from $u \rightsquigarrow v$
(get this from pre/post ordering)

$+ (u,v)$

Cycle detection in directed graph using topological sorting

Question
Given G, is it a DAG?

If it is, compute a topological sort.

If it fails, then output the cycle C.

Topological sort a graph using DFS

DFS based algorithm:

- Compute DFS(G)
- If there is a back edge $e = (v, u)$ then G is not a DAG. Output cycle C formed by path from u to v in T plus edge (v, u).
- Otherwise output nodes in decreasing post-visit order. Note: no need to sort, $DFS(G)$ can output nodes in this order.

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Computes topological ordering of the vertices.

Algorithm runs in $O(n + m)$ time.

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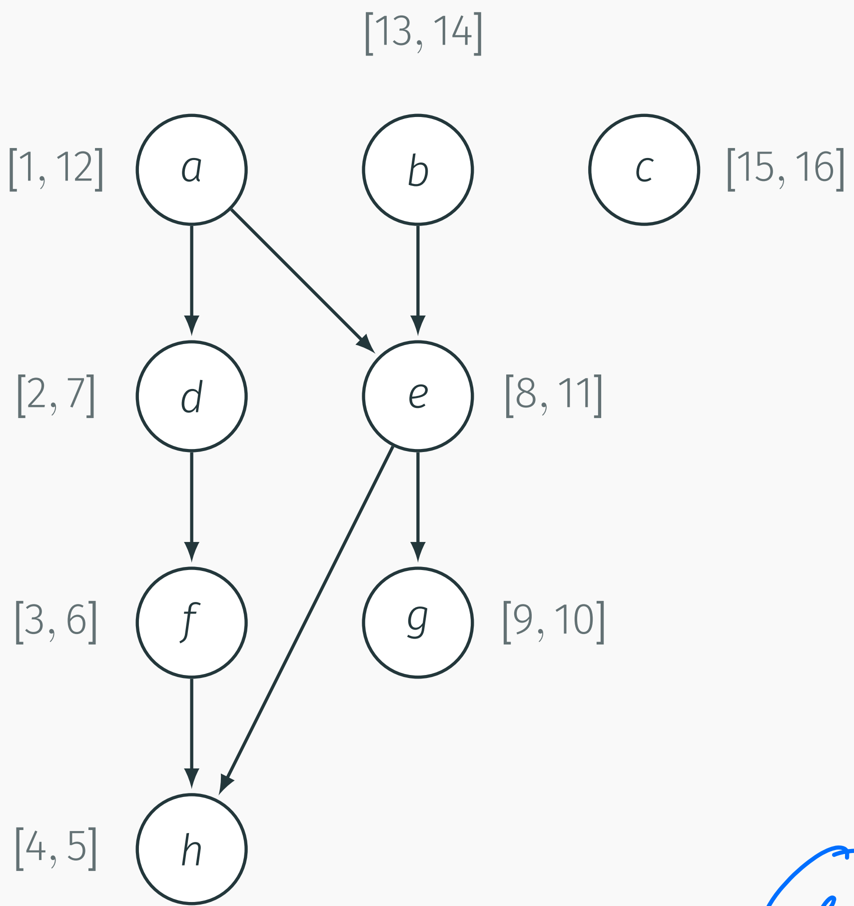
Computes topological ordering of the vertices.

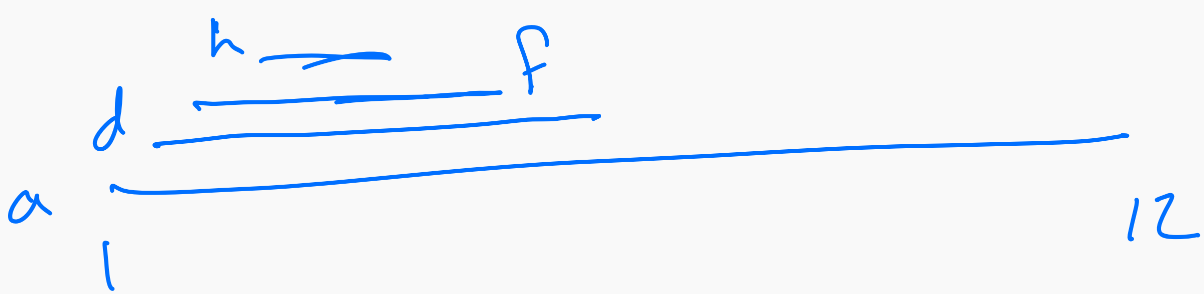
Algorithm runs in $O(n + m)$ time. Correctness is not so obvious.

See next two propositions.

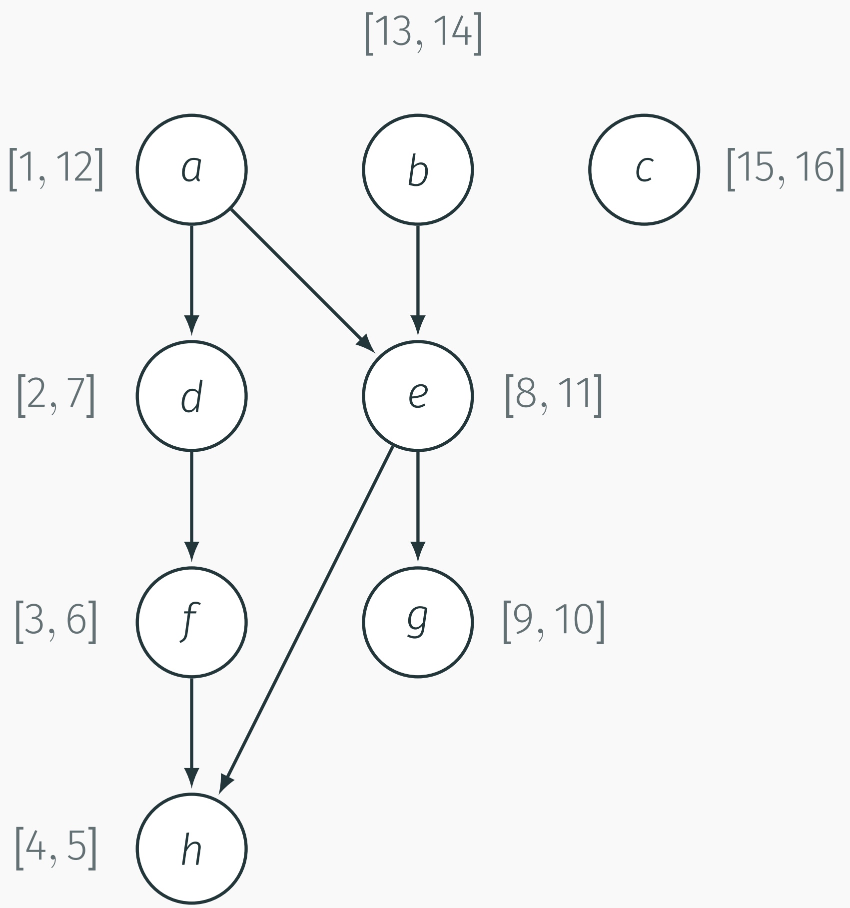
Example



a d f h



Example

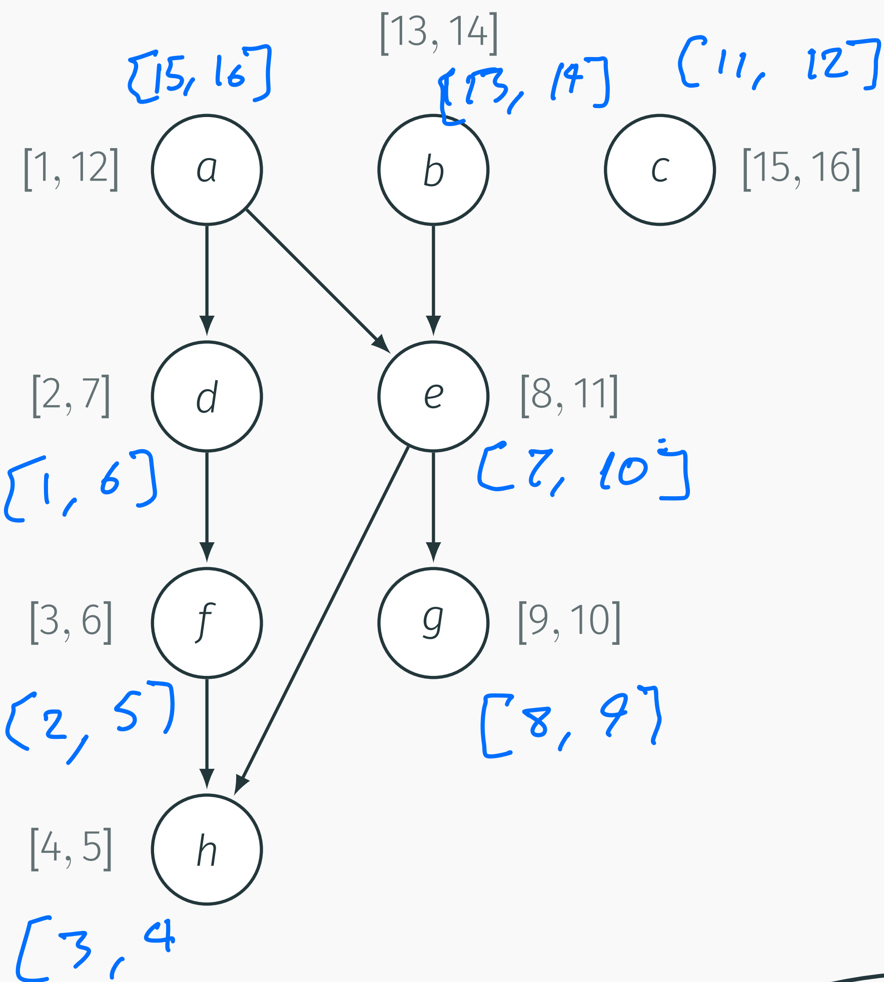


Listing out the vertices in post-number decreasing gives:

c,b,a,e,g,d,f,h

Remind you of anything?

Example



Listing out the vertices in post-number decreasing gives:

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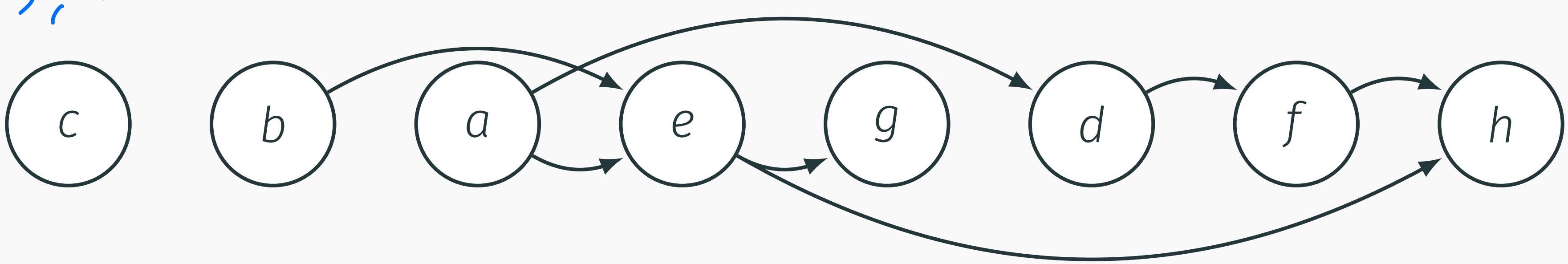
a, d, f, c, b, e, h, g

Remind you of anything?



Back edge and Cycles

Proposition
G has a cycle $\iff$ there is a back-edge in DFS(G).

Proof.
If: (u, v) is a back edge implies there is a cycle C consisting of the path from v to u in DFS search tree and the edge (u, v).

Only if: Suppose there is a cycle $C = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \rightarrow v_1$.

Let v_i be first node in C visited in DFS.

All other nodes in C are descendants of v_i since they are reachable from v_i.

Therefore, (v_{i-1}, v_i) (or (v_k, v_1) if $i = 1$) is a back edge. $\square$

Decreasing post numbering is valid

Proposition
If G is a DAG and $\mathrm{post}(v) > \mathrm{post}(u)$*, then* $(u \to v)$ *is not in* G*.*

Proof.
Assume $\mathrm{post}(u) < \mathrm{post}(v)$ and $(u \to v)$ is an edge in G.

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If G is a DAG and $\mathrm{post}(v) > \mathrm{post}(u)$, then $(u \to v)$ is not in G.

Proof.
Assume $\mathrm{post}(u) < \mathrm{post}(v)$ and $(u \to v)$ is an edge in G. One of two holds:

- Case 1: $[pre(u), post(u)]$ is contained in $[pre(v), post(v)]$.
- Case 2: $[pre(u), post(u)]$ is disjoint from $[pre(v), post(v)]$.

$\square$

Decreasing post numbering is valid

Proposition
If G is a DAG and $\mathrm{post}(v) > \mathrm{post}(u)$*, then* $(u \to v)$ *is not in G.*

Proof.
Assume $\mathrm{post}(u) < \mathrm{post}(v)$ and $(u \to v)$ is an edge in G. One of two holds:

- Case 1: $[pre(u), post(u)]$ is contained in $[pre(v), post(v)]$. Implies that u is explored during DFS(v) and hence is a descendent of v. Edge (u, v) implies a cycle in G but G is assumed to be DAG!
- Case 2: $[pre(u), post(u)]$ is disjoint from $[pre(v), post(v)]$. This cannot happen since v would be explored from u.

□

Translation

We just proved:

Proposition
If G is a DAG and $\mathrm{post}(v) > \mathrm{post}(u)$, then $(u \to v)$ is not in G.

$\implies$ sort the vertices of a DAG by decreasing post nubmering in decreasing order, then this numbering is valid.

Theorem
$G = (V, E)$: Graph with n vertices and m edges.

Comptue a topological sorting of G using DFS *in $O(n + m)$ time.*

That is, compute a numbering $\pi : V \to \{1, 2, \ldots, n\}$, such that

$$(u \to v) \in E(G) \implies \pi(u) < \pi(v).$$

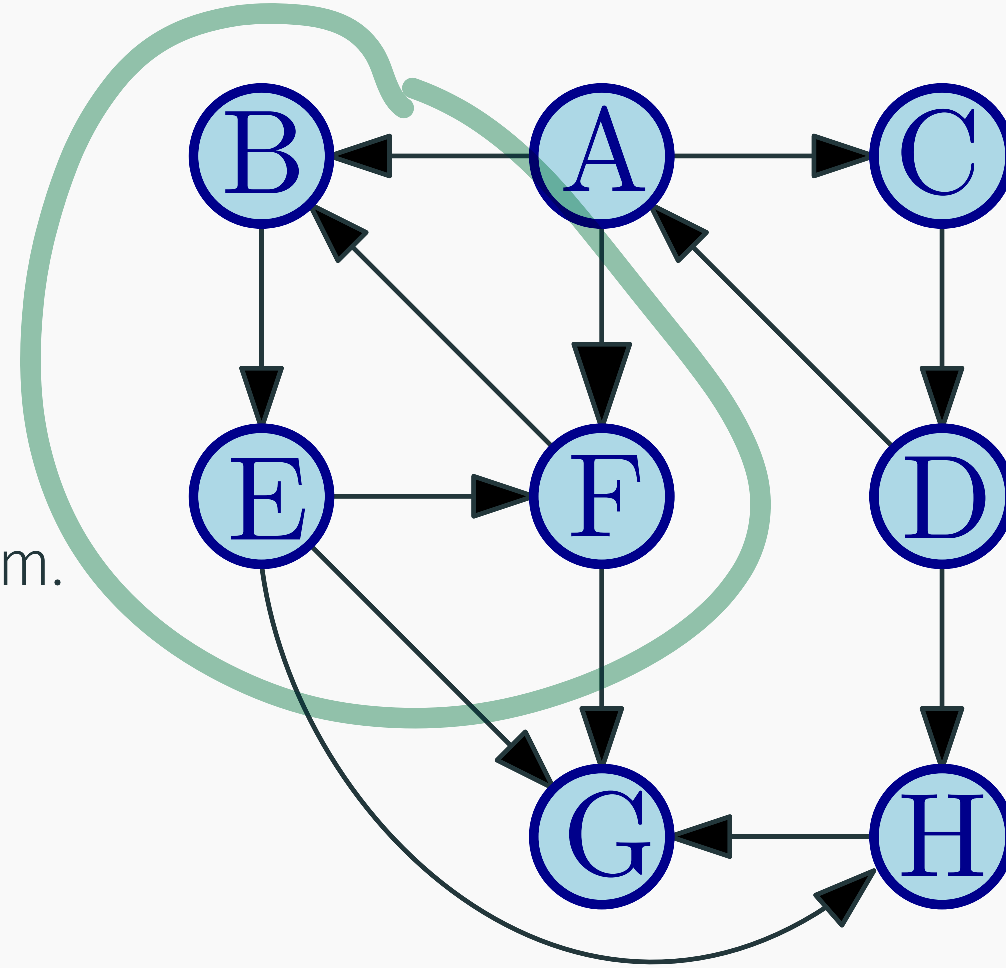
The meta graph of strong connected components

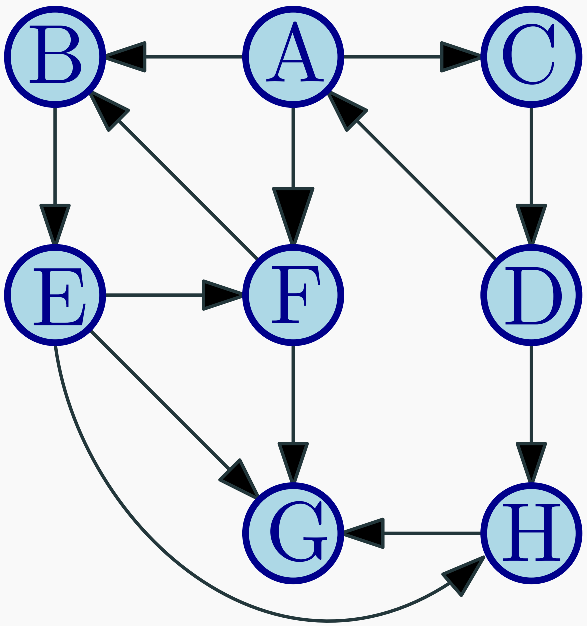
Algorithmic Problem
Find all SCCs of a given directed graph.

Previous lecture:

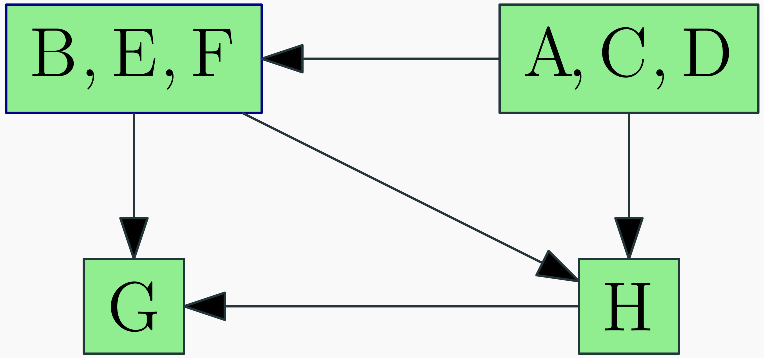
Saw an $O(n \cdot (n+m))$ time algorithm.

This lecture: sketch of a $O(n+m)$ time algorithm.





G:



Graph of SCCs G^{SCC}

Meta-graph of SCCs

Let $S_1, S_2, \ldots S_k$ be the strong connected components (i.e., SCCs) of G. The graph of SCCs is G^{SCC}

- Vertices are $S_1, S_2, \ldots S_k$
- There is an edge (S_i, S_j) if there is some $u \in S_i$ and $v \in S_j$ such that (u, v) is an edge in G.

The meta graph of SCCs is a DAG...

Proposition
For any graph G, the graph G^{SCC} has no directed cycle.

Proof.
If G^{SCC} has a cycle $S_1, S_2, \ldots, S_k$ then $S_1 \cup S_2 \cup \cdots \cup S_k$ should be in the same SCC in G. $\square$

To Remember: Structure of Graphs

Undirected graph: connected components of $G = (V, E)$ partition V and can be computed in $O(m + n)$ time.

Directed graph: the meta-graph $\mathrm{G}^{\mathrm{SCC}}$ of G can be computed in $O(m + n)$ time. $\mathrm{G}^{\mathrm{SCC}}$ gives information on the partition of V into strong connected components and how they form a DAG structure.

Above structural decomposition will be useful in several algorithms

Linear time algorithm for finding all SCCs

Finding all SCCs of a Directed Graph

Problem

Given a directed graph $G = (V, E)$, output all its strong connected components.

Finding all SCCs of a Directed Graph

Problem
Given a directed graph $G = (V, E)$, output all its strong connected components.

Straightforward algorithm:

```
Mark all vertices in V as not visited.
for each vertex u ∈ V not visited yet do
    find SCC(G, u) the strong component of u:
        Compute rch(G, u) using DFS(G, u)
        Compute rch(G^rev, u) using DFS(G^rev, u)
        SCC(G, u) ⇐ rch(G, u) ∩ rch(G^rev, u)
        ∀u ∈ SCC(G, u): Mark u as visited.
```

Running time: $O(n(n + m))$

Finding all SCCs of a Directed Graph

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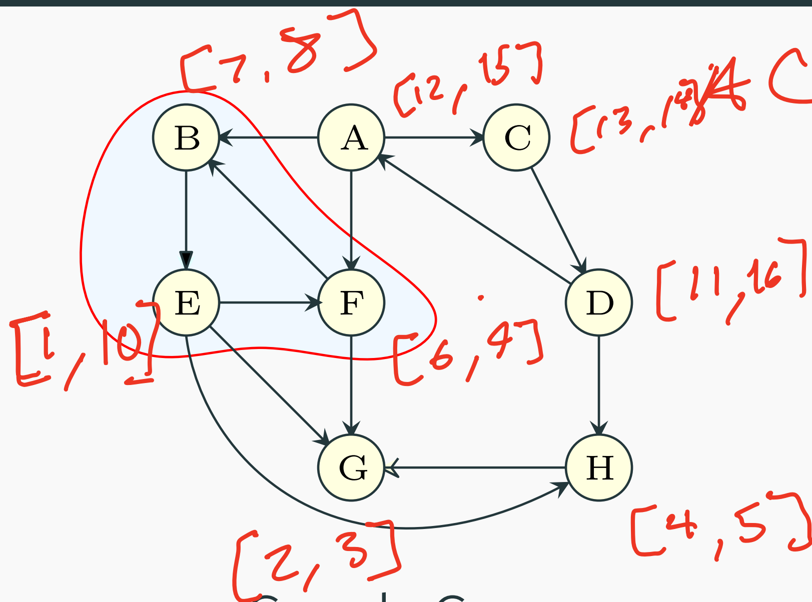
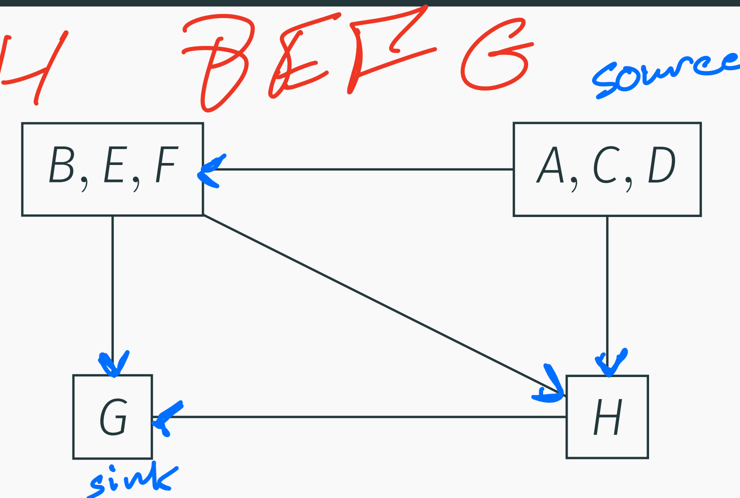
Running time: $O(n(n+m))$ Is there an $O(n+m)$ time algorithm?

Structure of a Directed Graph



Graph G

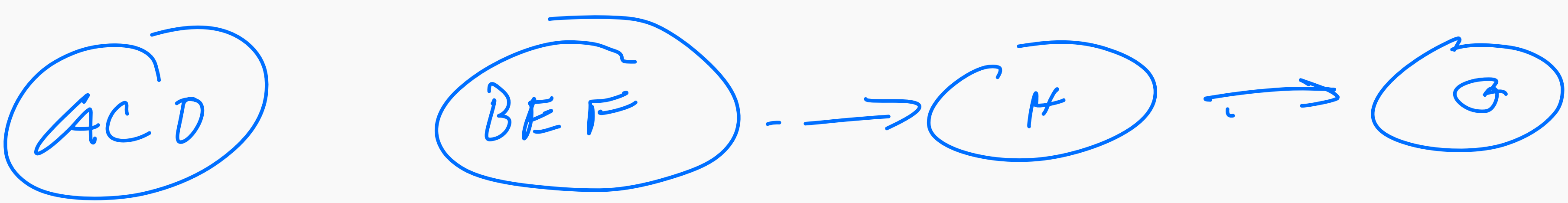


Graph of SCCs G^{SCC}

D, ~~A~~, C, E, . . .

Reminder G^{SCC} is created by collapsing every strong connected component to a single vertex.

Proposition
For a directed graph G, its meta-graph G^{SCC} is a DAG.



Linear-time Algorithm for SCCs: Ideas

Wishful Thinking Algorithm

- Let u be a vertex in a sink SCC of G^{SCC}
- Do DFS(u) to compute $\mathrm{SCC}(u)$
- Remove $\mathrm{SCC}(u)$ and repeat

Linear-time Algorithm for SCCs: Ideas

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- ... since there are no edges coming out a sink!
- DFS(u) takes time proportional to size of $SCC(u)$
- Therefore, total time $O(n + m)$!

Big Challenge(s)

How do we find a vertex in a sink SCC of G^{SCC}?

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How do we find a vertex in a sink SCC of G^{SCC}?

Can we obtain an implicit topological sort of G^{SCC} without computing G^{SCC}?

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Can we obtain an implicit topological sort of G^{SCC} without computing G^{SCC}?

Answer: **DFS**(G) gives some information!

Maximum post numbering and the source of the meta-graph

Post numbering and the meta graph

Claim
Let v be the vertex with maximum post numbering in DFS(G).
Then v is in a SCC S, such that S is a source of G^{SCC}.

Claim
Let v be the vertex with maximum post numbering in DFS(G^{rev}).
Then v is in a SCC S, such that S is a sink of G^{SCC}.

Reverse post numbering and the meta graph

Claim
Let v be the vertex with maximum post numbering in DFS(G^{rev}).
Then v is in a SCC S, such that S is a sink of G^{SCC}.

Holds even after we delete the vertices of S (i.e., the vertex with the maximum post numbering, is in a sink of the meta graph).

The linear-time SCC algorithm itself

```
do DFS(G^rev) and output vertices in decreasing post order.        O(n+m)
Mark all nodes as unvisited O(n)
for each u in the computed order do O(n)
    if u is not visited then
        DFS(u) O(n+m)
        Let S_u be the nodes reached by u
        Output S_u as a strong connected component
        Remove S_u from G
```

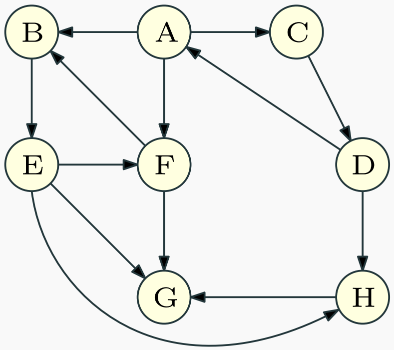
O(n+m)

Theorem

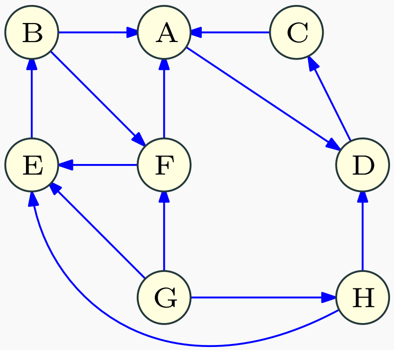
Algorithm runs in time $O(m + n)$ and correctly outputs all the SCCs of G.

Linear Time Algorithm: An Example - Initial steps 1

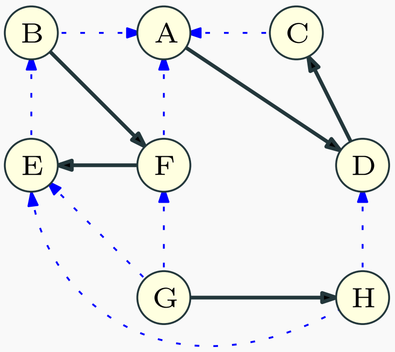
Graph G:



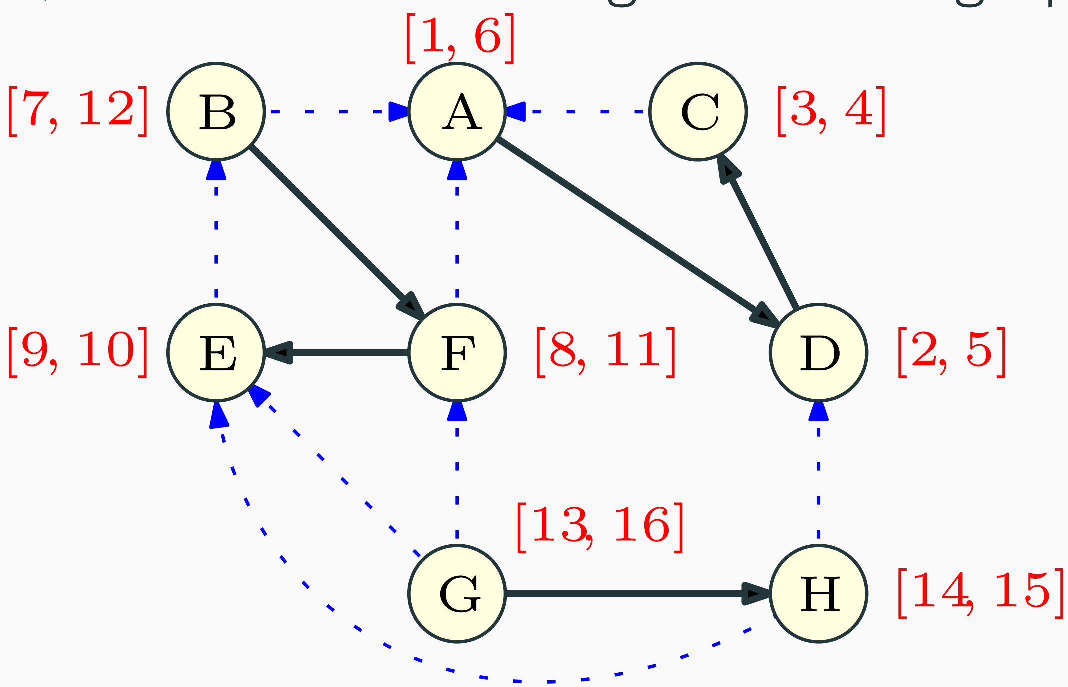
Reverse graph G^{rev}:



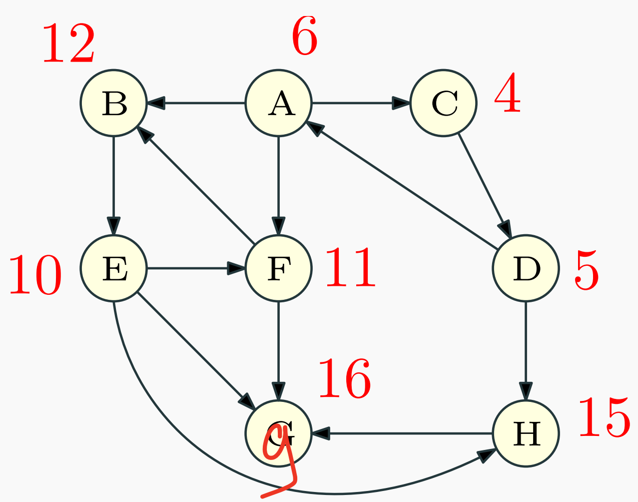
DFS of reverse graph:



Pre/Post DFS numbering of reverse graph:



Original graph G with rev post numbers:

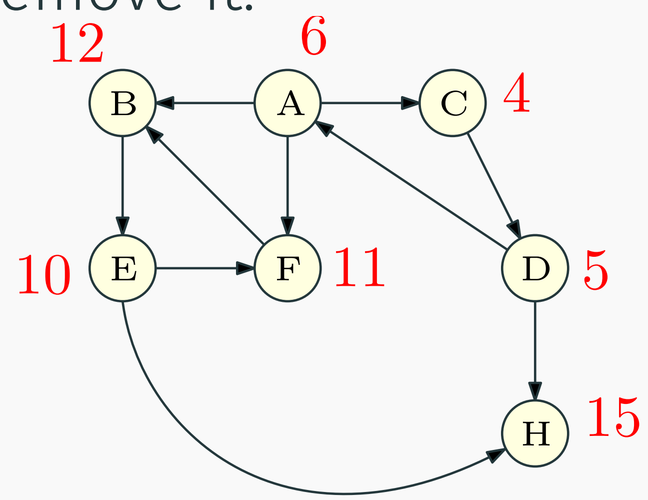


DFS(G, g)

DFS(G^{rev}, g)

$\Longrightarrow$

Do **DFS** from vertex G remove it.

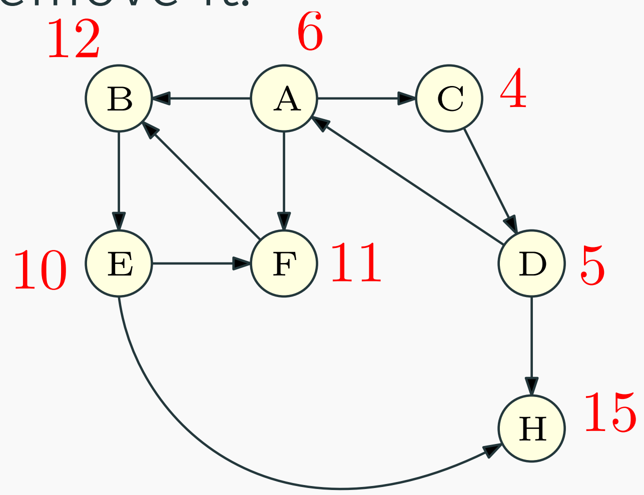


SCC computed:

$\{G\}$

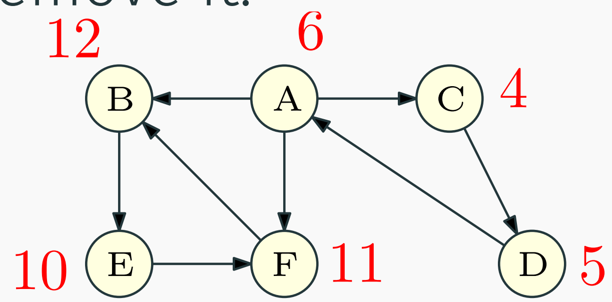
Linear Time Algorithm: An Example

Do **DFS** from vertex G remove it.



$\Longrightarrow$

Do **DFS** from vertex H, remove it.



SCC computed:

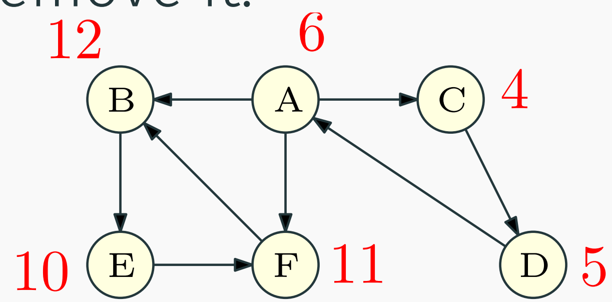
$\{G\}$

SCC computed:

$\{G\}, \{H\}$

Linear Time Algorithm: An Example

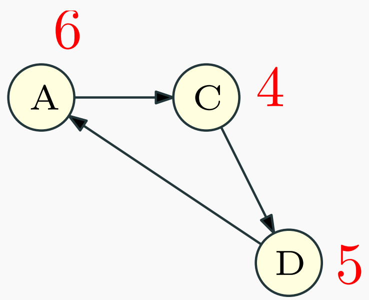
Do **DFS** from vertex H, remove it.



$\Longrightarrow$

Do **DFS** from vertex B

Remove visited vertices: $\{F, B, E\}$.



SCC computed:

$\{G\}, \{H\}$

SCC computed:

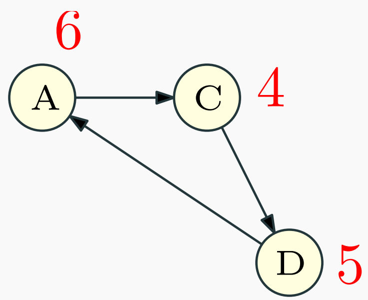
$\{G\}, \{H\}, \{F, B, E\}$

Linear Time Algorithm: An Example

Do **DFS** from vertex F

Remove visited vertices:

$\{F, B, E\}$.



$\Longrightarrow$

Do **DFS** from vertex A

Remove visited vertices:

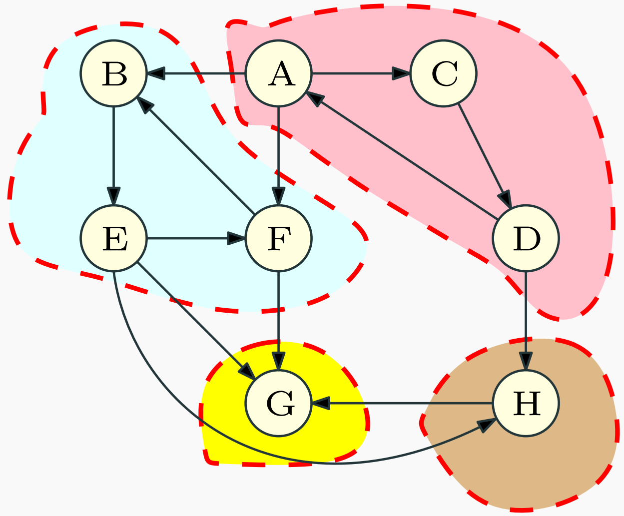
$\{A, C, D\}$.

SCC computed:

$\{G\}, \{H\}, \{F, B, E\}$

SCC computed:

$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$



SCC computed:

$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$

Which is the correct answer!

Obtaining the meta-graph...

Exercise:
Given all the strong connected components of a directed graph $G = (V, E)$ show that the meta-graph $\mathrm{G}^{\mathrm{SCC}}$ can be obtained in $O(m + n)$ time.

Solving Problems on Directed Graphs

A template for a class of problems on directed graphs:

- Is the problem solvable when G is strongly connected?
- Is the problem solvable when G is a DAG?
- If the above two are feasible then is the problem solvable in a general directed graph G by considering the meta graph G^{SCC}?

Summary

Take away Points

- DAGs
- Topological orderings.
- DFS: pre/post numbering.
- Given a directed graph G, its SCCs and the associated acyclic meta-graph G^{SCC} give a structural decomposition of G that should be kept in mind.
- There is a DFS based linear time algorithm to compute all the SCCs and the meta-graph. Properties of DFS crucial for the algorithm.
- DAGs arise in many application and topological sort is a key property in algorithm design. Linear time algorithms to compute a topological sort (there can be many possible orderings so not unique).

Scratch Figures