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Couple methods:
- Eliminate states which cannot reach an accept state.
- Run DFS with pre-post numbering
- Find all the backedges. Backedges form cycle.
- Use pre/post numbering to find if accept state is within cycle.

Bigger point: 
$[\text{Infinite?}]$ problem reduces to $[\text{Find cycle}]$!
You are given a DFA describing the regular language $L$. Want to know if $|L|$ is infinite. How can we do this?

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- If so, the language is infinite
Pre-lecture brain teaser

You are given a DFA describing the regular language $L$. Want to know if $|L|$ is infinite. How can we do this?

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**Bigger point:** [Infinite?] problem reduces to [Find cycle]!
Last part of the course!
Finishing touches!

- Part I: models of computation (reg exps, DFA/NFA, CFGs, TMs)
- Part II: (efficient) algorithm design
- Part III: intractability via reductions
  - Undecidablity: problems that have no algorithms
  - NP-Completeness: problems unlikely to have efficient algorithms unless $P = NP$
Turing defined TMs as a machine model of computation

**Church-Turing thesis:** any function that is computable can be computed by TMs

**Efficient Church-Turing thesis:** any function that is computable can be computed by TMs with only a polynomial slow-down
Computability and Complexity Theory

• What functions can and **cannot** be computed by TMs?
• What functions/problems can and cannot be solved **efficiently**?

Why?

• Foundational questions about computation
• Pragmatic: Can we solve our problem or not?
• Are we not being clever enough to find an efficient algorithm or should we stop because there isn’t one or likely to be one?
Reductions to Prove Intractability

A general methodology to prove impossibility results.

- Start with some known hard problem $X$
- Reduce $X$ to your favorite problem $Y$

If $Y$ can be solved then so can $X \Rightarrow Y$. But we know $X$ is hard to $Y$ has to be hard too.
A general methodology to prove impossibility results.

- Start with some **known** hard problem $X$
- **Reduce** $X$ to your favorite problem $Y$

If $Y$ can be solved then so can $X \Rightarrow Y$. But we know $X$ is hard to $Y$ has to be hard too.

**Caveat:** In algorithms we reduce new problem to known solved one!
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**Caveat**: In algorithms we reduce new problem to known solved one!

Who gives us the initial hard problem?

- Some clever person (Cantor/Gödel/Turing/Cook/Levin ...) who establish hardness of a fundamental problem
- Assume some core problem is hard because we haven’t been able to solve it for a long time. This leads to
A general methodology to prove impossibility results.

- Start with some known hard problem $X$
- Reduce $X$ to your favorite problem $Y$

If $Y$ can be solved then so can $X \Rightarrow Y$ is also hard

What if we want to prove a problem is easy?
When proving hardness we limit attention to decision problems

- A decision problem $\Pi$ is a collection of instances (strings)
- For each instance $I$ of $\Pi$, answer is YES or NO
- Equivalently: boolean function $f_\Pi : \Sigma^* \rightarrow \{0, 1\}$ where $f(I) = 1$ if $I$ is a YES instance, $f(I) = 0$ if NO instance
- Equivalently: language $L_\Pi = \{I \mid I$ is a YES instance$\}$
When proving hardness we limit attention to decision problems

- A decision problem $\Pi$ is a collection of instances (strings)
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**Notation about encoding:** distinguish $I$ from encoding $\langle I \rangle$

- $n$ is an integer. $\langle n \rangle$ is the encoding of $n$ in some format (could be unary, binary, decimal etc)
- $G$ is a graph. $\langle G \rangle$ is the encoding of $G$ in some format
- $M$ is a TM. $\langle M \rangle$ is the encoding of TM as a string according to some fixed convention
Aside: Different problems can be formulated differently. Example: Traveling Salesman

Common Formulation: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Decision Formulation: Given a list of cities and the distances between each pair of cities, is there a route route that visits each city exactly once and returns to the origin city while having a shorter length than integer $k$. 
Examples

- Given directed graph $G$, is it strongly connected? $\langle G \rangle$ is a YES instance if it is, otherwise NO instance.
- Given number $n$, is it a prime number?
  
  $$L_{PRIMES} = \{\langle n \rangle \mid n \text{ is prime}\}$$

- Given number $n$ is it a composite number?
  
  $$L_{COMPOSITE} = \{\langle n \rangle \mid n \text{ is a composite}\}$$

- Given $G = (V, E)$, $s, t, B$ is the shortest path distance from $s$ to $t$ at most $B$? Instance is $\langle G, s, t, B \rangle$
Reductions: Overview
For languages $L_X, L_Y$, a reduction from $L_X$ to $L_Y$ is:

- An algorithm ...
- Input: $w \in \Sigma^*$
- Output: $w' \in \Sigma^*$
- Such that:

$$w \in L_X \iff w' \in L_Y$$
For decision problems $X, Y$, a reduction from $X$ to $Y$ is:

- An algorithm ...
- Input: $I_X$, an instance of $X$.
- Output: $I_Y$, an instance of $Y$.
- Such that:
  \[ I_Y \text{ is YES instance of } Y \iff I_X \text{ is YES instance of } X \]
Using reductions to solve problems

- $\mathcal{R}$: Reduction $X \rightarrow Y$
- $\mathcal{A}_Y$: algorithm for $Y$:
Using reductions to solve problems

- $\mathcal{R}$: Reduction $X \rightarrow Y$
- $\mathcal{A}_Y$: algorithm for $Y$:
- $\implies$ New algorithm for $X$:

\[
\mathcal{A}_X(l_X):
\begin{align*}
// & \quad l_X: \text{ instance of } X. \\
l_Y & \leftarrow \mathcal{R}(l_X) \\
\text{return } & \mathcal{A}_Y(l_Y)
\end{align*}
\]
Using reductions to solve problems

- \( R \): Reduction \( X \rightarrow Y \)
- \( A_Y \): algorithm for \( Y \):
- \( \Longrightarrow \) New algorithm for \( X \):

\[
A_X(l_X):
\]

// \( l_X \): instance of \( X \).

\[
l_Y \leftarrow R(l_X)
\]

return \( A_Y(l_Y) \)

In particular, if \( R \) and \( A_Y \) are polynomial-time algorithms, \( A_X \) is also polynomial-time.
Reductions and running time

$R(n)$: running time of $\mathcal{R}$

$Q(n)$: running time of $\mathcal{A}_Y$

**Question:** What is running time of $\mathcal{A}_X$?

Example: If $R(n) = n^2$ and $Q(n) = n^{1.5}$ then $\mathcal{A}_X$ is $O(n^2 + n^{3.5})$. 

$\mathcal{R}$ $\mathcal{A}_Y$

$I_X$ $I_Y$

YES

NO

$\mathcal{A}_X$
**Question:** What is running time of $\mathcal{A}_X$? $O(Q(R(n)))$. Why?

- If $I_X$ has size $n$, $\mathcal{R}$ creates an instance $I_Y$ of size at most $R(n)$
- $\mathcal{A}_Y$’s time on $I_Y$ is by definition at most $Q(|I_Y|) \leq Q(R(n))$.

**Example:** If $R(n) = n^2$ and $Q(n) = n^{1.5}$ then $\mathcal{A}_X$ is $O(n^2 + n^3)$
Comparing Problems

- Reductions allow us to formalize the notion of “Problem X is no harder to solve than Problem Y”.
- If Problem X reduces to Problem Y (we write $X \leq Y$), then X cannot be harder to solve than Y.
- More generally, if $X \leq Y$, we can say that X is no harder than Y, or Y is at least as hard as X. $X \leq Y$:
  - X is no harder than Y, or
  - Y is at least as hard as X.
Examples of Reductions
Given a graph $G$, a set of vertices $V'$ is:
Given a graph $G$, a set of vertices $V'$ is:

- An **independent set**: if no two vertices of $V'$ are connected by an edge of $G$. 
Independent Sets and Cliques

Given a graph $G$, a set of vertices $V'$ is:

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- An **independent set**: if no two vertices of $V'$ are connected by an edge of $G$.
- **clique**: every pair of vertices in $V'$ is connected by an edge of $G$. 
Problem: **Independent Set**

**Instance:** A graph $G$ and an integer $k$.

**Question:** Does $G$ have an independent set of size $\geq k$?
Problem: **Independent Set**

**Instance:** A graph $G$ and an integer $k$.

**Question:** Does $G$ has an independent set of size $\geq k$?

Problem: **Clique**

**Instance:** A graph $G$ and an integer $k$.

**Question:** Does $G$ has a clique of size $\geq k$?
Recall

For decision problems $X, Y$, a reduction from $X$ to $Y$ is:

- An algorithm ...
- that takes $I_X$, an instance of $X$ as input ...
- and returns $I_Y$, an instance of $Y$ as output ...
- such that the solution (YES/NO) to $I_Y$ is the same as the solution to $I_X$. 

Reducing Independent Set to Clique

An instance of **Independent Set** is a graph $G$ and an integer $k$. 

![Graph diagram](image-url)
Reducing Independent Set to Clique

An instance of **Independent Set** is a graph $G$ and an integer $k$. 
An instance of **Independent Set** is a graph $G$ and an integer $k$.

Reduction given $\langle G, k \rangle$ outputs $\langle \overline{G}, k \rangle$ where $\overline{G}$ is the complement of $G$. $\overline{G}$ has an edge $uv \iff uv$ is **not** an edge of $G$. 
Reducing Independent Set to Clique

An instance of **Independent Set** is a graph $G$ and an integer $k$.

Reduction given $\langle G, k \rangle$ outputs $\langle \overline{G}, k \rangle$ where $\overline{G}$ is the complement of $G$. $\overline{G}$ has an edge $uv \iff uv$ is **not** an edge of $G$.

A independent set of size $k$ in $G \iff$ A clique of size $k$ in $\overline{G}$
Lemma

$G$ has an independent set of size $k \iff \overline{G}$ has a clique of size $k$.

Proof.

Need to prove two facts:

$G$ has independent set of size at least $k$ implies that $\overline{G}$ has a clique of size at least $k$.

$\overline{G}$ has a clique of size at least $k$ implies that $G$ has an independent set of size at least $k$.

Since $S \subseteq V$ is an independent set in $G \iff S$ is a clique in $\overline{G}$. □
Independent Set and Clique

- Independent Set $\leq_P$ Clique.
Independent Set and Clique

- **Independent Set \( \leq_p \) Clique.**
  What does this mean?
  - If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
Independent Set and Clique

- Independent Set \( \leq_p \) Clique. What does this mean?
- If have an algorithm for Clique, then we have an algorithm for Independent Set.
- Clique is at least as hard as Independent Set.
Independent Set and Clique

• \textbf{Independent Set} \leq_p \textbf{Clique}.
  What does this mean?

• If have an algorithm for \textbf{Clique}, then we have an algorithm for \textbf{Independent Set}.

• \textbf{Clique} is at least as hard as \textbf{Independent Set}.

• Also... \textbf{Clique} \leq_p \textbf{Independent Set}. Why? Thus \textbf{Clique} and \textbf{Independent Set} are polynomial-time equivalent.
I want to show Independent Set is at least as hard as Clique.
I want to show **Independent Set** is at least as hard as **Clique**. Write out the equality: \( \text{Clique} \leq_p \text{Independent Set} \)
I want to show **Independent Set** is at least as hard as **Clique**.
Write out the equality: $\text{Clique} \leq_p \text{Independent Set}$
Draw reduction figure:

\[
\begin{array}{c}
\mathcal{R} \\
\mathcal{A}_X \\
\mathcal{A}_Y \\
\end{array}
\]

- $I_X \rightarrow \mathcal{R}$
- $I_Y \rightarrow \mathcal{A}_Y$
- $\mathcal{R} \rightarrow \mathcal{A}_X$

YES
NO
I want to show \textbf{Independent Set} is at least as hard as \textbf{Clique}.

Write out the equality: \textbf{Clique} \ \leq_p \textbf{Independent Set}

Draw reduction figure:

Fill in the blanks:

- \( I_X = \langle G \rangle \)
- \( A_X = \text{Clique} \)
- \( I_Y = \langle G \rangle \)
- \( A_Y = \text{Independent Set} \)
- \( R : \overline{G} = \{V, \overline{E}\} \)
Assume you can solve the **Clique** problem in $T(n)$ time. Then you can solve the **Independent Set** problem in

(A) $O(T(n))$ time.
(B) $O(n \log n + T(n))$ time.
(C) $O(n^2 T(n^2))$ time.
(D) $O(n^4 T(n^4))$ time.
(E) $O(n^2 + T(n^2))$ time.
(F) Does not matter - all these are polynomial if $T(n)$ is polynomial, which is good enough for our purposes.
Independent Set and Vertex Cover
Given a graph $G = (V, E)$, a set of vertices $S$ is:
Vertex Cover

Given a graph $G = (V, E)$, a set of vertices $S$ is:

- A vertex cover if every $e \in E$ has at least one endpoint in $S$. 
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Vertex Cover

Given a graph $G = (V, E)$, a set of vertices $S$ is:

- A **vertex cover** if every $e \in E$ has at least one endpoint in $S$. 
Problem (Vertex Cover)

Input: A graph $G$ and integer $k$.  
Goal: Is there a vertex cover of size $\leq k$ in $G$?
The Vertex Cover Problem

Problem (Vertex Cover)

Input: A graph $G$ and integer $k$.

Goal: Is there a vertex cover of size $\leq k$ in $G$?

Can we relate Independent Set and Vertex Cover?
Lemma

Let $G = (V, E)$ be a graph. $S$ is an Independent Set $\iff V \setminus S$ is a vertex cover.
**Lemma**

Let $G = (V, E)$ be a graph. $S$ is an Independent Set $\iff V \setminus S$ is a vertex cover.

**Proof.**

$(\Rightarrow)$ Let $S$ be an independent set

- Consider any edge $uv \in E$.
- Since $S$ is an independent set, either $u \notin S$ or $v \notin S$.
- Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
- $V \setminus S$ is a vertex cover.

$(\Leftarrow)$ Let $V \setminus S$ be some vertex cover:

- Consider $u, v \in S$.
- $uv$ is not an edge of $G$, as otherwise $V \setminus S$ does not cover $uv$.
- Thus, $S$ is thus an independent set.
Lemma
Let $G = (V, E)$ be a graph. $S$ is an Independent Set $\iff V \setminus S$ is a vertex cover.

Proof.

($\Rightarrow$) Let $S$ be an independent set
- Consider any edge $uv \in E$.
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($\Leftarrow$) Let $V \setminus S$ be some vertex cover:
- Consider $u, v \in S$
- $uv$ is not an edge of $G$, as otherwise $V \setminus S$ does not cover $uv$.
- $\implies S$ is thus an independent set. \qed
Independent Set $\leq_p$ Vertex Cover

- $G$: graph with $n$ vertices, and an integer $k$ be an instance of the Independent Set problem.
Independent Set $\leq_P$ Vertex Cover

- $G$: graph with $n$ vertices, and an integer $k$ be an instance of the **Independent Set** problem.
- $G$ has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
Independent Set $\leq_p$ Vertex Cover

- $G$: graph with $n$ vertices, and an integer $k$ be an instance of the **Independent Set** problem.
- $G$ has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
- $(G, k)$ is an instance of **Independent Set**, and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
Independent Set $\leq_P$ Vertex Cover

- $G$: graph with $n$ vertices, and an integer $k$ be an instance of the **Independent Set** problem.
- $G$ has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
- $(G, k)$ is an instance of **Independent Set**, and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
- Therefore, **Independent Set $\leq_P$ Vertex Cover**. Also **Vertex Cover $\leq_P$ Independent Set**.
Independent Set $\leq_P$ Vertex Cover

• $G$: graph with $n$ vertices, and an integer $k$ be an instance of the Independent Set problem.
• $G$ has an independent set of size $\geq k$ $\iff$ $G$ has a vertex cover of size $\leq n - k$

- $I_X = \langle G \rangle$
- $A_X = \text{Independent Set}(G, k)$
- $I_Y = \langle G \rangle$
- $A_Y = \text{Vertex Cover}(G, n - k)$
- $R : G' = G$
NFAs | DFAs and Universality
Given DFA $M$ and string $w \in \Sigma^*$, does $M$ accept $w$?

- Instance is $\langle M, w \rangle$
- Algorithm: given $\langle M, w \rangle$, output YES if $M$ accepts $w$, else NO

Does above DFA accept 0010110?
Given DFA $M$ and string $w \in \Sigma^*$, does $M$ accept $w$?

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**Question:** Is there an (efficient) algorithm for this problem?
DFA Accepting a String

Given DFA $M$ and string $w \in \Sigma^*$, does $M$ accept $w$?

- Instance is $\langle M, w \rangle$
- Algorithm: given $\langle M, w \rangle$, output YES if $M$ accepts $w$, else NO

**Question:** Is there an (efficient) algorithm for this problem?

Yes. Simulate $M$ on $w$ and output YES if $M$ reaches a final state.

**Exercise:** Show a linear time algorithm. Note that linear is in the input size which includes both encoding size of $M$ and $|w|$.
Given NFA $N$ and string $w \in \Sigma^*$, does $N$ accept $w$?

- Instance is $\langle N, w \rangle$
- Algorithm: given $\langle N, w \rangle$, output YES if $N$ accepts $w$, else NO

Does above NFA accept 0010110?
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NFA Accepting a String

Given NFA $N$ and string $w \in \Sigma^*$, does $N$ accept $w$?

- Instance is $\langle N, w \rangle$
- Algorithm: given $\langle N, w \rangle$, output YES if $N$ accepts $w$, else NO

**Question:** Is there an algorithm for this problem?

- Convert $N$ to equivalent DFA $M$ and use previous algorithm!
- Hence a reduction that takes $\langle N, w \rangle$ to $\langle M, w \rangle$
- Is this reduction efficient?
Given NFA \( N \) and string \( w \in \Sigma^* \), does \( N \) accept \( w \)?

- Instance is \( \langle N, w \rangle \)
- Algorithm: given \( \langle N, w \rangle \), output YES if \( N \) accepts \( w \), else NO

**Question:** Is there an algorithm for this problem?

- Convert \( N \) to equivalent DFA \( M \) and use previous algorithm!
- Hence a reduction that takes \( \langle N, w \rangle \) to \( \langle M, w \rangle \)
- Is this reduction efficient? No, because \( |M| \) is exponential in \( |N| \) in the worst case.

**Exercise:** Describe a polynomial-time algorithm.

Hence reduction may allow you to see an easy algorithm but not necessarily best algorithm!
A DFA $M$ is universal if it accepts every string. That is, $L(M) = \Sigma^*$, the set of all strings.

Problem (DFA universality)

Input: A DFA $M$.

Goal: Is $M$ universal?

How do we solve DFA Universality?

We check if $M$ has any reachable non-final state.
An NFA $N$ is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

Problem (NFA universality)

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How do we solve NFA Universality?

Reduce it to DFA Universality?
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**Problem (NFA universality)**

**Input:** A NFA $M$.

**Goal:** Is $M$ universal?

How do we solve NFA Universality?

Reduce it to DFA Universality?

Given an NFA $N$, convert it to an equivalent DFA $M$, and use the DFA Universality Algorithm.

What is the problem with this reduction?
An NFA $N$ is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

**Problem (NFA universality)**

**Input:** A NFA $M$.

**Goal:** Is $M$ universal?

How do we solve NFA Universality?

Reduce it to DFA Universality?

Given an NFA $N$, convert it to an equivalent DFA $M$, and use the DFA Universality Algorithm.

What is the problem with this reduction? The reduction takes exponential time!

NFA Universality is known to be PSPACE-Complete.
Polynomial time reductions
We say that an algorithm is efficient if it runs in polynomial-time.
Polynomial-time reductions

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To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.
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To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

If we have a polynomial-time reduction from problem $X$ to problem $Y$ (we write $X \leq_P Y$), and a poly-time algorithm $A_Y$ for $Y$, we have a polynomial-time/efficient algorithm for $X$. 


We say that an algorithm is efficient if it runs in polynomial-time.

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If we have a polynomial-time reduction from problem \( X \) to problem \( Y \) (we write \( X \leq_P Y \)), and a poly-time algorithm \( A_Y \) for \( Y \), we have a polynomial-time/efficient algorithm for \( X \).

\[
\begin{array}{c}
\mathcal{R} \\
I_X \rightarrow \quad I_Y \rightarrow \quad A_Y \rightarrow \quad \text{YES, NO} \\
\mathcal{A}_X
\end{array}
\]
Polynomial-time Reduction

A polynomial time reduction from a decision problem $X$ to a decision problem $Y$ is an algorithm $A$ that has the following properties:

- given an instance $I_X$ of $X$, $A$ produces an instance $I_Y$ of $Y$
- $A$ runs in time polynomial in $|I_X|$.
- Answer to $I_X$ YES $\iff$ answer to $I_Y$ is YES.
A polynomial time reduction from a decision problem $X$ to a decision problem $Y$ is an algorithm $\mathcal{A}$ that has the following properties:

- given an instance $I_X$ of $X$, $\mathcal{A}$ produces an instance $I_Y$ of $Y$
- $\mathcal{A}$ runs in time polynomial in $|I_X|$.
- Answer to $I_X$ YES $\iff$ answer to $I_Y$ is YES.

**Lemma**

If $X \leq_P Y$ then a polynomial time algorithm for $Y$ implies a polynomial time algorithm for $X$.

Such a reduction is called a Karp reduction. Most reductions we will need are Karp reductions. Karp reductions are the same as mapping reductions when specialized to polynomial time for the reduction step.
Let $X$ and $Y$ be two decision problems, such that $X$ can be solved in polynomial time, and $X \leq_P Y$. Then

(A) $Y$ can be solved in polynomial time.
(B) $Y$ can NOT be solved in polynomial time.
(C) If $Y$ is hard then $X$ is also hard.
(D) None of the above.
(E) All of the above.
Be careful about reduction direction

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM $X$ TO $Y$
That is, show that an algorithm for $Y$ implies an algorithm for $X$. 
The Satisfiability Problem (SAT)
Propositional Formulas

Definition
Consider a set of boolean variables $x_1, x_2, \ldots, x_n$.

- A literal is either a boolean variable $x_i$ or its negation $\neg x_i$.
- A clause is a disjunction of literals. For example, $x_1 \lor x_2 \lor \neg x_4$ is a clause.
- A formula in conjunctive normal form (CNF) is propositional formula which is a conjunction of clauses. $(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5$ is a CNF formula.
Propositional Formulas

**Definition**
Consider a set of boolean variables $x_1, x_2, \ldots, x_n$.

- A **literal** is either a boolean variable $x_i$ or its negation $\neg x_i$.
- A **clause** is a disjunction of literals. For example, $x_1 \lor x_2 \lor \neg x_4$ is a clause.
- A **formula in conjunctive normal form (CNF)** is propositional formula which is a conjunction of clauses. For example, $(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5$ is a CNF formula.
- A formula $\varphi$ is a **3CNF**: A CNF formula such that every clause has **exactly** 3 literals.
  - $(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3 \lor x_1)$ is a 3CNF formula, but $(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5$ is not.
Every boolean formula $f : \{0, 1\}^n \rightarrow \{0, 1\}$ can be written as a CNF formula.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$f(x_1, x_2, \ldots, x_6)$</th>
<th>$\overline{x}_1 \lor x_2 \overline{x}_3 \lor x_4 \lor \overline{x}_5 \lor x_6$</th>
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<tbody>
<tr>
<td>0</td>
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<td>$f(0, \ldots, 0, 0)$</td>
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<td>$f(1, \ldots, 0, 0)$</td>
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<td>$f(1, \ldots, 1)$</td>
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</table>

For every row that $f$ is zero compute corresponding CNF clause. Take the and ($\land$) of all the CNF clauses computed.
Problem: **SAT**

**Instance:** A CNF formula $\varphi$.

**Question:** Is there a truth assignment to the variable of $\varphi$ such that $\varphi$ evaluates to true?

Problem: **3SAT**

**Instance:** A 3CNF formula $\varphi$.

**Question:** Is there a truth assignment to the variable of $\varphi$ such that $\varphi$ evaluates to true?
Satisfiability

**SAT**
Given a CNF formula \( \varphi \), is there a truth assignment to variables such that \( \varphi \) evaluates to true?

**Example**

- \((x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5\) is satisfiable; take \(x_1, x_2, \ldots, x_5\) to be all true
- \((x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2)\) is not satisfiable.

**3SAT**
Given a 3CNF formula \( \varphi \), is there a truth assignment to variables such that \( \varphi \) evaluates to true?

(More on **2SAT** in a bit...)
Importance of **SAT** and **3SAT**

- **SAT** and **3SAT** are basic constraint satisfaction problems.
- Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
- Arise naturally in many applications involving hardware and software verification and correctness.
- As we will see, it is a fundamental problem in theory of NPCompleteness.
Given two bits $x, z$ which of the following SAT formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \lor x) \land (z \lor \overline{x})$.

(B) $(z \lor x) \land (\overline{z} \lor \overline{x})$.

(C) $(\overline{z} \lor x) \land (\overline{z} \lor \overline{x}) \land (\overline{z} \lor \overline{x})$.

(D) $z \oplus x$.

(E) $(z \lor x) \land (\overline{z} \lor \overline{x}) \land (z \lor \overline{x}) \land (\overline{z} \lor x)$.
Given two bits $x, z$ which of the following SAT formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \lor x) \land (z \lor \overline{x})$.

(B) $(z \lor x) \land (\overline{z} \lor \overline{x})$.

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(D) $z \oplus x$.

(E) $(z \lor x) \land (\overline{z} \lor \overline{x}) \land (z \lor \overline{x}) \land (\overline{z} \lor x)$.

\[
\begin{array}{c|c|c|c}
  x & y & z = \overline{x} \\
\hline
  0 & 0 & 0 \\
\hline
  0 & 1 & 1 \\
\hline
  1 & 0 & 1 \\
\hline
  1 & 1 & 0 \\
\end{array}
\]
Given three bits $x, y, z$ which of the following SAT formulas is equivalent to the formula $z = x \land y$:

(A) $(\overline{z} \lor x \lor y) \land (z \lor \overline{x} \lor \overline{y})$.

(B) $(\overline{z} \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y)$.

(C) $(\overline{z} \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y)$.

(D) $(z \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y)$.

(E) $(z \lor x \lor y) \land (z \lor x \lor \overline{y}) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y}) \land (\overline{z} \lor x \lor y) \land (\overline{z} \lor x \lor \overline{y}) \land (\overline{z} \lor \overline{x} \lor y) \land (\overline{z} \lor \overline{x} \lor \overline{y})$. 
Given three bits $x, y, z$ which of the following \textbf{SAT} formulas is equivalent to the formula $z = x \land y$:

(A) $(\overline{z} \lor x \lor y) \land (z \lor \overline{x} \lor \overline{y})$.

(B) $(\overline{z} \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y})$.

(C) $(\overline{z} \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y})$.

(D) $(z \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y})$.

(E) $(z \lor x \lor y) \land (z \lor x \lor \overline{y}) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y}) \land (\overline{z} \lor x \lor y) \land (\overline{z} \lor x \lor \overline{y}) \land (\overline{z} \lor x \lor \overline{y})$.

<table>
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<tr>
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<th>$y$</th>
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Exercise

What is a non-satisfiable SAT assignment?
Fin