Pre-lecture brain teaser

What do each of the reductions prove?

1. $u - v$ shortest path $\leq_P$ All pairs shortest path

2. SAT $\leq_P$ Longest path [1]

3. Shortest path $\leq_P$ SAT [2]

[1] Given a graph $G = (V, E)$ and integer k, is there a simple path that uses at least k vertices.

[2] `http://www.aloul.net/Papers/faloul_iceee06.pdf`.

ECE-374-B: Lecture 22 - Decidability I

Instructor: Abhishek Kumar Umrawal

November 14, 2023

University of Illinois at Urbana-Champaign

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What do each of the reductions prove?

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Cantor’s diagonalization argument

Diagonalization Intro

Published in 1891 by George Cantor, is a proof that sought to answer the following question.

Are all infinite sets $(\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{C})$ the same size?

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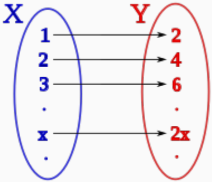
Let's say two sets are the same size if there is a 1-1 mapping between them.



First we need an anchor point $(\mathbb{N})$. Let's say the set of natural numbers has a particular size $\aleph_0$.

Countable Sets I

We say the set $\mathbb{N}$ is countable because you can list out all it's elements systematically, i.e., enumerate them.

$$1, 2, 3, 4, 5, 6, \ldots \tag{1}$$

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Set of integers is also countable.

Countable Sets II

Set of rational numbers is also countable.

	1	2	3	4	5	6	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	
$\vdots$							

Focus on ordering numbers based on the diagonals.

Countable Sets III

Is the set of complex *integers* countable?

Countable Sets IV

Is $\mathbb{R}$ countable?

1	0.	9	8	2	1	2	...
2	0.	4	8	6	8	5	...
3	0.	1	7	3	7	9	
4	0.	0	6	7	2	7	
5	0.	3	2	3	4	8	
6	0.	0	3	2	7	0	
$\vdots$							

How do we draw a 1-1 mapping between $\mathbb{N}$ and $\mathbb{R}$?

Countable Sets IV

Is $\mathbb{R}$ countable?

1	0.	9	8	2	1	2	...
2	0.	4	8	6	8	5	...
3	0.	1	7	3	7	9	
4	0.	0	6	7	2	7	
5	0.	3	2	3	4	8	
6	0.	0	3	2	7	0	
⋮							
D							

You can not count the real numbers II

$I = (0,1)$, $\mathbb{N} = \{1,2,3,\ldots\}$.

Claim (Cantor)
$|\mathbb{N}| \neq |I|$, where $I = (0,1)$.

Proof.
Write every number in $(0,1)$ in its decimal expansion. E.g., $1/3 = 0.33333333333333333333\ldots$.

Assume that $|\mathbb{N}| = |I|$. Then there exists a one-to-one mapping $f : \mathbb{N} \to I$. Let β_i be the i^{th} digit of $f(i) \in (0,1)$.

$d_i =$ any number in $\{0,1,2,3,4,5,6,7,8,9\} \setminus \{d_{i-1}, \beta_i\}$

$D = 0.d_1 d_2 d_3 \ldots \in (0,1)$.

D is a well defined unique number in $(0,1)$,

But there is no j such that $f(j) = D$. A contradiction. $\square$

“Most General” computer?

- DFAs are simple model of computation.
- Accept only the regular languages.
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages: $\{L \mid L \subseteq \{0,1\}^*\}$ is ~~countably infinite~~ / uncountably infinite
- Set of all programs: $\{P \mid P \text{ is a finite length computer program}\}$: is countably infinite / ~~uncountably infinite~~.

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- Set of all programs: $\{P \mid P \text{ is a finite length computer program}\}$: is countably infinite / $\xcancel{\text{uncountably infinite}}$.
- **Conclusion:** There are languages for which there are no programs.

Program Diagonalization

How do we know that there are languages that cannot be represented by programs? Use Cantor!

Program Diagonalization

How do we know that there are languages that cannot be represented by programs? Use Cantor! Recall a program can be represented by a string where:

- M is the Turing machine (program), and
- $\langle M \rangle$ is the string representation of the TM M.

Program Diagonalization

Define $f(i, j) = 1$ if M_i accepts $\langle M_j \rangle$, else 0.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	$\langle M_6 \rangle$	$\dots$
M_1	0	1	1	1	1	1	
M_2	1	1	0	0	0	0	
M_3	0	0	0	1	0	0	
M_4	1	1	1	0	1	1	
M_5	1	0	0	0	1	0	
M_6	0	1	0	1	1	0	
$\vdots$							

Program Diagonalization

Let's define a new program as follows.

$$D = \{\langle M \rangle | M \text{ does not accept } \langle M \rangle\}$$

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M_1	0	1	1	1	1	1		1
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M_3	0	0	0	1	0	0		1
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M_6	0	1	0	1	1	0		1
$\vdots$								
M_D	1	0	1	1	0	1		$\square$

Recap of decidability

Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages:

$$L = \{L(M) \mid M \text{ some Turing machine}\}\,.$$

- Recursive / decidable languages:

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}\,.$$

Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages: (bad)

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- Fundamental questions:
  - What languages are RE?
  - Which are recursive?
  - What is the difference?
  - What makes a language decidable?

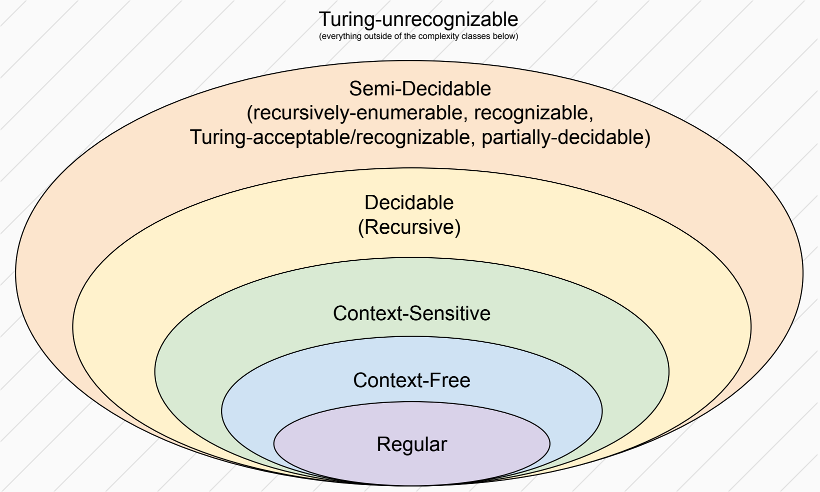
Decidable vs recursively-enumerable

A semi-decidable problem (equivalent of recursively enumerable) could be:

- **Decidable** - equivalent of recursive (TM always accepts or rejects).
- **Undecidable** - Problem is not recursive (doesn't always halt on negative)

There are undecidable problem that are not semi-decidable (recursively enumerable).

Problem (Language) Space



Un-/Decidable anchor

Like in the case of NP-complete-ness, we need an anchor point to compare languages to to determine whether they are decidable (or not)!

Introduction to the halting theorem

The halting problem

Halting problem: Given a program Q, if we run it would it stop?

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Halting problem: Given a program Q, if we run it would it stop?

Q: Can one build a program P, that always stops, and solves the halting problem.

Theorem ("Halting theorem")
There is no program that always stops and solves the halting problem.

Intuition, why solving the Halting problem is really hard

Definition
An integer number n is a <u>weird number</u> if

- *the sum of the proper divisors (including 1 but not itself) of n the number is $> n$,*
- *no subset of those divisors sums to the number itself.*

70 is weird. Its divisors are $1, 2, 5, 7, 10, 14, 35$.
$1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$. No subset of them adds up to 70.

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Write a program P that tries all odd numbers in order, and check if they are weird. The programs stops if it found such number.

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If can solve halting problem $\implies$ can resolve this open problem.

If you can halt, you can prove or disprove anything...

- Consider any math claim C.
- **Prover** algorithm P_C:
  (A) Generate sequence of all possible proofs (sequence of strings) into a pipe/queue.

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  (B) $\langle p \rangle \leftarrow$ pop top of queue.
  (C) Feed $\langle p \rangle$ and $\langle C \rangle$, into a proof verifier ("easy").
  (D) If $\langle p \rangle$ valid proof of $\langle C \rangle$, then stop and accept.
  (E) Go to (B).
- P_C halts $\iff$ C is true and has a proof.
- If halting is decidable, then can decide if any claim in math is true.

Turing machines...

TM = Turing machine = program.

Reminder: Undecidability

Definition
Language $L \subseteq \Sigma^*$ is undecidable if no program P, given $w \in \Sigma^*$ as input, can **always stop** and output whether $w \in L$ or $w \notin L$.

(Usually defined using TM not programs. But equivalent.)

Reminder: The following language is undecidable.

Decide if given a program M, and an input w, does M accepts w.
Formally, the corresponding language is

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

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Turing proved the following.

Theorem
A_{TM} *is undecidable.*

The halting problem

A_{TM} is not TM decidable!

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Theorem (The halting theorem.)
A_{TM} *is not Turing decidable.*

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Halt: TM deciding A_{TM}. **Halt** always halts, and works as follows.

$$\textbf{Halt}\Big(\langle M, w \rangle \Big) = \begin{cases} \text{accept} & M \text{ accepts } w \\ \text{reject} & M \text{ does not accept } w. \end{cases}$$

Halting theorem proof continued 1

We build the following new function.

Flipper($\langle M \rangle$)

  res $\leftarrow$ **Halt**($\langle M, \langle M \rangle \rangle$)

  if res is accept **then**

    reject

  else

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Flipper always stops.

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Halting theorem proof continued 2

$$\textbf{Flipper}\Big(\langle M\rangle\Big) = \begin{cases} \text{reject} & M \text{ accepts } \langle M\rangle \\ \text{accept} & M \text{ does not accept } \langle M\rangle \,. \end{cases}$$

Flipper is a TM (duh!), and as such it has an encoding $\langle\textbf{Flipper}\rangle$.
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This is can't be correct.

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Assumption that **Halt** exists is false. $\implies \mathrm{A}_{TM}$ is not TM decidable. □

Unrecognizable

TM recognizable

Definition
Language L is TM decidable if there exists M that always stops, such that $L(M) = L$.

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Theorem (Halting)
$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}$ *is TM recognizable, but not decidable.*

TM recognizable

Lemma
If L and $\overline{L} = \Sigma^* \setminus L$ are both TM recognizable, then L and $\overline{L}$ are decidable.

TM recognizable

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If L and $\overline{L} = \Sigma^* \setminus L$ are both TM recognizable, then L and $\overline{L}$ are decidable.

Proof.

M: TM recognizing L.

M_c: TM recognizing $\overline{L}$.

Given input x, using UTM simulating running M and M_c on x in parallel. One of them must stop and accept. Return result.

$\implies$ L is decidable. $\square$

Complement language for A_{TM}

$$\overline{\mathrm{A}_{\mathit{TM}}} = \Sigma^* \setminus \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } \mathit{TM} \text{ and } M \text{ accepts } w \right\}.$$

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But don't really care about invalid inputs. So, really:

$$\overline{\mathrm{A}_{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ does } \textbf{not} \text{ accept } w \right\}.$$

Complement language for A_{TM} is not TM-recognizable

Theorem
The language

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is not TM recognizable.

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Proof.
A_{TM} is TM-recognizable.

If $\overline{\mathrm{A}_{TM}}$ is TM-recognizable

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Proof.
A_{TM} is TM-recognizable.

If $\overline{\mathrm{A}_{TM}}$ is TM-recognizable

$\implies$ (by Lemma)

A_{TM} is decidable. A contradiction. □

Reductions

Reduction

Meta definition: Problem **X** reduces to problem **B**, if given a solution to **B**, then it implies a solution for **X**. Namely, we can solve **Y** then we can solve **X**. We will done this by **X** $\implies$ **Y**.

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Definition
oracle ORAC for language L is a function that receives as a word w, returns TRUE $\iff w \in L$.

Lemma
A language X reduces to a language Y, if one can construct a TM decider for X using a given oracle ORAC_Y *for Y.*

We will denote this fact by $X \implies Y$.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.

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- Assume L is decided by TM M.
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- Contradiction **X** is not decidable.
- Thus, L must be not decidable.

Reduction implies decidability

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If Y is decidable then X is decidable.

Proof.
Let $\mathtt{T}$ be a decider for Y (i.e., a program or a TM). Since X reduces to Y, it follows that there is a procedure $\mathtt{T}_{X|Y}$ (i.e., decider) for X that uses an oracle for Y as a subroutine. We replace the calls to this oracle in $\mathtt{T}_{X|Y}$ by calls to $\mathtt{T}$. The resulting program $\mathtt{T}_X$ is a decider and its language is X. Thus X is decidable (or more formally TM decidable). $\square$

The countrapositive...

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If X is undecidable then Y is undecidable.

Halting

The halting problem

Language of all pairs $\langle M, w \rangle$ such that M halts on w:

$$A_{\mathrm{Halt}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ stops on } w \right\}.$$

Similar to language already known to be undecidable:

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

On way to proving that Halting is undecidable...

Lemma

The language A_{TM} *reduces to* A_{Halt}*. Namely, given an oracle for* A_{Halt} *one can build a decider (that uses this oracle) for* A_{TM}*.*

On way to proving that Halting is undecidable...

Proof.
Let ORAC_{Halt} be the given oracle for A_{Halt}. We build the following decider for A_{TM}.

AnotherDecider-A_{TM} $\Big(\langle M, w\rangle\Big)$
    $res \leftarrow \mathrm{ORAC}_{Halt}\Big(\langle M, w\rangle\Big)$
    `// if` M `does not halt on` w `then reject.`
    if $res =$ `reject` **then**
        `halt and reject.`
    `//` M `halts on` w `since` $res = \mathrm{accept}$.
    `// Simulating` M `on` w `terminates in finite time.`
    $res_2 \leftarrow$`Simulate` M `on` w.
    return res_2.

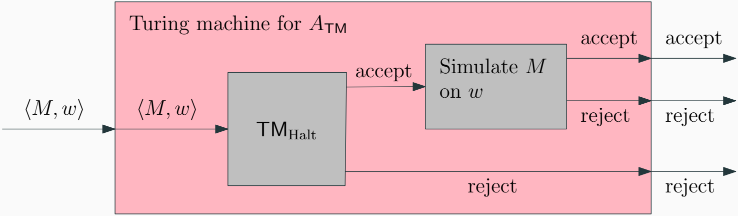
This procedure always return and as such its a decider for A_{TM}. □

The Halting problem is not decidable

Theorem
The language A_{Halt} is not decidable.

Proof.
Assume, for the sake of contradiction, that A_{Halt} is decidable. As such, there is a TM, denoted by TM_{Halt}, that is a decider for A_{Halt}. We can use TM_{Halt} as an implementation of an oracle for A_{Halt}, which would imply that one can build a decider for A_{TM}. However, A_{TM} is undecidable. A contradiction. It must be that A_{Halt} is undecidable. $\square$

The same proof by figure...



... if A_{Halt} is decidable, then A_{TM} is decidable, which is impossible.

More reductions next time