Pre-lecture brain teaser

Consider the problem of a n-input AND function. The input (x) is a string n-digits long with $\Sigma = \{0, 1\}$ and has an output (y) which is the logical AND of all the elements of x.

Formulate a **language** that describes the above problem.

ECE-374-B: Lecture 1 - Regular Languages

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Decision : Is y the logical "AND" of the bits of x
(Yes/No)

$L_{AND01} = \{ 11 = 1,\ 1 = 1,\ 111 = 1,\ 1111 = 1,\ 11111 = 1,\ 0 = 0,\ 01 = 0,\ 10 = 0,\ 00 = 0,\ 0110 = 0, \ldots \}$

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$$L_{AND_N} = \left\{ \begin{array}{cccc} 0|0, & 1|1, & & \\ 0 \cdot 0|0, & 0 \cdot 1|0, & 1 \cdot 0|0, & 1 \cdot 1|1 \\ \vdots & \vdots & \vdots & \vdots \\ (0\cdot)^n|0, & (0\cdot)^{n-1}1|0, & \ldots & (1\cdot)^n|1\ldots \end{array} \right\} \quad (1)$$

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This is an example of a regular language which we'll be discussing today.

Refresh on strings

Rapid-fire questions -strings

Answer the following questions taking $\Sigma = \{0, 1\}$.

1. What is Σ^0? ε
2. How many elements are there in Σ^n? 2^n
3. If $|u| = 2$ and $|v| = 3$ then what is $|u \bullet v|$? 5
4. Let u be an arbitrary string in Σ^*. What is ϵu? What is $u\epsilon$?

Languages

Definition
A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$

All possible strings

Languages

Definition
A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

Standard set operations apply to languages.

- For languages A, B the concatenation of A, B is $AB = \{xy \mid x \in A, y \in B\}$.
- For languages A, B, their union is $A \cup B$, intersection is $A \cap B$, and difference is $A \setminus B$ (also written as $A - B$).
- For language $A \subseteq \Sigma^*$ the complement of A is $\bar{A} = \Sigma^* \setminus A$.

$A = \{00, 11\}$

$\bar{A} = \{\epsilon, 0, 1, 01, 10, 000, 001, 010, \ldots\}$

Set Concatenation

Definition
Given two sets X and Y of strings (over some common alphabet Σ) the concatenation of X and Y is

$$XY = \{xy \mid x \in X, y \in Y\} \tag{2}$$

Question: $X = \{ECE, CS, \}, Y = \{340, 374\} \implies$
$XY = .$

$\{ECE340, CS340, ECE374, CS374\}$

Σ^* and languages

Definition

1. Σ^n is the set of all strings of length n. Defined inductively:
   $\Sigma^n = \{\epsilon\}$ if $n = 0$
   $\Sigma^n = \Sigma\Sigma^{n-1}$ if $n > 0$
2. $\Sigma^* = \cup_{n \geq 0} \Sigma^n$ is the set of all finite length strings
3. $\Sigma^+ = \cup_{n \geq 1} \Sigma^n$ is the set of non-empty strings.

Definition
A language L is a set of strings over Σ. In other words $L \subseteq \Sigma^*$.

Question: Does Σ^* have strings of infinite length?

Problem

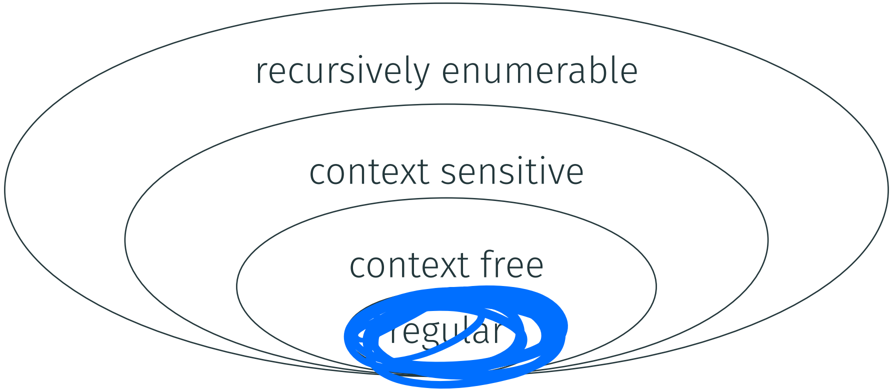
Consider languages over $\Sigma = \{0, 1\}$.

1. *What is* $\emptyset^0$?
2. *If* $|L| = 2$, *then what is* $|L^4|$?
3. *What is* $\emptyset^*$, $\{\epsilon\}^*$?
4. *For what* L *is* L^* *finite?*
5. *What is* $\emptyset^+$?
6. *What is* $\{\epsilon\}^+$?

Terminology Review

- A character(a, b, c, x) is a unit of information represented by a symbol: (letters, digits, whitespace)
- A alphabet(Σ) is a set of characters
- A string(w) is a sequence of characters
- A language(A, B, C, L) is a set of strings

Chomsky Hierarchy



Grammar	Languages	Production Rules	Automation	Examples
Type-0	Recursively enumerable	$\gamma \rightarrow \alpha$ (no constraints)	Turing machine	$L = \{\langle M, w \rangle \mid M$ is a TM which halts on $w\}$
Type-1	Context-sensitive	$\alpha A \beta \rightarrow \alpha \gamma \beta$	Linear bounded Non-deterministic Turing machine	$L = \{a^n b^n c^n \mid n > 0\}$
Type-2	Context-free	$A \rightarrow \alpha$	Non-deterministic Push-down automata	$L = \{a^n b^n \mid n > 0\}$
Type-3	Regular	$A \rightarrow aB$	Finite State Machine	$L = \{a^n \mid n > 0\}$

Meaning of symbols: • a = terminal • A, B = variables • α, β, γ = string of $\{a \cup A\}^*$ • α, β = maybe empty —– γ = never empty

• Table borrowed from wikipedia: `https://en.wikipedia.org/wiki/Chomsky_hierarchy`

Regular Languages

Theorem (Kleene's Theorem)

A language is regular if and only if it can be obtained from finite languages by applying the three operations:

- *Union*
- *Concatenation*
- *Repetition*

a finite number of times.

Regular Languages

A class of simple but useful languages.

The set of regular languages over some alphabet Σ is defined inductively.

Base Case

- $\emptyset$ is a regular language.
- $\{\epsilon\}$ is a regular language.
- $\{a\}$ is a regular language for each $a \in \Sigma$. Interpreting a as string of length 1.

Inductive step:

We can build up languages using a few basic operations:

- If L_1, L_2 are regular then $L_1 \cup L_2$ is regular.
- If L_1, L_2 are regular then $L_1 L_2$ is regular.
- If L is regular, then $L^* = \cup_{n \geq 0} L^n$ is regular.
  The $\cdot^*$ operator name is Kleene star.
- If L is regular, then so is $\overline{L} = \Sigma^* \setminus L$.

Regular languages are closed under operations of union, concatenation and Kleene star.

Some simple regular languages

Lemma
If w is a string then $L = \{w\}$ is regular.

Example: $\{aba\}$ or $\{abbabbab\}$. Why?

$L_a = \{a\}$

$L_b = \{b\}$

$\{aba\} = L_a \cdot L_b \cdot L_a = L_{aba}$

Some simple regular languages

Lemma
If w is a string then $L = \{w\}$ is regular.

Example: $\{aba\}$ or $\{abbabbab\}$. Why?

Lemma
Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

L_a

L_{aba}

L_{abaab}

$\rightarrow L_a \cup L_{aba} \cup L_{abaab}$

Regular Languages

Have basic operations to build regular languages.

Important: Any language generated by a finite sequence of such operations is regular.

Lemma
Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

Regular Languages

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Lemma

Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

Note: Kleene star (repetition) is a **single** operation!

$L_{a^*} = \{a\}^*$

$= \{a, aa, aaa, \ldots\}$

Regular Languages - Example

Example: The language $L_{01} = 0^i1^j|$ for all $i, j \geq 0$ is regular:

$L_{0s} = L_0^* = \{0, 00, 000, 0000, \varepsilon\}$

$L_{1s} =$

$L_{0s} \cdot L_{1s} = L_{0^i1^j}$

Rapid-fire questions - regular languages

1. $L_1 = \left\{ 0^i \;\middle|\; i = 0, 1, \ldots, \infty \right\}$. The language L_1 is regular. T/F?

$L_1 =$

$L_0 = \{0\}$

L_0^*

Rapid-fire questions - regular languages

1. $L_1 = \left\{ 0^i \;\middle|\; i = 0, 1, \ldots, \infty \right\}$. The language L_1 is regular. T/F?
2. $L_2 = \left\{ 0^{17i} \;\middle|\; i = 0, 1, \ldots, \infty \right\}$. The language L_2 is regular. T/F?

$L_0 = \{0\}$

$L_{0^{17}} = L_0 \cdot L_0 \cdot \ldots$ 17 time

$L_{0^{17}}^*$

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4. $L_4 = \{w \in \{0, 1\}^* \mid w \text{ has at most } 2 \text{ 1s}\}$. L_4 is regular. T/F?

Regular Expressions

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star names after him [1].

Inductive Definition

A regular expression $\mathbf{r}$ over an alphabet Σ is one of the following:

Base cases:

- $\emptyset$ denotes the language $\emptyset$
- ϵ denotes the language $\{\epsilon\}$.
- a denote the language $\{a\}$.

Inductive cases: If $\mathbf{r}_1$ and $\mathbf{r}_2$ are regular expressions denoting languages R_1 and R_2 respectively then,

- $(\mathbf{r}_1 + \mathbf{r}_2)$ denotes the language $R_1 \cup R_2$
- $(\mathbf{r}_1 \bullet \mathbf{r}_2) = r_1 \bullet r_2 = (\mathbf{r}_1\mathbf{r}_2)$ denotes the language R_1R_2
- $(\mathbf{r}_1)^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages	Regular Expressions
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	**a** denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r_1} + \mathbf{r_2}$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r_1} \bullet \mathbf{r_2}$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

Notation and Parenthesis

- For a regular expression r, $L(\mathrm{r})$ is the language denoted by r. Multiple regular expressions can denote the same language!
  Example: $(0+1)$ and $(1+0)$ denotes same language $\{0,1\}$

Notation and Parenthesis

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  Example: $r^*s + t = ((r^*)s) + t$

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- Superscript $+$. For convenience, define $\mathbf{r}^+ = \mathbf{r}\mathbf{r}^*$. Hence if $L(\mathbf{r}) = R$ then $L(\mathbf{r}^+) = R^+$.
- Other notation: $r + s$, $r \cup s$, $r|s$ all denote union. rs is sometimes written as $r \bullet s$.

Some examples of regular expressions

Creating regular expressions

1. All strings that end in 1011?

Creating regular expressions

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2. All strings except 11?

Creating regular expressions

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2. All strings except 11?
3. All strings that do not contain 000 as a subsequence?

Creating regular expressions

1. All strings that end in 1011?
2. All strings except 11?
3. All strings that do not contain 000 as a subsequence?
4. All strings that do not contain the substring 10?

Interpreting regular expressions

1. $(0 + 1)^*$:

Interpreting regular expressions

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2. $(0 + 1)^*001(0 + 1)^*$:

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Interpreting regular expressions

1. $(0 + 1)^*$:
2. $(0 + 1)^*001(0 + 1)^*$:
3. $0^* + (0^*10^*10^*10^*)^*$:
4. $(\epsilon + 1)(01)^*(\epsilon + 0)$:

Tying everything together

Consider the problem of a n-input AND function. The input (x) is a string n-digits long with an input alphabet $\Sigma_i = \{0, 1\}$ and has an output (y) which is the logical AND of all the elements of x. We knwo the language used to describe it is:

$$L_{AND_N} = \begin{Bmatrix} 0 \cdot |0, & 1 \cdot |1, & & \\ 0 \cdot 0 \cdot |0, & 0 \cdot 1 \cdot |0, & 1 \cdot 0 \cdot |0, & 1 \cdot 1 \cdot |1 \\ \vdots & \vdots & \vdots & \vdots \\ (0\cdot)^n|0, & (0\cdot)^{n-1}1|0, & \dots & (1\cdot)^n|1\dots \end{Bmatrix}$$

Formulate the regular expression which describes the above language:

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Formulate the regular expression which describes the above language:

$$\Sigma = \{0, 1, `\cdot', `|'\}\ r_{AND_N} = \underbrace{(\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* \text{“}0\cdot\text{”} (\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* \text{“}|0\text{”}}_{\text{all output 0 instances}} + \overbrace{(\text{“}1\cdot\text{”})^* \text{“}|1\text{”}}^{\text{all output 1 instances}}$$

Regular expressions in programming

One last expression....

Bit strings with odd number of 0s and 1s

Bit strings with odd number of 0s and 1s

The regular expression is

$$(00 + 11)^*(01 + 10)\left(00 + 11 + (01 + 10)(00 + 11)^*(01 + 10)\right)^*$$

Bit strings with odd number of 0s and 1s

The regular expression is

$$(00+11)^*(01+10)\Big(00+11+(01+10)(00+11)^*(01+10)\Big)^*$$

(Solved using techniques to be presented in the following lectures…)