Pre-lecture brain teaser

What do each of the reductions prove?

1. All-pairs-shortest $\leq_P$ u-v shortest path
2. SAT $\leq_P$ Longest-path [1]
3. Shortest-path $\leq_P$ SAT [2]

[1] Given a graph $G(V, E)$ and integer k, is there a simple path that uses atleast k vertices
[2] `http://www.aloul.net/Papers/faloul_iceee06.pdf`

ECE-374-B: Lecture 22 - Decidability I

Instructor: Nickvash Kani

November 18, 2025

University of Illinois Urbana-Champaign

Pre-lecture brain teaser

What do each of the reductions prove?

Assume $E \geq 0$ Floyd-Warshall $O(n^3)$ Dijkstra's algorithm

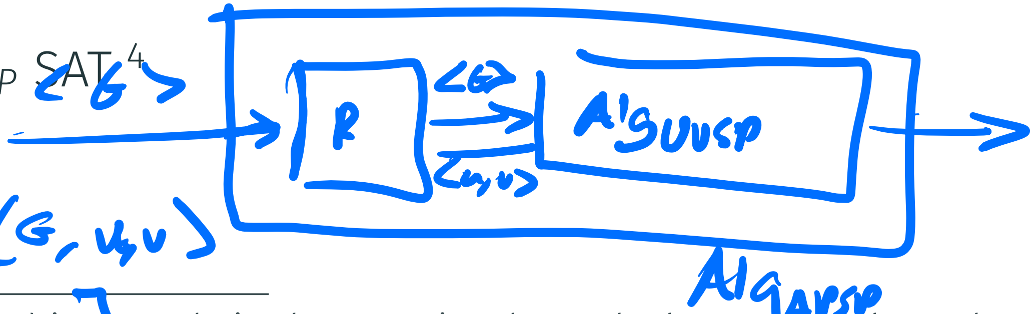
1. All-pairs-shortest $\leq_P$ u-v shortest path

   u-v sp is atleast as hard as AP-sp

   AP sp is no more hard than uv-sp

2. SAT $\leq_P$ Longest-path [3]

3. Shortest-path $\leq_P$ SAT [4]



for all u/v
Run $Alg_{UVSP}(G, u, v)$
record SP[u,v]
return SP

[3] Given a graph G(V, E) and integer k, is there a simple path that uses atleast k vertices

[4] http://www.aloul.net/Papers/faloul_iceee06.pdf

Pre-lecture brain teaser

What do each of the reductions prove?

1. All-pairs-shortest $\leq_P$ u-v shortest path
2. SAT $\leq_P$ Longest-path [3]
3. Shortest-path $\leq_P$ SAT [4]

is it possible in a fully connected graph for a LP < 10 vertices

Yes if some are negative

LP is NP-hard

Longest Path greater to

$G=(V,E)$ $|V|=n$ $|E|=m$ $\langle n\ n-1\ n-2\ n-3 \ldots 1 \rangle$

n vertices

[3]Given a graph $G(V, E)$ and integer k, is there a simple path that uses atleast k vertices

[4]http://www.aloul.net/Papers/faloul_iceee06.pdf

Assuming a fully connected graph $n!$ paths of n vertices $(n-1)!$ that contain n-1 vertices $\sum_{i=1}^{n} (i)!$

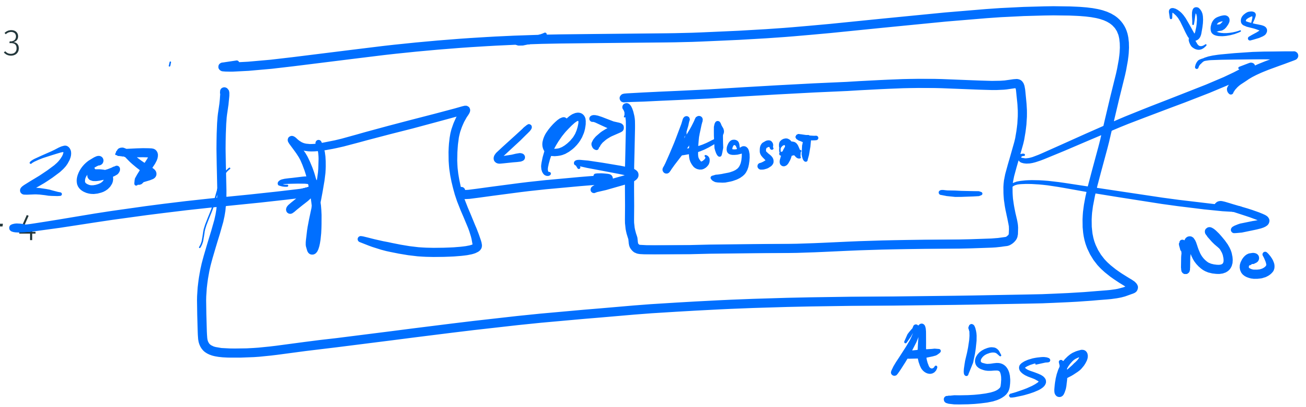
Pre-lecture brain teaser

What do each of the reductions prove?

1. All-pairs-shortest $\leq_P$ u-v shortest path
2. SAT $\leq_P$ Longest-path [3]
3. Shortest-path $\leq_P$ SAT [4]

SP is in NP

SAT $\leq$ SP



[3] Given a graph $G(V, E)$ and integer k, is there a simple path that uses atleast k vertices
n vertices

[4] http://www.aloul.net/Papers/faloul_iceee06.pdf

Assuming a fully connected graph

$n!$ paths of n vertices

$(n-1)!$ that contain n-1 vertices

$$\sum_{i=1}^{n} (i)!$$

Cantor’s diagonalization argument

Diagonalization Intro

Published in 1891 by George Cantor, is the proof that sought to answer a single question:

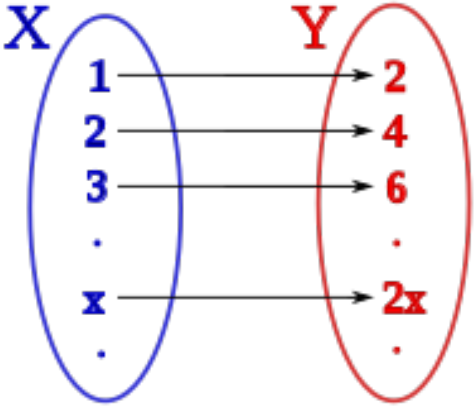
Are all infinite sets ($\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$) the same size?

Diagonalization Intro

Published in 1891 by George Cantor, is the proof that sought to answer a single question:

Are all infinite sets $(\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{C})$ the same size?

Let's say a set is the same size if there is a 1-1 mapping between the two sets:



First we need an anchor point $(\mathbb{N})$. Let's say the set of natural numbers has a particular size $\aleph_0$

Countable Sets I

We say the set $\mathbb{N}$ is countable because you can list out all it's elements systematically:

$$1, 2, 3, 4, 5, 6, \ldots \tag{1}$$

Countable Sets I

We say the set $\mathbb{N}$ is countable because you can list out all it's elements systematically:

$$1, 2, 3, 4, 5, 6, \ldots \tag{1}$$

Set of integers is also countable

Countable Sets II

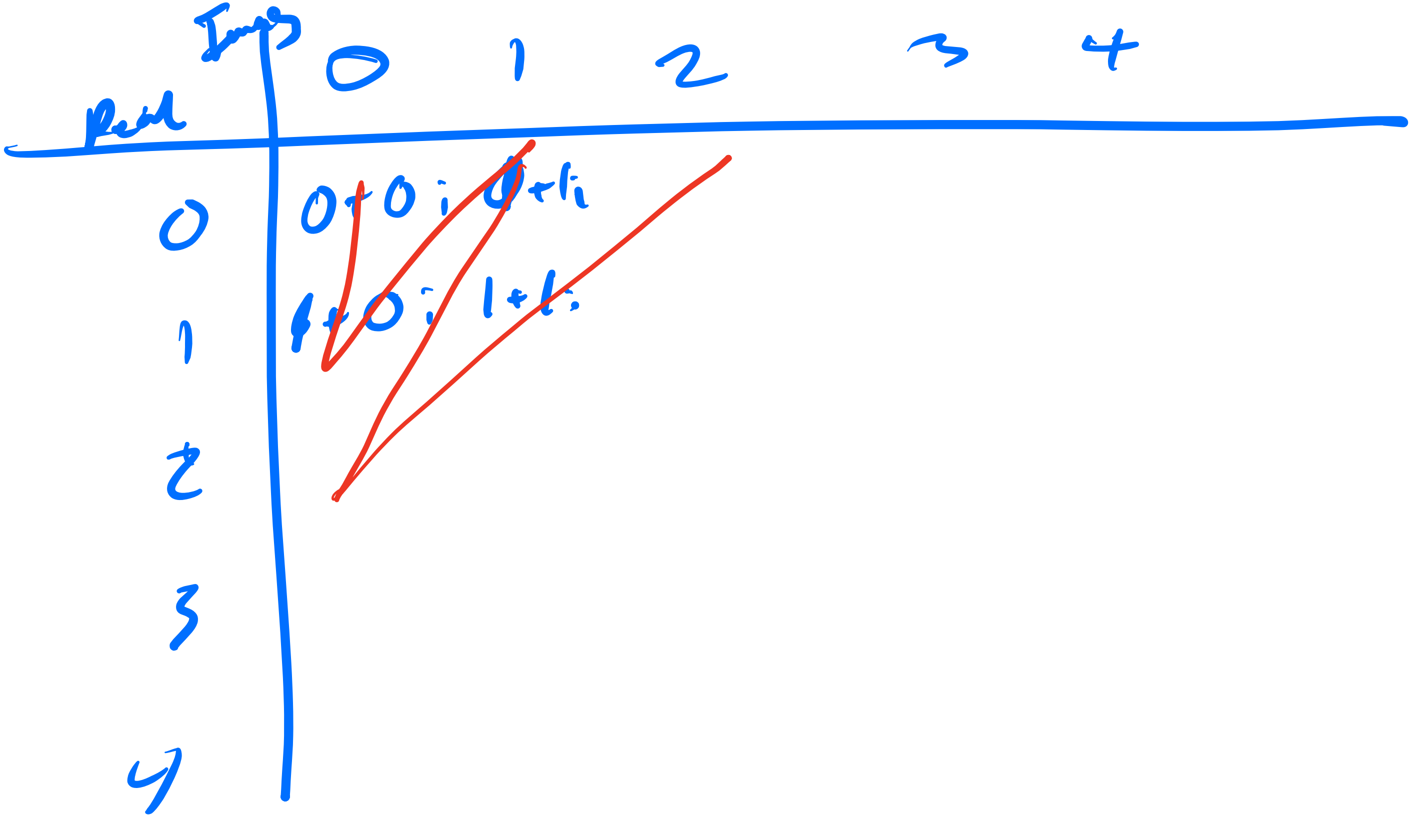
Set of rational numbers is also countable:

	1	2	3	4	5	6	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	
$\vdots$							

Focus on ordering numbers based on the diagonals.

Countable Sets III

Is the set of complex *integers* countable?



Countable Sets IV

Is $\mathbb{R}$ countable?

$[0, 1)$

1	0.	9	8	2	1	2	...
2	0.	4	8	6	8	5	...
3	0.	1	7	3	7	9	
4	0.	0	6	7	2	7	
5	0.	3	2	3	4	8	
6	0.	0	3	2	7	0	
$\vdots$							

How do we draw a 1-1 mapping between $\mathbb{N}$ and $\mathbb{R}$

Countable Sets IV

Is $\mathbb{R}$ countable?

1. Assume we have a mapping from $\mathbb{N}$ to $\mathbb{R}$

1	0.	9	8	2	1	2	...
2	0.	4	8	6	8	5	...
3	0.	1	7	3	7	9	
4	0.	0	6	7	2	7	
5	0.	3	2	3	4	8	
6	0.	0	3	2	7	0	
⋮							
D	0.	5	8	8	5	1	

How do we draw a 1-1 mapping between $\mathbb{N}$ and $\mathbb{R}$

You can not count the real numbers II

$I = (0,1)$, $\mathbb{N} = \{1,2,3,\ldots\}$.

Claim (Cantor)
$|\mathbb{N}| \neq |I|$, *where* $I = (0,1)$.

Proof.
Write every number in $(0,1)$ in its decimal expansion. E.g.,
$1/3 = 0.33333333333333333333\ldots$.

Assume that $|\mathbb{N}| = |I|$. Then there exists a one-to-one mapping $f : \mathbb{N} \to I$. Let β_i be the i^{th} digit of $f(i) \in (0,1)$.

$d_i =$ any number in $\{0,1,2,3,4,5,6,7,8,9\} \setminus \{d_{i-1}, \beta_i\}$

$D = 0.d_1 d_2 d_3 \ldots \in (0,1)$.

D is a well defined unique number in $(0,1)$,

But there is no j such that $f(j) = D$. A contradiction. $\square$

“Most General” computer?

TM

- ~~DFA~~s are simple model of computation.
- ~~Accept only the regular languages~~
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages:
  $\{L \mid L \subseteq \{0,1\}^*\}$ is ~~countably infinite~~ / uncountably infinite
- Set of all programs:
  $\{P \mid P$ is a finite length computer program$\}$:
  is countably infinite / ~~uncountably infinite.~~

All possible strings

	ε	0	1	00	01	10	11	...
L0	1	0	0	0	0	0	0	
L1	0	1	0	1	0	1	...	

P0	0
P1	0 1
P2	0 0
...	9 9 .

“Most General” computer?

- DFAs are simple model of computation.
- Accept only the regular languages.
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages:
  $\{L \mid L \subseteq \{0, 1\}^*\}$ is ~~countably infinite~~ / uncountably infinite
- Set of all programs:
  $\{P \mid P$ is a finite length computer program$\}$:
  is countably infinite / ~~uncountably infinite.~~
- **Conclusion:** There are languages for which there are no programs.

Program Diagonalization

How do we know that there are languages that cannot be represented by programs? Use Cantor!

Program Diagonalization

How do we know that there are languages that cannot be represented by programs? Use Cantor! Recall a program can be represented by a string where:

- M is the Turing machine (program)
- $\langle M \rangle$ is the string representation of the TM M

Program Diagonalization

Define $f(i,j) = 1$ if M_i accepts $\langle M_j \rangle$, else 0

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	$\langle M_6 \rangle$	$\dots$
M_1	0	1	1	1	1	1	
M_2	1	1	0	0	0	0	
M_3	0	0	0	1	0	0	
M_4	1	1	1	0	1	1	
M_5	1	0	0	0	1	0	
M_6	0	1	0	1	1	0	
$\vdots$							

Program Diagonalization

Let's define a new program:

$$D = \{\langle M \rangle | M \text{ does not accept } \langle M \rangle\}$$

Program Diagonalization

Let’s define a new program:

$$D = \{\langle M \rangle | M \text{ does not accept } \langle M \rangle\}$$

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	$\langle M_6 \rangle$	...	$\langle M_D \rangle$
M_1	0	1	1	1	1	1		1
M_2	1	1	0	0	0	0		1
M_3	0	0	0	1	0	0		1
M_4	1	1	1	0	1	1		0
M_5	1	0	0	0	1	0		0
M_6	0	1	0	1	1	0		1
$\vdots$								
M_D	1	0	~~0~~ 1	~~0~~ 1	~~1~~ 0	~~0~~ 1		□

Recap of decidability

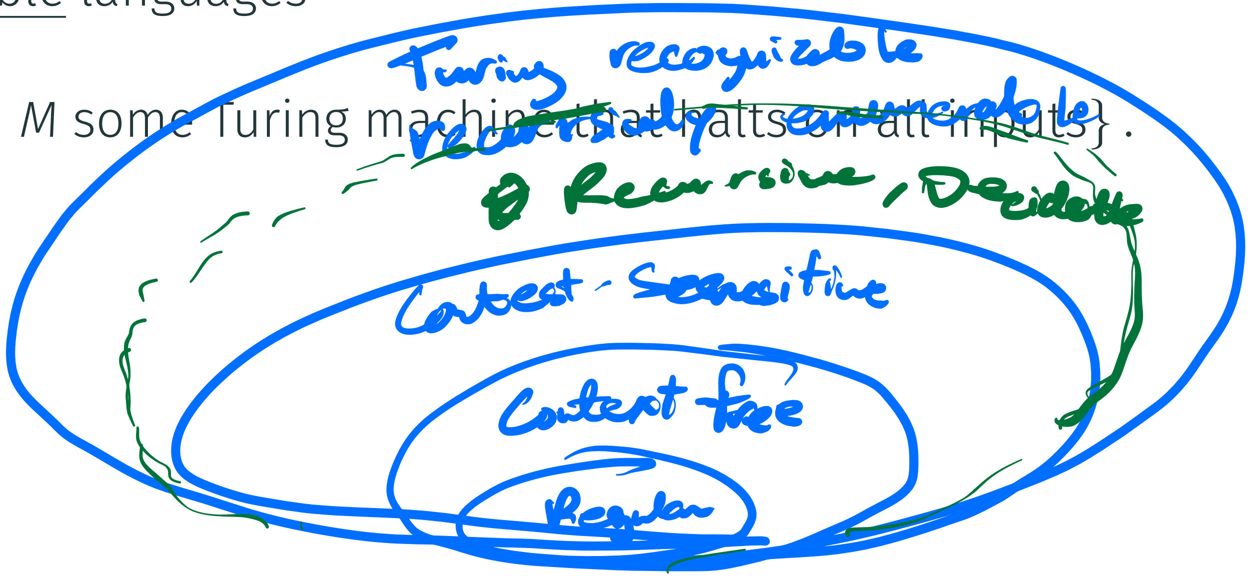
Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages

$$L = \{L(M) \mid M \text{ some Turing machine}\}\,.$$

- Recursive / decidable languages

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}\,.$$



Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages (bad)

$$L = \{L(M) \mid M \text{ some Turing machine}\}.$$

- Recursive / decidable languages (good)

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}.$$

Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages (bad)

$$L = \{L(M) \mid M \text{ some Turing machine}\}\,.$$

- Recursive / decidable languages (good)

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}\,.$$

- Fundamental questions:
  - What languages are RE?
  - Which are recursive?
  - What is the difference?
  - What makes a language decidable?

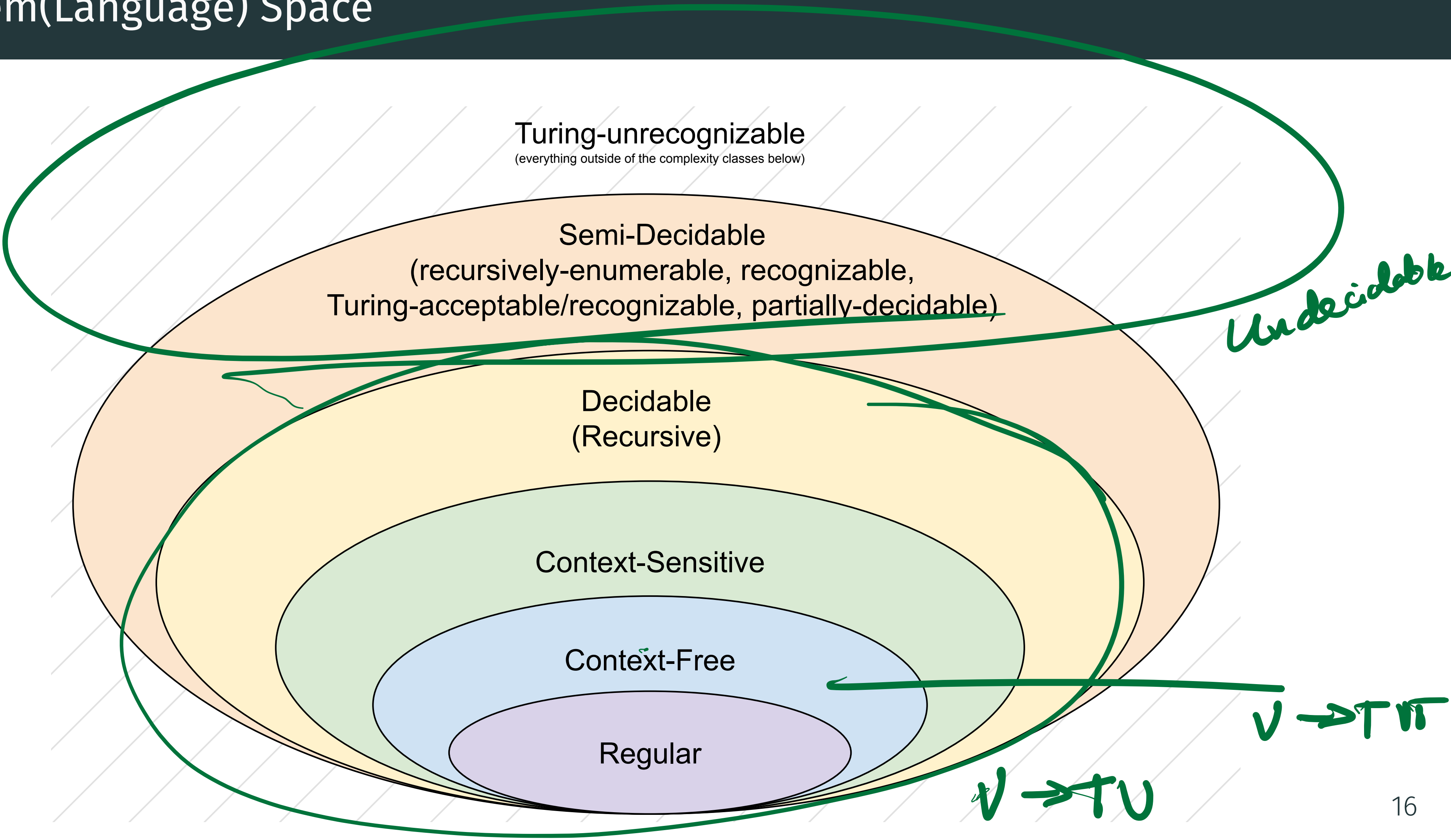
Decidable vs recursively-enumerable

A semi-decidable problem (equivalent of recursively enumerable) could be:

- **Decidable** - equivalent of recursive (TM always accepts or rejects).
- **Undecidable** - Problem is not recursive (doesn't always halt on negative)

There are undecidable problem that are not semi-decidable (recursively enumerable).

Problem(Language) Space



Un-/Decidable anchor

Like in the case of NP-complete-ness, we need an anchor point to compare languages to to determine whether they are decidable (or not)!

Introduction to the halting theorem

The halting problem

Halting problem: Given a program Q, if we run it would it stop?

The halting problem

Halting problem: Given a program Q, if we run it would it stop?

Q: Can one build a program P, that always stops, and solves the halting problem.

Theorem ("Halting theorem")
There is no program that always stops and solves the halting problem.

Intuition, why solving the Halting problem is really hard

Definition
An integer number n is a weird number if

- *the sum of the proper divisors (including 1 but not itself) of n the number is $> n$,*
- *no subset of those divisors sums to the number itself.*

70 is weird. Its divisors are $1, 2, 5, 7, 10, 14, 35$. $1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$. No subset of them adds up to 70.

Intuition, why solving the Halting problem is really hard

Definition
An integer number n is a <u>weird number</u> if

- *the sum of the proper divisors (including 1 but not itself) of n the number is $> n$,*
- *no subset of those divisors sums to the number itself.*

70 is weird. Its divisors are $1, 2, 5, 7, 10, 14, 35$. $1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$. No subset of them adds up to 70.

Open question: Are there are any odd weird numbers?

Intuition, why solving the Halting problem is really hard

Definition
An integer number n is a <u>*weird number*</u> *if*

- *the sum of the proper divisors (including 1 but not itself) of n the number is $> n$,*
- *no subset of those divisors sums to the number itself.*

70 is weird. Its divisors are $1, 2, 5, 7, 10, 14, 35$. $1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$. No subset of them adds up to 70.

Open question: Are there are any odd weird numbers?

Write a program P that tries all odd numbers in order, and check if they are weird. The programs stops if it found such number.

Intuition, why solving the Halting problem is really hard

Definition
An integer number n is a <u>*weird number*</u> *if*

- *the sum of the proper divisors (including 1 but not itself) of n the number is $> n$,*
- *no subset of those divisors sums to the number itself.*

70 is weird. Its divisors are $1, 2, 5, 7, 10, 14, 35$. $1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$. No subset of them adds up to 70.

Open question: Are there are any odd weird numbers?

Write a program P that tries all odd numbers in order, and check if they are weird. The programs stops if it found such number.

If can solve halting problem $\implies$ can resolve this open problem.

If you can halt, you can prove or disprove anything...

- Consider any math claim C.
- **Prover** algorithm P_C:
  (A) Generate sequence of all possible proofs (sequence of strings) into a pipe/queue.

If you can halt, you can prove or disprove anything...

- Consider any math claim C.
- **Prover** algorithm P_C:
  (A) Generate sequence of all possible proofs (sequence of strings) into a pipe/queue.
  (B) $\langle p \rangle \leftarrow$ pop top of queue.

If you can halt, you can prove or disprove anything...

- Consider any math claim C.
- **Prover** algorithm P_C:
  (A) Generate sequence of all possible proofs (sequence of strings) into a pipe/queue.
  (B) $\langle p \rangle \leftarrow$ pop top of queue.
  (C) Feed $\langle p \rangle$ and $\langle C \rangle$, into a proof verifier ("easy").

If you can halt, you can prove or disprove anything...

- Consider any math claim C.
- **Prover** algorithm P_C:
  (A) Generate sequence of all possible proofs (sequence of strings) into a pipe/queue.
  (B) $\langle p \rangle \leftarrow$ pop top of queue.
  (C) Feed $\langle p \rangle$ and $\langle C \rangle$, into a proof verifier ("easy").
  (D) If $\langle p \rangle$ valid proof of $\langle C \rangle$, then stop and accept.
  (E) Go to (B).
- P_C halts $\iff$ C is true and has a proof.
- If halting is decidable, then can decide if any claim in math is true.

Turing machines...

TM = Turing machine = program.

Reminder: Undecidability

Definition
Language $L \subseteq \Sigma^*$ is undecidable if no program P, given $w \in \Sigma^*$ as input, can **always stop** and output whether $w \in L$ or $w \notin L$.

(Usually defined using TM not programs. But equivalent.

Reminder: The following language is undecidable

Decide if given a program M, and an input w, does M accepts w. Formally, the corresponding language is

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Reminder: The following language is undecidable

Decide if given a program M, and an input w, does M accepts w. Formally, the corresponding language is

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Definition
A decider for a language L, is a program (or a TM) that always stops, and outputs for any input string $w \in \Sigma^*$ whether or not $w \in L$.

A language that has a decider is decidable.

Reminder: The following language is undecidable

Decide if given a program M, and an input w, does M accepts w. Formally, the corresponding language is

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Definition
A <u>decider</u> for a language L, is a program (or a TM) that always stops, and outputs for any input string $w \in \Sigma^*$ whether or not $w \in L$.

A language that has a decider is <u>decidable</u>.

Turing proved the following:

Theorem
A_{TM} *is undecidable.*

The halting problem

A_{TM} is not TM decidable!

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Theorem (The halting theorem.)
A_{TM} is not Turing decidable.

A_{TM} is not TM decidable!

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Theorem (The halting theorem.)
A_{TM} *is not Turing decidable.*

Proof: Assume A_{TM} is TM decidable...

A_{TM} is not TM decidable!

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Theorem (The halting theorem.)
A_{TM} is not Turing decidable.

Proof: Assume A_{TM} is TM decidable...

Halt: TM deciding A_{TM}. Halt always halts, and works as follows:

$$\mathrm{Halt}\big(\langle M, w \rangle\big) = \begin{cases} \text{accept} & M \text{ accepts } w \\ \text{reject} & M \text{ does not accept } w. \end{cases}$$

Halting theorem proof continued 1

We build the following new function:

```
Flipper(⟨M⟩)
    res ← Halt(⟨M, M⟩)
    if res is accept then
            reject
    else
            accept
```

Halting theorem proof continued 1

We build the following new function:

```
Flipper(⟨M⟩)
    res ← Halt(⟨M, M⟩)
    if res is accept then
            reject
    else
            accept
```

Flipper always stops:

$$\text{Flipper}\Big(\langle M \rangle\Big) = \begin{cases} \text{reject} & M \text{ accepts } \langle M \rangle \\ \text{accept} & M \text{ does not accept } \langle M \rangle. \end{cases}$$

Halting theorem proof continued 2

$$\text{Flipper}\big(\langle M\rangle\big) = \begin{cases} \text{reject} & M \text{ accepts } \langle M\rangle \\ \text{accept} & M \text{ does not accept } \langle M\rangle\,. \end{cases}$$

Flipper is a TM (duh!), and as such it has an encoding $\langle \text{Flipper}\rangle$. Run Flipper on itself:

$$\text{Flipper}\big(\langle \text{Flipper}\rangle\big) = \begin{cases} \text{reject} & \text{Flipper accepts } \langle \text{Flipper}\rangle \\ \text{accept} & \text{Flipper does not accept } \langle \text{Flipper}\rangle\,. \end{cases}$$

Halting theorem proof continued 2

$$\text{Flipper}\Big(\langle M\rangle\Big) = \begin{cases} \text{reject} & M \text{ accepts } \langle M\rangle \\ \text{accept} & M \text{ does not accept } \langle M\rangle\,. \end{cases}$$

Flipper is a TM (duh!), and as such it has an encoding $\langle\text{Flipper}\rangle$. Run Flipper on itself:

$$\text{Flipper}\Big(\langle \text{Flipper}\rangle\Big) = \begin{cases} \text{reject} & \text{Flipper accepts } \langle \text{Flipper}\rangle \\ \text{accept} & \text{Flipper does not accept } \langle \text{Flipper}\rangle\,. \end{cases}$$

This is can’t be correct

Halting theorem proof continued 2

$$\text{Flipper}\Big(\langle M\rangle\Big) = \begin{cases} \text{reject} & M \text{ accepts } \langle M\rangle \\ \text{accept} & M \text{ does not accept } \langle M\rangle\,. \end{cases}$$

Flipper is a TM (duh!), and as such it has an encoding $\langle\text{Flipper}\rangle$. Run Flipper on itself:

$$\text{Flipper}\Big(\langle\text{Flipper}\rangle\Big) = \begin{cases} \text{reject} & \text{Flipper accepts } \langle\text{Flipper}\rangle \\ \text{accept} & \text{Flipper does not accept } \langle\text{Flipper}\rangle\,. \end{cases}$$

This is can't be correct

Assumption that Halt exists is false. $\implies$ A_{TM} is not TM decidable. $\square$

Unrecognizable

TM recognizable

Definition
Language L is TM *decidable if there exists M that always stops, such that $L(M) = L$.*

TM recognizable

Definition
Language L is TM decidable if there exists M that always stops, such that $L(M) = L$.

Definition
Language L is TM recognizable if there exists M that stops on some inputs, such that $L(M) = L$.

Definition
Language L is TM decidable if there exists M that always stops, such that $L(M) = L$.

Definition
Language L is TM recognizable if there exists M that stops on some inputs, such that $L(M) = L$.

Theorem (Halting)
$A_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and } M \text{ accepts } w \right\}$. *is TM recognizable, but not decidable.*

$\overline{A_{TM}} = \{ \langle M, w \rangle \mid M$ is a TM and M does $\underline{\text{not}}$ accept w $\}$

TM recognizable

Lemma
If L and $\overline{L} = \Sigma^* \setminus L$ are both TM recognizable, then L and $\overline{L}$ are decidable.

TM recognizable

Lemma
If L and $\overline{L} = \Sigma^* \setminus L$ are both TM recognizable, then L and $\overline{L}$ are decidable.

Proof.
M: TM recognizing L.

M_c: TM recognizing $\overline{L}$.

Given input x, using UTM simulating running M and M_c on x in parallel. One of them must stop and accept. Return result.

$\implies$ L is decidable. $\square$

Complement language for A_{TM}

$$\overline{\mathrm{A}_{TM}} = \Sigma^* \setminus \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Complement language for A_{TM}

$$\overline{\mathrm{A}_{TM}} = \Sigma^* \setminus \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

But don't really care about invalid inputs. So, really:

$$\overline{\mathrm{A}_{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ does } \textbf{not} \text{ accept } w \right\}.$$

Complement language for A_{TM} is not TM-recognizable

Theorem
The language

$$\overline{\mathrm{A}_{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and } M \text{ does } \textbf{not} \text{ accept } w \right\}.$$

is not TM recognizable.

Complement language for A_{TM} is not TM-recognizable

Theorem
The language

$$\overline{\mathrm{A}_{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ does } \textbf{not} \text{ accept } w \right\}.$$

is not TM recognizable.

Proof.
A_{TM} is TM-recognizable.

If $\overline{\mathrm{A}_{TM}}$ is TM-recognizable

Complement language for A_{TM} is not TM-recognizable

Theorem
The language

$$\overline{\mathrm{A}_{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and } M \text{ does } \textbf{not} \text{ accept } w \right\}.$$

is not TM recognizable.

Proof.
A_{TM} is TM-recognizable.

If $\overline{\mathrm{A}_{TM}}$ is TM-recognizable

$\implies$ (by Lemma)

A_{TM} is decidable. A contradiction. $\square$

Reductions

Reduction

Meta definition: Problem X <u>reduces</u> to problem B, if given a solution to B, then it implies a solution for X. Namely, we can solve Y then we can solve X. We will done this by $X \implies Y$.

Reduction

Meta definition: Problem X reduces to problem B, if given a solution to B, then it implies a solution for X. Namely, we can solve Y then we can solve X. We will done this by X $\implies$ Y.

Definition
oracle ORAC for language L is a function that receives as a word w, returns TRUE $\iff w \in L$.

Reduction

Meta definition: Problem X reduces to problem B, if given a solution to B, then it implies a solution for X. Namely, we can solve Y then we can solve X. We will done this by $X \implies Y$.

Definition

oracle ORAC for language L is a function that receives as a word w, returns TRUE $\iff w \in L$.

Lemma

A language X reduces to a language Y, if one can construct a TM decider for X using a given oracle ORAC_Y for Y.

We will denote this fact by $X \implies Y$.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of Y.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of Y.
- Assume L is decided by TM M.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of Y.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem X using M.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of Y.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem X using M.
- Result in decider for X (i.e., A_{TM}).

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of Y.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem X using M.
- Result in decider for X (i.e., A_{TM}).
- Contradiction X is not decidable.

Reduction proof technique

- Y: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of Y.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem **X** using M.
- Result in decider for **X** (i.e., A_{TM}).
- Contradiction **X** is not decidable.
- Thus, L must be not decidable.

Reduction implies decidability

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If Y is decidable then X is decidable.

Proof.
Let T be a decider for Y (i.e., a program or a TM). Since X reduces to Y, it follows that there is a procedure $\mathsf{T}_{X|Y}$ (i.e., decider) for X that uses an oracle for Y as a subroutine. We replace the calls to this oracle in $\mathsf{T}_{X|Y}$ by calls to T. The resulting program T_X is a decider and its language is X. Thus X is decidable (or more formally TM decidable). $\square$

The countrapositive...

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If X is undecidable then Y is undecidable.

Halting

The halting problem

Language of all pairs $\langle M, w \rangle$ such that M halts on w:

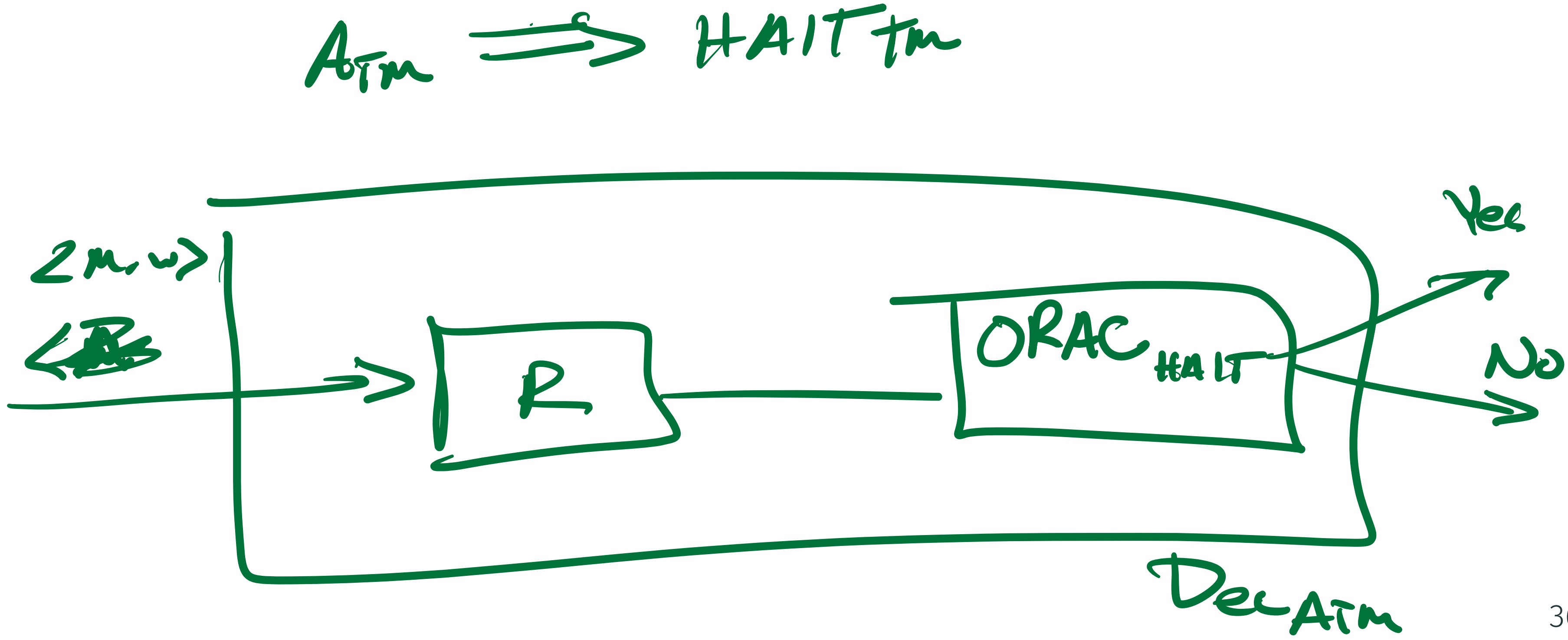
$$A_{\mathrm{Halt}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ stops on } w \right\}.$$

Similar to language already known to be undecidable:

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

On way to proving that Halting is undecidable...

Lemma
The language A_{TM} reduces to $\mathrm{A}_{\mathrm{Halt}}$. Namely, given an oracle for $\mathrm{A}_{\mathrm{Halt}}$ one can build a decider (that uses this oracle) for A_{TM}.



On way to proving that Halting is undecidable...

Proof.
Let ORAC_{Halt} be the given oracle for A_{Halt}. We build the following decider for A_{TM}.

AnotherDecider-$\mathrm{A}_{TM}\Big(\langle M, w\rangle\Big)$

```
    res ← ORAC_Halt(⟨M, w⟩)
    // if M does not halt on w then reject.
    if res = reject then
            halt and reject.
    // M halts on w since res =accept.
    // Simulating M on w terminates in finite time.
    res_2 ←Simulate M on w.
    return res_2 .
```

This procedure always return and as such its a decider for A_{TM}. $\square$

The Halting problem is not decidable

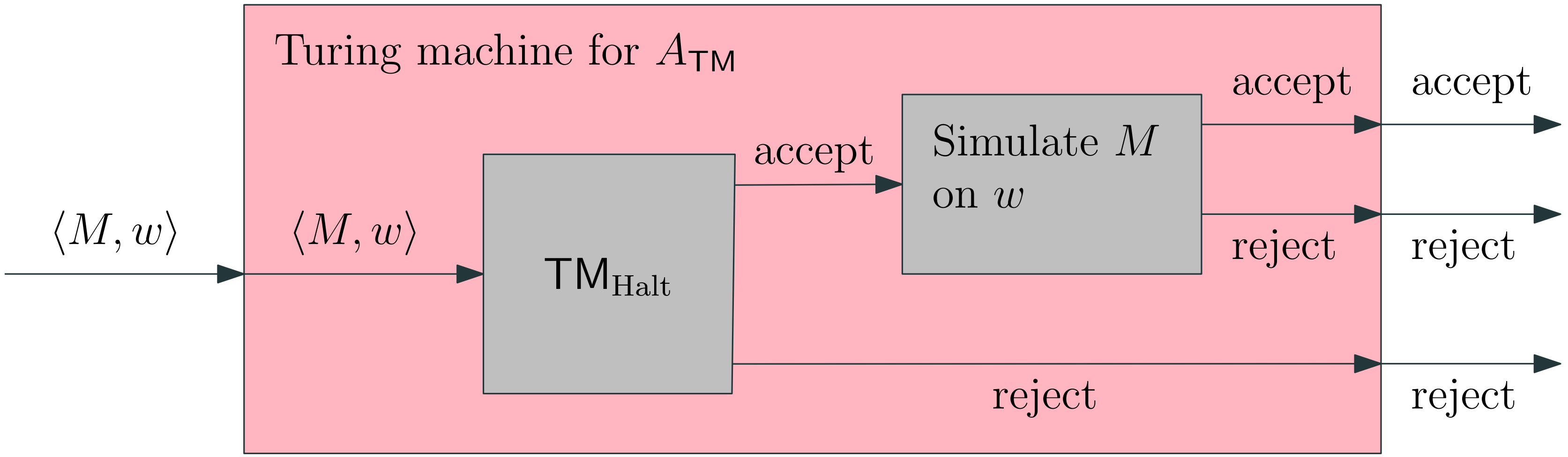
Theorem

The language A_{Halt} is not decidable.

Proof.

Assume, for the sake of contradiction, that A_{Halt} is decidable. As such, there is a TM, denoted by TM_{Halt}, that is a decider for A_{Halt}. We can use TM_{Halt} as an implementation of an oracle for A_{Halt}, which would imply that one can build a decider for A_{TM}. However, A_{TM} is undecidable. A contradiction. It must be that A_{Halt} is undecidable. $\square$

The same proof by figure...



... if A_{Halt} is decidable, then A_{TM} is decidable, which is impossible.

More reductions next time