Pre-lecture brain teaser

In the following languages, three are decidable and three are undecidable. Which are which?

- $A_{CFG} = \left\{ \langle G, w \rangle \;\middle|\; G \text{ is a CFG that generates string } w \right\}$.
- $E_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a CFG and } L(G) = \emptyset \right\}$.
- $ALL_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a CFG and } L(G) = \Sigma^* \right\}$.
- $A_{LBA} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a LBA that generates string } w \right\}$.
- $E_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a LBA where } L(M) = \emptyset \right\}$.
- $ALL_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a LBA where } L(M) = \Sigma^* \right\}$.

ECE-374-B: Lecture 24 - Midterm 3 Review

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December 02, 2025

University of Illinois Urbana-Champaign

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$A_{TM} = \{ \langle M, w \rangle \mid \text{———} \}$

$E_{TM} = \{ \langle M \rangle \mid \text{———} \}$

$ALL_{TM} = \{ \langle M \rangle \mid L(M) = \Sigma^* \}$

$A_{DFA} = \{ \langle A, w \rangle \mid \text{———} \}$

$E_{DFA} = \{ \langle A \rangle \mid L(A) = \emptyset \}$

$ALL_{DFA} = \{ \langle A \rangle \mid L(A) = \Sigma^* \}$

A_{CFG} decidable?

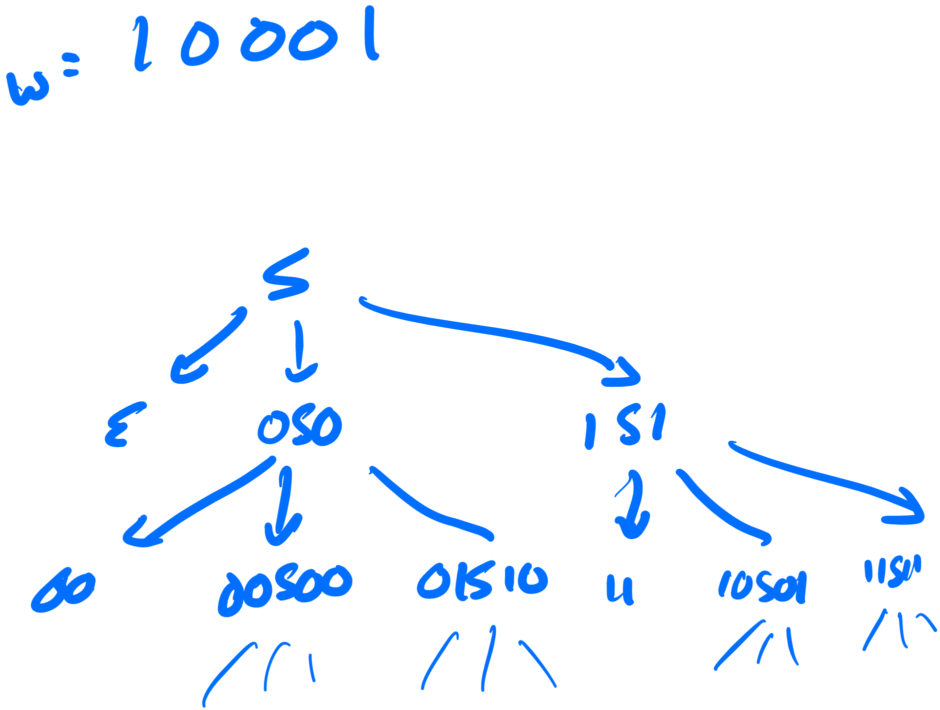
$A_{CFG} = \{ \langle G, w \rangle \mid G$ is a context-free grammar, G accepts $w \}$

A_{CFG} decidable?

YES!

A_{CFG} decidable?

YES!



- $V = \{S\}$
- $T = \{0, 1\}$
- $P = \{S \rightarrow \epsilon \mid 0S0 \mid 1S1\}$
  (abbrev. for $S \rightarrow \epsilon, S \rightarrow 0S0, S \rightarrow 1S1$)

A_{CFG} decidable?

YES!

$|w| = n$

$V \to TVT$

Lemma

A CFG in Chomsky normal form can derive a string w in at most 2^n *steps!*

Knowing this, we can just simulate all the possible rule combinations for 2^n steps and see if any of the resulting strings matches w.

E_{CFG} decidable?

$$E_{CFG} = \{ \langle G \rangle \mid L(G) = \emptyset \}$$

E_{CFG} decidable?

YES!

E_{CFG} decidable?

YES!

In this case, we just need to know if we can get from the start variable to a string with only terminal symbols.

1. Mark all terminal symbols in G
2. Repeat until no new variables get marked:
    2.1 Mark any variable A where G has the rule $A \to U_1 U_2 \ldots U_k$ where U_i is a marked terminal/variable
3. If start variable is not marked, accept. Otherwise reject.

- $V = \{S\}$
- $T = \{0, 1\}$
- $P = \{S \to \epsilon \mid 0S0 \mid 1S1\}$ (abbrev. for $S \to \epsilon, S \to 0S0, S \to 1S1$)

ALL_{CFG} decidable?

ALL_{CFG} decidable?

Nope

ALL_{CFG} decidable?

Nope

Proof requires computation histories which are outside the scope of this course.

A_{LBA} decidable?

$|w| = n$

$A_{LBA} = \{ \langle M, w \rangle \mid M \text{ is LBA}, M \text{ accepts } w \}$

$|\text{tape}| = n^k$

A_{LBA} decidable?

YES!

A_{LBA} decidable?

YES!

Remember a LBA has a finite tapes. Therefore we know:

1. A tape of length n where each cell can contain g symbols, you have g^n possible configurations.
2. The tape head can be in one of n positions and has q states yielding a tape head that can be in qn configurations.
3. Therefore the machine can be in qng^n configurations.

A_{LBA} decidable?

YES!

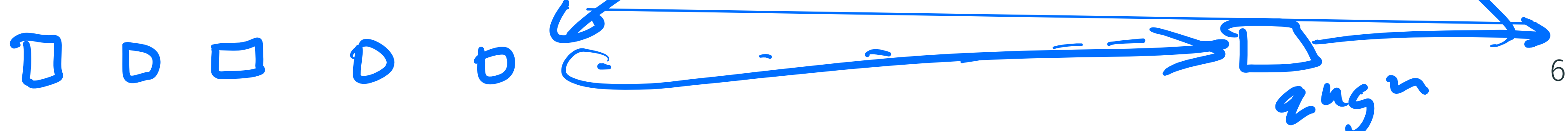
$A_{374TM} = \{\langle M, w\rangle \mid M$ is a TM, M accept w in 374 steps $\}$

Remember a LBA has a finite tapes. Therefore we know:

1. A tape of length n where each cell can contain g symbols, you have g^n possible configurations.
2. The tape head can be in one of n positions and has q states yielding a tape that can be in qn configurations.
3. Therefore the machine can be in qng^n configurations.

Lemma

If an LBA does not accept or reject in qng^n then it is stuck in a loop forever.



A_{LBA} decidable?

Decider for A_{LBA} will:

1. Simulate $\langle M \rangle$ on w for qng^n steps.
   1.1 if accepts, then accept
   1.2 if rejects, then reject
2. If neither accepts or rejects, means it's in a loop in whihc case, reject.

E_{LBA} decidable?

E_{LBA} decidable?

Nope

E_{LBA} decidable?

Nope

Proof requires computational history trick, a story for another time......

ALL_{LBA} decidable?

ALL_{LBA} decidable?

Nope

ALL_{LBA} decidable?

Nope

No standard proof for this, but let's look at a pattern:

Decidability across grammar complexities

	DFA	*CFG*	*PDA*	*LBA*	*TM*
A	D	D	D	D	U
E	D	D	D	U	U
ALL	D	U	U	U	U

Eventually problems get too tough....

ALL_{LBA} decidable?

Nope

No standard proof for this, but let's look at a pattern:

So we sort've know that ALL_{LBA} isn't decidable because we knew ALL_{CFG} wasn't (though intuition is never sufficient evidence).

Un-/decidability practice problems

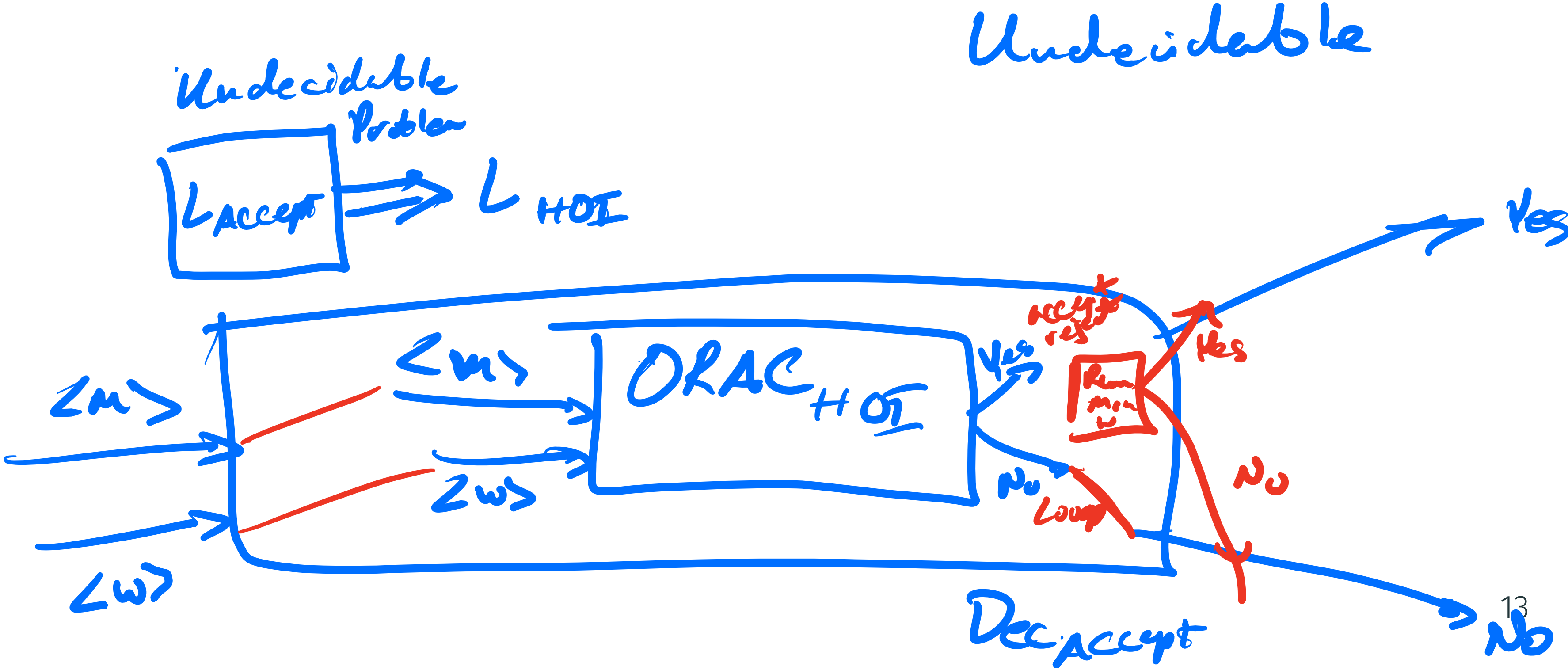
Available Undecidable languages

- $L_{Accept} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and accepts } w \right\}.$
- $L_{HALT} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and halts on } \varepsilon \right\}.$

Practice 1: Halt on Input

Is the language:

$$L_{HaltOnInput} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and halts on } w \right\}.$$

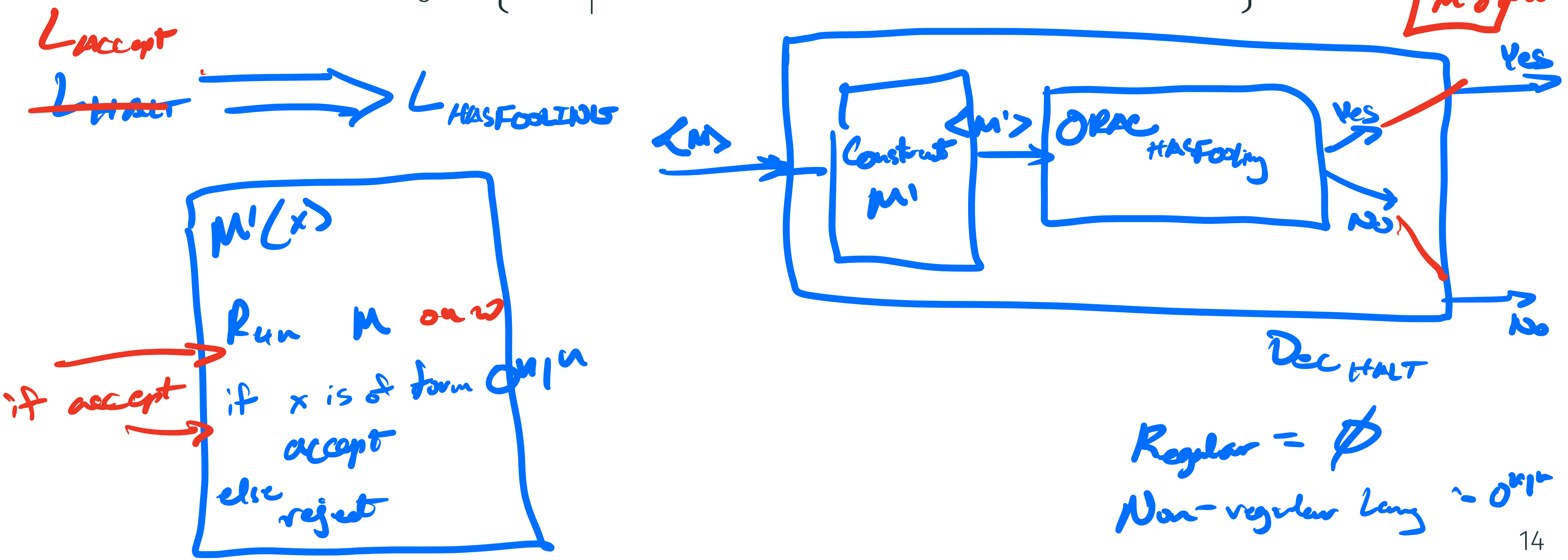


Practice 2: L has fooling set

Is the language:

- $L_{HALT} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and halts on } \varepsilon \right\}$.

$$L_{HasFooling} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and } L(M) \text{ has a fooling set } \right\}.$$



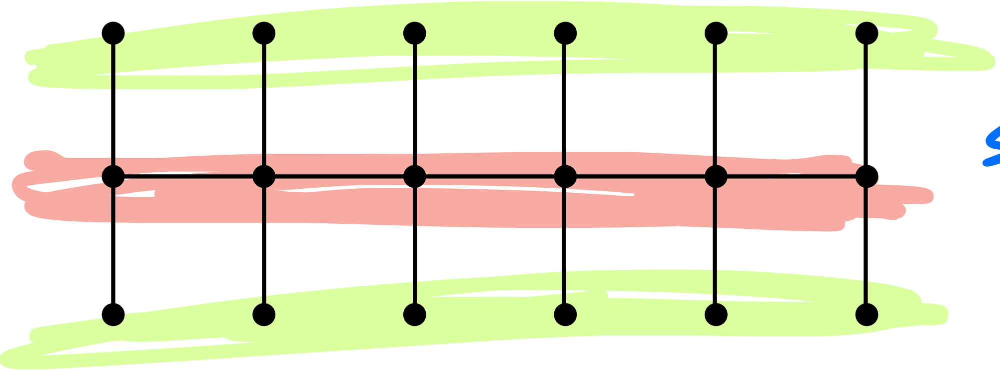
NP-Complete practice problems

Practice: NP-Complete Reduction I

A centipede is an undirected graph formed by a path of length k with two edges (legs) attached to each node on the path as shown in the below figure. Hence, the centipede graph has $3k$ vertices. The **CENTIPEDE** problem is the following: given an undirected graph $G = (V, E)$ and an integer k, does G contain a centipede of $3k$ vertices as a subgraph? Prove that **CENTIPEDE** is **NP-Complete**.

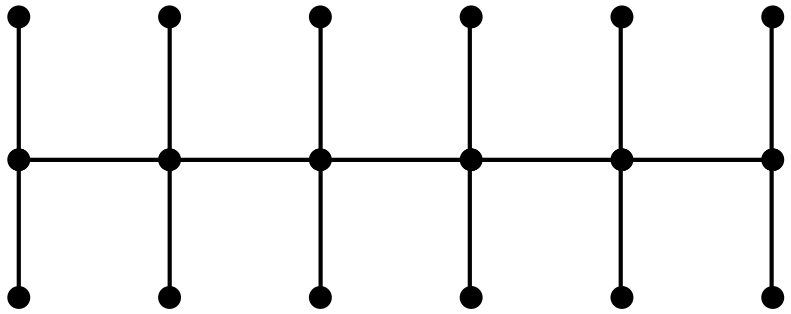
SAT ≤ Centipede

SAT ≤ HP ≤ Centipede



Practice: NP-Complete Reduction

What do we need to do to prove Centipede is NP-Complete?

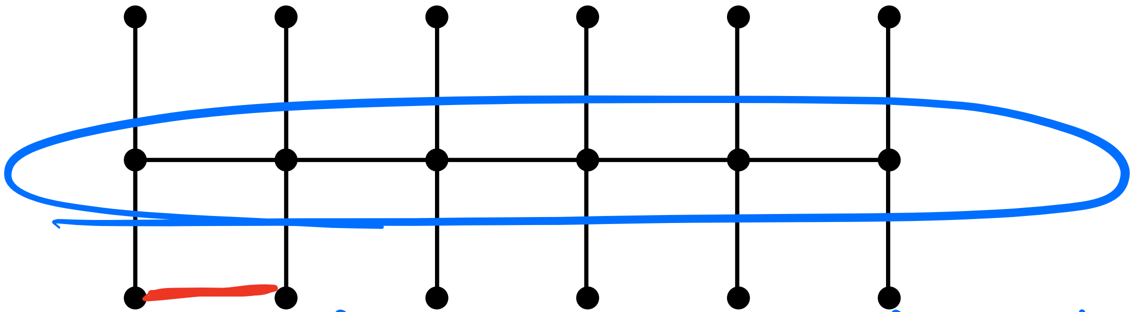


NP

NP-hard

Practice: NP-Complete Reduction

Prove Centipede is in **NP**:



Certificate: a path of size k that forms back $B[1..k]$

a array of two tuples that are node attached to the corresponding node in the back $Legs[1..2]$ $[1..k]$

Certifier: Check $|B| == k$

Check for $i = 1..k-1$ $(B[i], B[i+1]) \in G$

for $i = 1..k$ $\langle B[i], Legs[i][1] \rangle \in G$

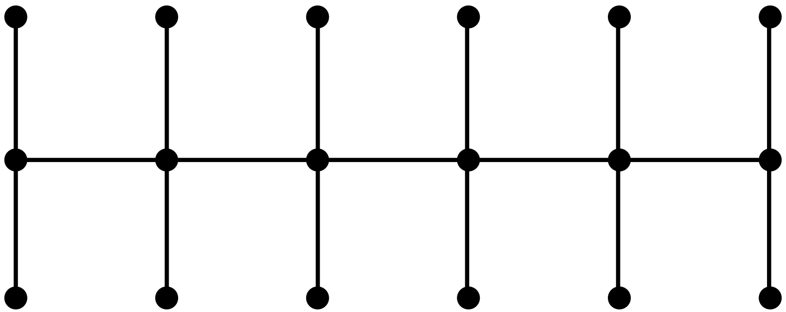
$Legs[i][2]) \in G$

Certifier runs in poly time

$\therefore$ Centipede is in NP

Practice: NP-Complete Reduction

Prove Centipede is in **NP-hard**:



$|V| = n$

Hamiltonian Path $\le_p$ Centipede

R: Form G' by copying G

add vertices L_1^i, L_2^i for all $i = 1 \dots n$

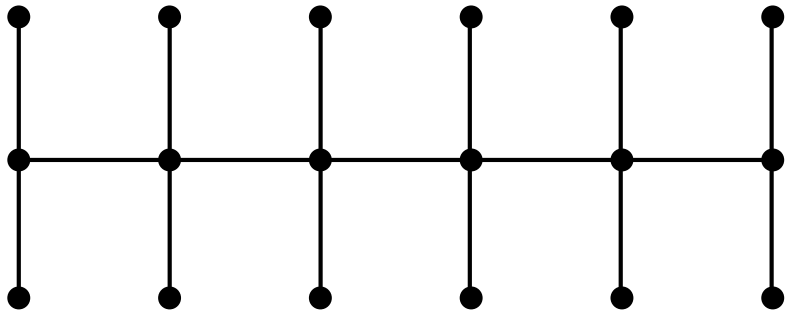
add edge (v_i, L_1^i) & (v_i, L_2^i) for all $i = 1 \dots n$

<G> → R → <G'>, k := n → Alg centipede → yes / no

Alg_{HP}

Practice: NP-Complete Reduction

Prove Centipede is in **NP-hard**:



Hamiltonian Path: Given a graph G (either directed or undirected), is there a path that visits every vertex exactly once

$HC \leq_P Centipede$

Practice: NP-Complete Reduction I

A quasi-satisfying assignment for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

Practice: NP-Complete Reduction I

A quasi-satisfying assignment for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

Prove quasiSAT is in NP

Practice: NP-Complete Reduction I

A quasi-satisfying assignment for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

Prove quasiSAT is NP-hard

Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard

Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard

3SAT: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment.

Good luck on the exam