# Pre-lecture brain teaser

In the following languages, three are decidable and three are undecidable. Which are which?

- $A_{CFG} = \left\{ \langle G, w \rangle \;\middle|\; G \text{ is a } CFG \text{ that accepts string } w \right\}$.
- $E_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a } CFG \text{ and } L(G) = \emptyset \right\}$.
- $ALL_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a } CFG \text{ and } L(G) = \Sigma^* \right\}$.
- $A_{LBA} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } LBA \text{ that generates string } w \right\}$.
- $E_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } LBA \text{ where } L(M) = \emptyset \right\}$.
- $ALL_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } LBA \text{ where } L(M) = \Sigma^* \right\}$.

# ECE-374-B: Lecture 25 - Midterm 3 Review

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April 26, 2022

University of Illinois at Urbana-Champaign

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# $A_{CFG}$ decidable?

$V = \{S\}$

$T = \{0, 1\}$

$$P = \left\{ \begin{array}{l} S \to \varepsilon \\ S \to 0S0 \\ S \to 1S1 \end{array} \right\}$$

# $A_{CFG}$ decidable?

YES!

# $A_{CFG}$ decidable?

YES!

- $V = \{S\}$
- $T = \{0, 1\}$
- $P = \{S \to \epsilon \mid 0S0 \mid 1S1\}$
  (abbrev. for $S \to \epsilon, S \to 0S0, S \to 1S1$)

$|w| = n$

# $A_{CFG}$ decidable?

YES!

**Lemma**
*A CFG in Chomsky normal form can derive a string $w$ in at most $2n - 1$ steps! (Shown in Sipser textbook)*

Knowing this, we can just simulate all the possible rule combinations for $2^n$ steps and see if any of the resulting strings matches $w$.

# $E_{CFG}$ decidable?

## $E_{CFG}$ decidable?

YES!

# $E_{CFG}$ decidable?

YES!

In this case, we just need to know if we can get from the start variable to a string with only terminal symbols.

1. Mark all terminal symbols in $G$
2. Repeat until no new variables get marked:
   2.1 Mark any variable $A$ where G has the rule $A \to U_1 U_2 \dots U_k$ where $U_i$ is a marked terminal/variable
3. If start variable is not marked, accept. Otherwise reject.

- $V = \{S\}$
- $T = \{0, 1\}$
- $P = \{S \to \epsilon \mid 0S0 \mid 1S1\}$
  (abbrev. for $S \to \epsilon, S \to 0S0, S \to 1S1$)

$A \to S$

$S \to \varepsilon$

# $ALL_{CFG}$ decidable?

# $ALL_{CFG}$ decidable?

Nope

# $ALL_{CFG}$ decidable?

Nope

Proof requires computation histories which are outside the scope of this course.

# $A_{LBA}$ decidable?

# $A_{LBA}$ decidable?

YES!

# $A_{LBA}$ decidable?

YES!

Remember a LBA has a finite tapes. Therefore we know:

1. A tape of length $n$ where each cell can contain $g$ symbols, you have $g^n$ possible configurations.
2. The tape head can be in one of $n$ positions and has $q$ states yielding a tape that can be in $qn$ configurations.
3. Therefore the machine can be in $qng^n$ configurations.

# $A_{LBA}$ decidable?

YES!

Remember a LBA has a finite tapes. Therefore we know:

1. A tape of length $n$ where each cell can contain $g$ symbols, you have $g^n$ possible configurations.
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3. Therefore the machine can be in $qng^n$ configurations.

**Lemma**
*If an LBA does not accept or reject in $qng^n$ then it is stuck in a loop forever.*

# $A_{LBA}$ decidable?

Decider for $A_{LBA}$ will:

1. Simulate $\langle M \rangle$ on $w$ for $qng^n$ steps.
   1.1 if accepts, then accept
   1.2 if rejects, then reject
2. If neither accepts or rejects, means it's in a loop in which case, reject.

# $E_{LBA}$ decidable?

# $E_{LBA}$ decidable?

Nope

# $E_{LBA}$ decidable?

Nope

Proof requires computational history trick, a story for another time......

# $ALL_{LBA}$ decidable?

# $ALL_{LBA}$ decidable?

Nope

# $ALL_{LBA}$ decidable?

Nope

No standard proof for this, but let's look at a pattern:

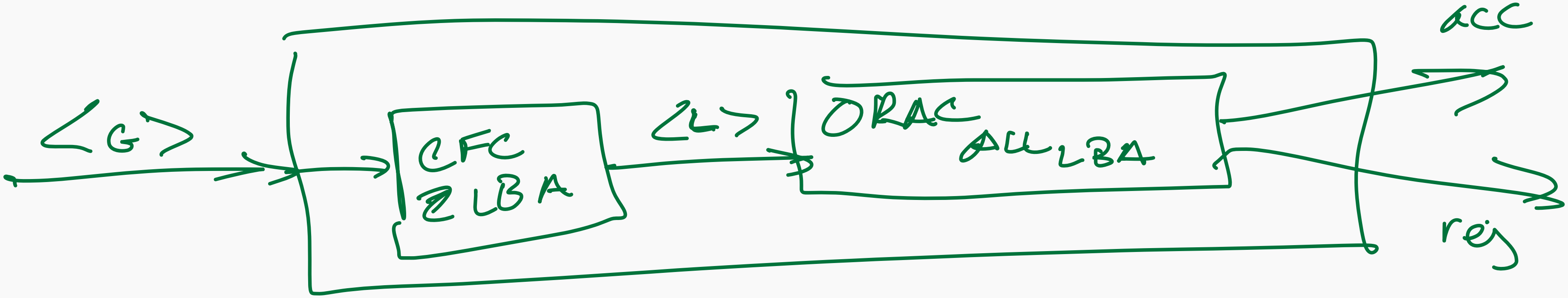
# $ALL_{LBA}$ decidable?

Nope

No standard proof for this, but let's look at a pattern:

So we sort've figure that $ALL_{LBA}$ isn't decidable because we know (assuming you believe me) $ALL_{CFG}$ wasn't (though intuition is never sufficient evidence).

$ALL_{CFG} \Rightarrow ALL\ LBA$



# Decidability across grammar complexities

| | *DFA* | *CFG* | *PDA* | *LBA* | *TM* |
|---|---|---|---|---|---|
| A | D | D | D | D | U |
| E | D | D | D | U | U |
| ALL | D | U | U | U | U |

INF

Eventually problems get too tough....

# Rice theorem

The above proofs were somewhat repetitious...

...they imply a more general result.

**Theorem (Rice's Theorem.)**
*Suppose that* $\mathsf{L}$ *is a language of Turing machines; that is, each word in* $\mathsf{L}$ *encodes a TM. Furthermore, assume that the following two properties hold.*

(a) *Membership in* $\mathsf{L}$ *depends only on the Turing machine's language, i.e. if* $L(M) = L(N)$ *then* $\langle M \rangle \in \mathsf{L} \Leftrightarrow \langle N \rangle \in \mathsf{L}$.

(b) *The set* $\mathsf{L}$ *is "non-trivial," i.e.* $\mathsf{L} \neq \emptyset$ *and* $\mathsf{L}$ *does not contain all Turing machines.*

*Then* $\mathsf{L}$ *is a undecidable.* (***Note: In an exam, you can't just say undecidable because of Rice's theorem.***)

# Un-/decidability practice problems

# Available Undecidable languages

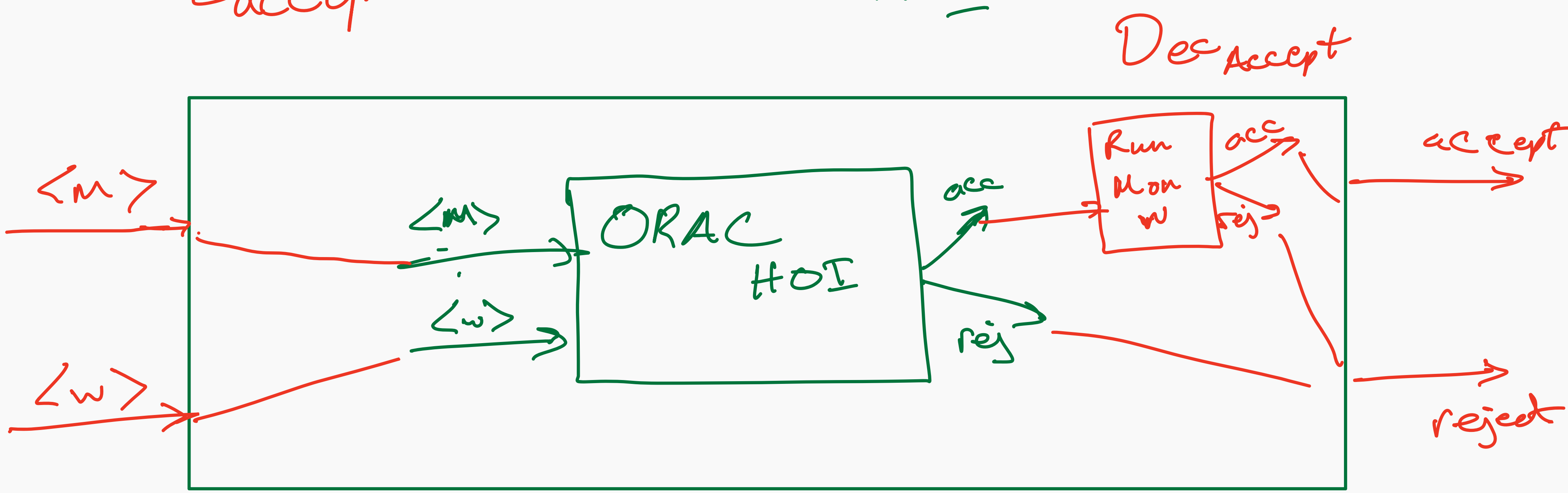
- $L_{Accept} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and accepts } w \right\}$.
- $L_{HALT} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and halts on } \varepsilon \right\}$.

# Practice 1: Halt on Input

Is the language: $L_{Accept} = \{ \langle M, w \rangle \mid M \text{ is a TM } \& \text{ accepts } w \}$

$$L_{HaltOnInput} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and halts on } w \right\}.$$

$L_{accept} \Longrightarrow L_{HOt}$



# Practice 1: Halt on Input

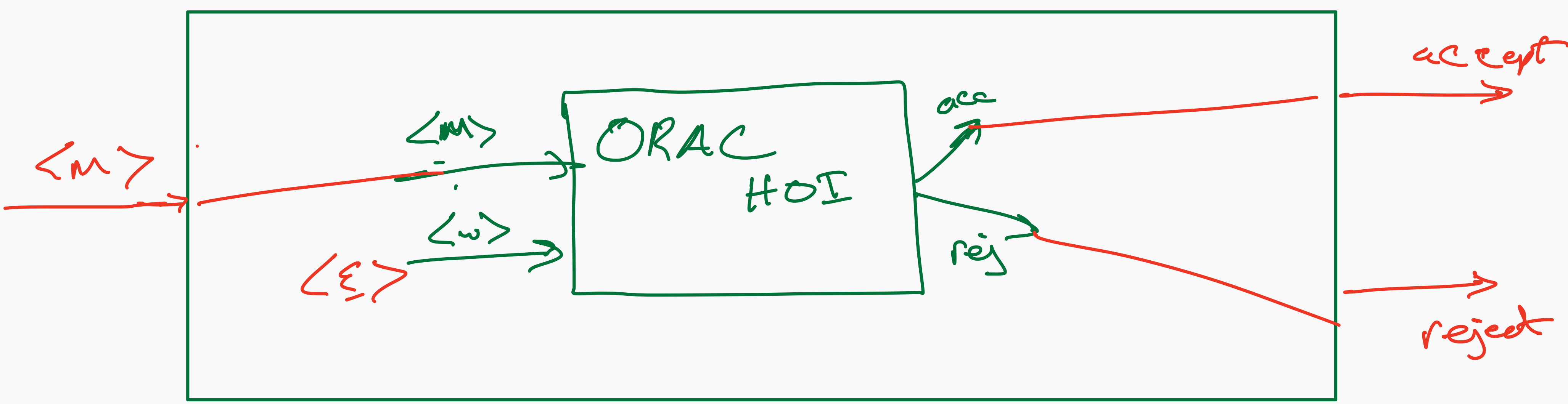
Is the language:

$L_{HALT_{onB}} = \{\langle M \rangle \mid M \text{ is a TM \& halts on } \varepsilon\}$

$$L_{HaltOnInput} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and halts on } w \right\}.$$

$L_{HALT} \Longrightarrow L_{HOI}$

$Dec_{HALT\,onB}$



# Practice 2: L has fooling set

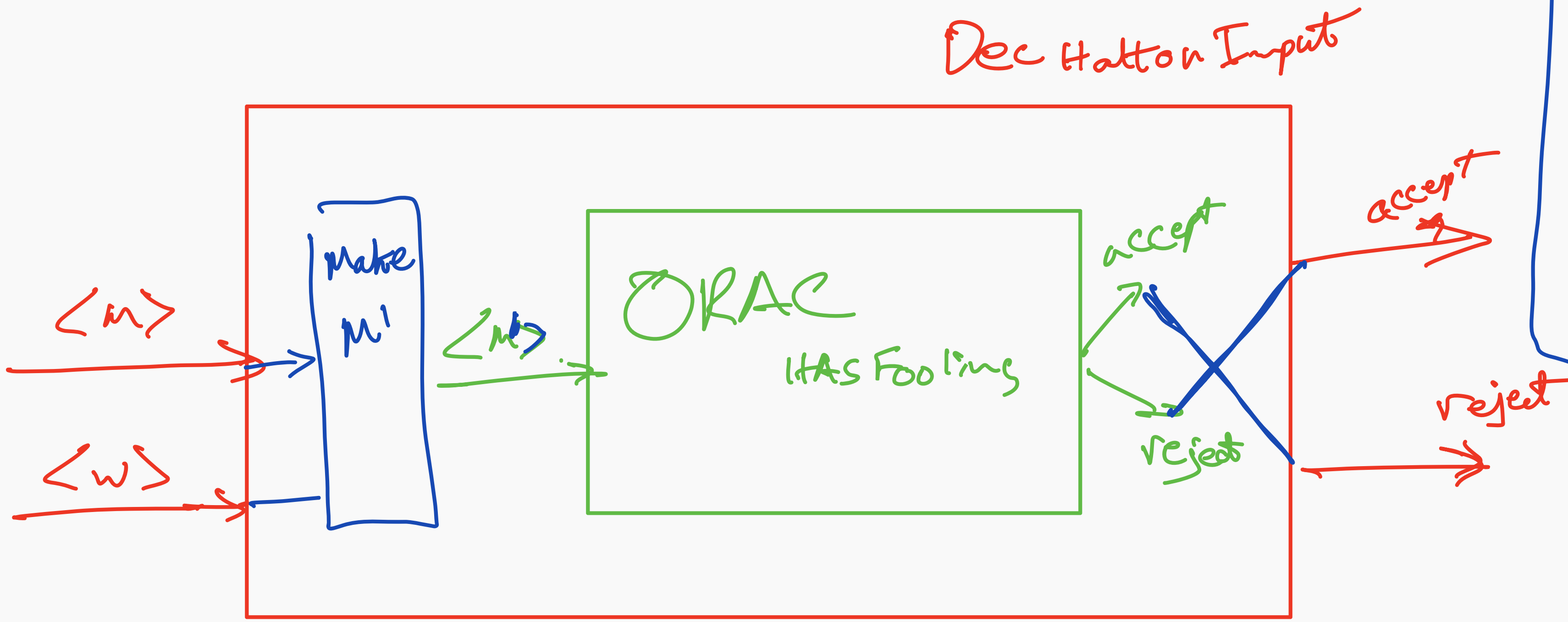
Is the language: $L_{HALTonInput} = \{ \langle M, w \rangle \mid M \text{ is a TM and halts on } w \}$

$$L_{HasFooling} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ has a fooling set} \}$$

infinite

$L_{HoI} \Longrightarrow L_{HasFooling}$

$M'(x)$
if $x$ is of form $0^n 1^n$ accept
Run $M$ on $w$
accept



# Practice 2: L has fooling set

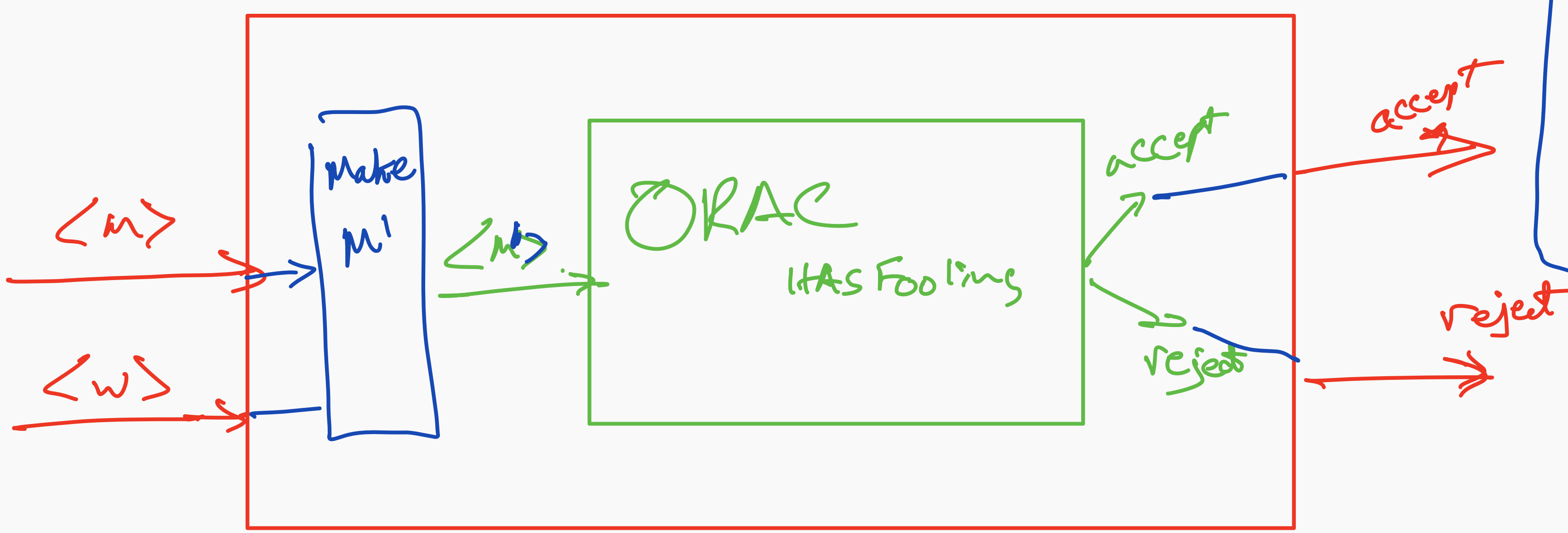
Is the language: $L_{HALTonInput} = \{\langle M, w\rangle \mid M \text{ is a TM } \& \text{ halts on } w\}$

$$L_{HasFooling} = \{\langle M\rangle \mid M \text{ is a } \textcolor{orange}{TM} \text{ and } L(M) \text{ has a infinite fooling set}\}$$

$M'(x)$
- Run $M$ on $w$
- if $x$ has form $0^n1^n$ accept
- else reject

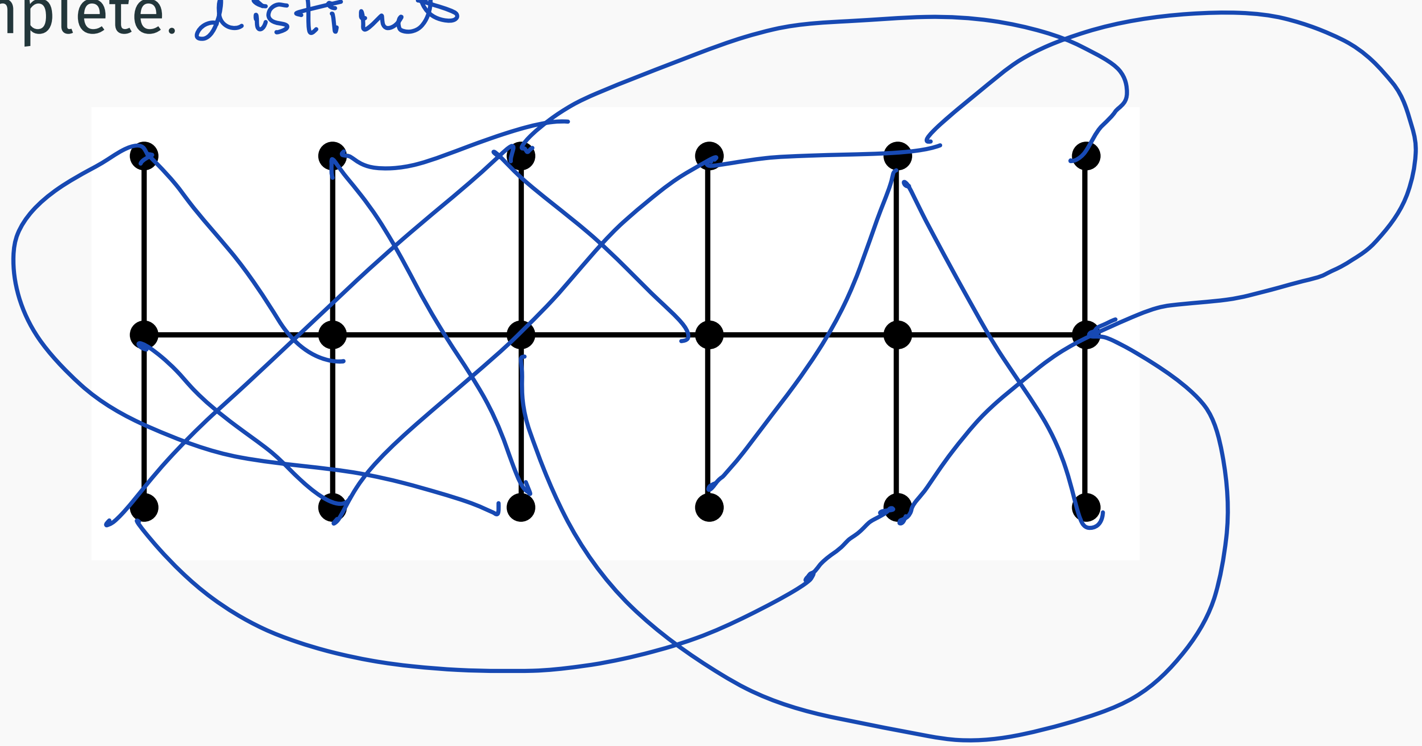
$L_{HoI} \Longrightarrow L_{HasFooling}$

Dec HaltonInput



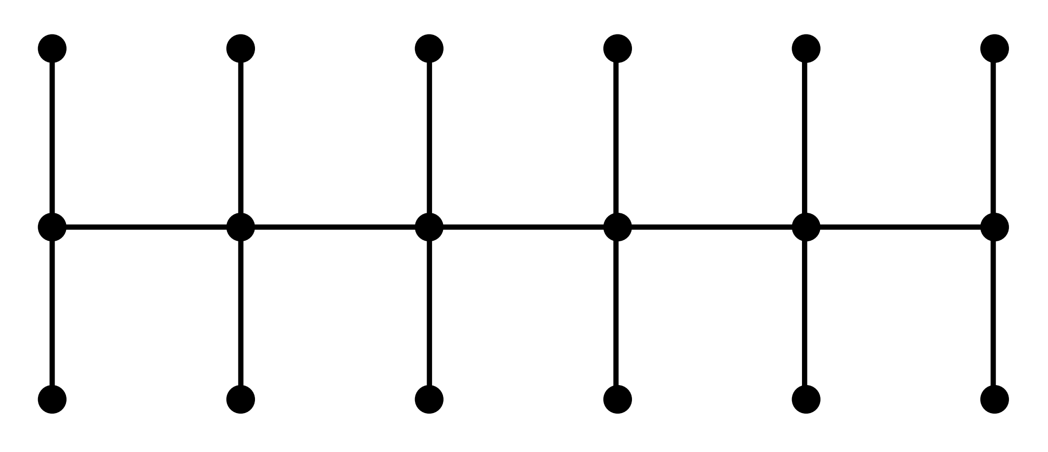
# NP-Complete practice problems

A centipede is an undirected graph formed by a path of length $k$ with two edges (legs) attached to each node on the path as shown in the below figure. Hence, the centipede graph has $3k$ vertices. The **CENTIPEDE** problem is the following: given an undirected graph $G = (V, E)$ and an integer $k$, does G contain a centipede of $3k$ distinct vertices as a subgraph? Prove that **CENTIPEDE** is **NP-Complete**.



# Practice: NP-Complete Reduction

What do we need to do to prove Centipede is NP-Complete?

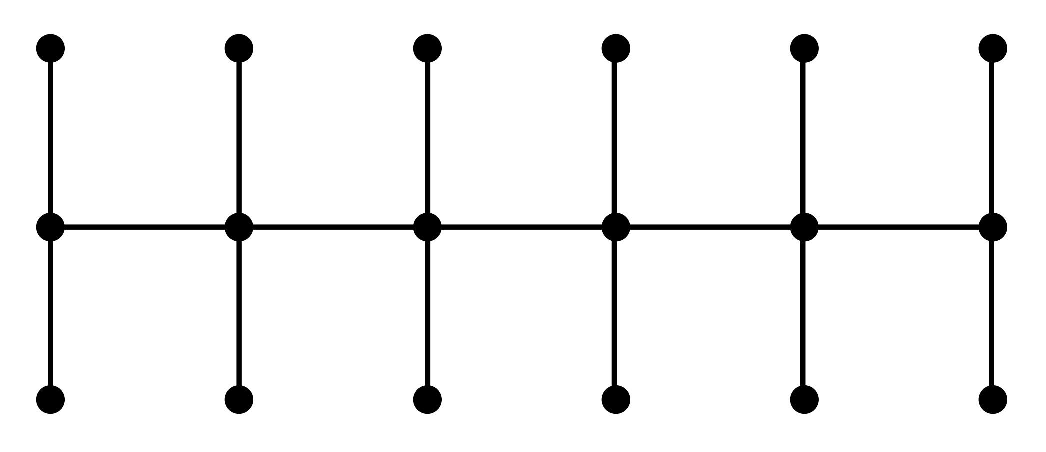


Prove CENTIPEDE $\in$ NP

CENTIPEDE $\in$ NP-hard

# Practice: NP-Complete Reduction

Prove Centipede is in **NP**:



Certificate: 3 vertex arrays

backbone[1 ⋯ k]

leg1[1 ⋯ k] leg2[1 ⋯ k]

Certifier: check for legs

for i = 1 ⋯ k

(leg[i], back[i]) ∈ G

leg2[i] . . . ∈ G

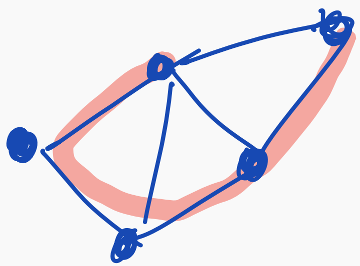
(back[i], back[i+1]) ∈ G

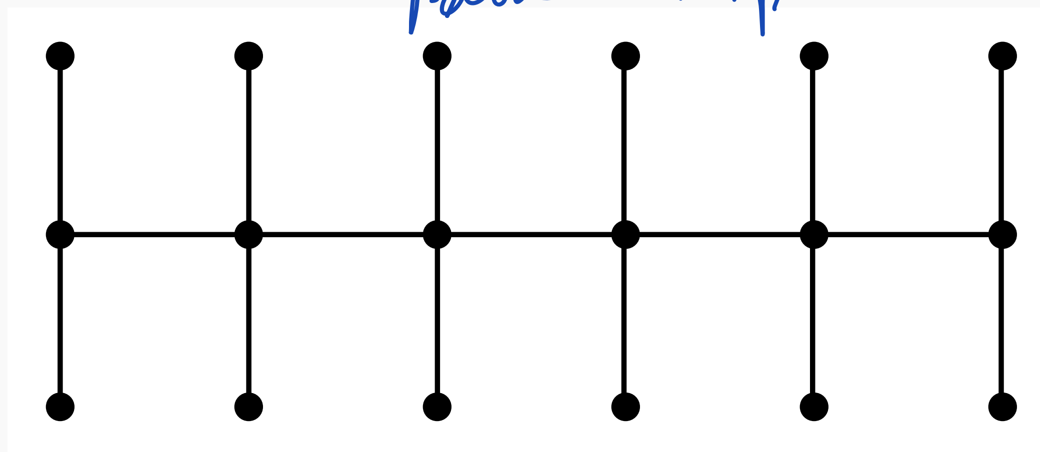
Make sure all vertices are distinct

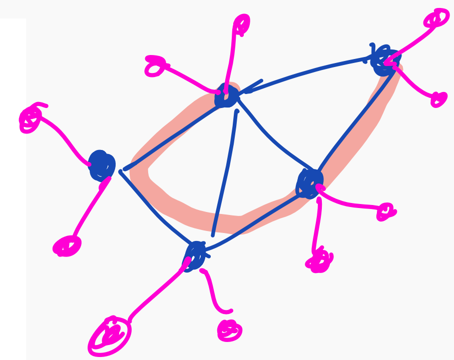
CENTI $\leq_p$ IndSet      HAM$_{PATH}$ $\leq_p$ $\boxed{A_{TM}}$

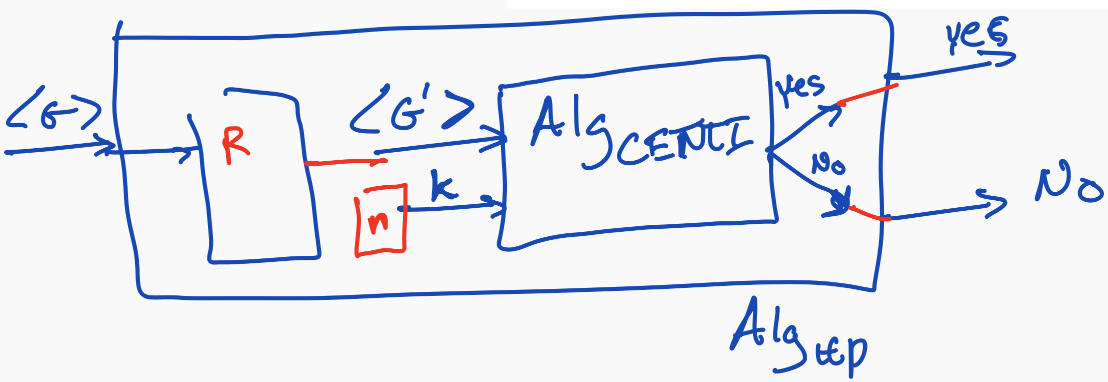
Prove Centipede is in **NP-hard**:

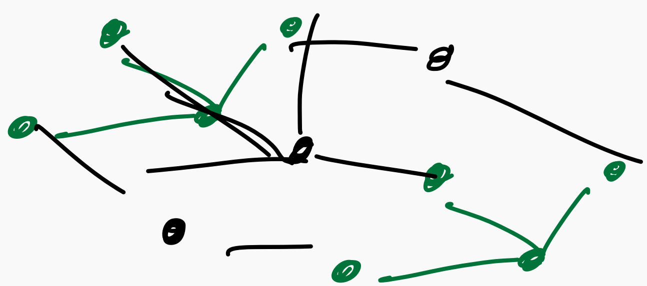
HAM Path $\leq_p$ CENTIPEDE









Reduction

Take $\langle G\rangle = (V, E)$

Copy $\langle G\rangle = \langle G'\rangle$

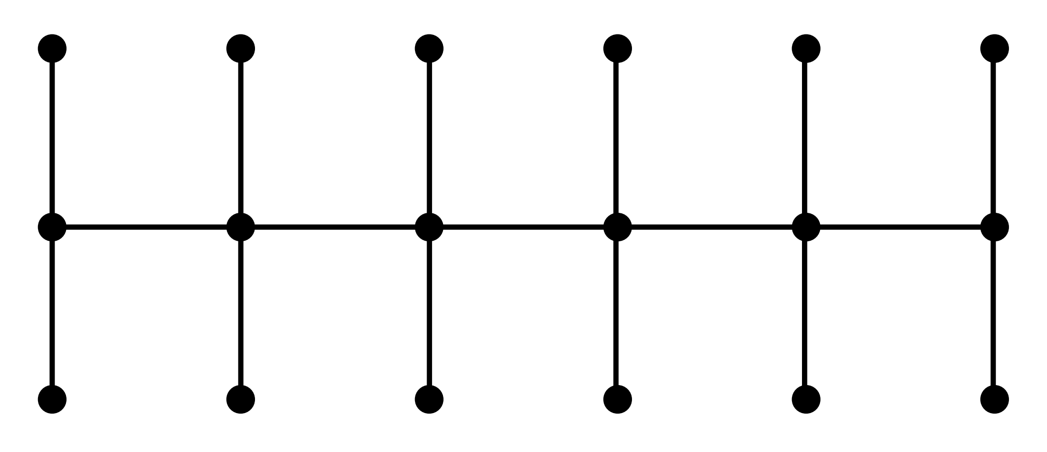
Add $2n$ vertices to $\langle G'\rangle$

for $\exists v$ in $V$

add edge between $(v', v_1')$ $(v', v_2')$

# Practice: NP-Complete Reduction

Prove Centipede is in **NP-hard**:



**Hamiltonian Path**: Given a graph G (either directed or undirected), is there a path that visits every vertex exactly once

$HC \leq_P Centipede$

# Practice: NP-Complete Reduction I

A quasi-satisfying assignment for a 3CNF boolean formula $\Phi$ is an assignment of truth values to the variables such that at most one clause in $\Phi$ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

# Practice: NP-Complete Reduction I

A quasi-satisfying assignment for a 3CNF boolean formula $\Phi$ is an assignment of truth values to the variables such that at most one clause in $\Phi$ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

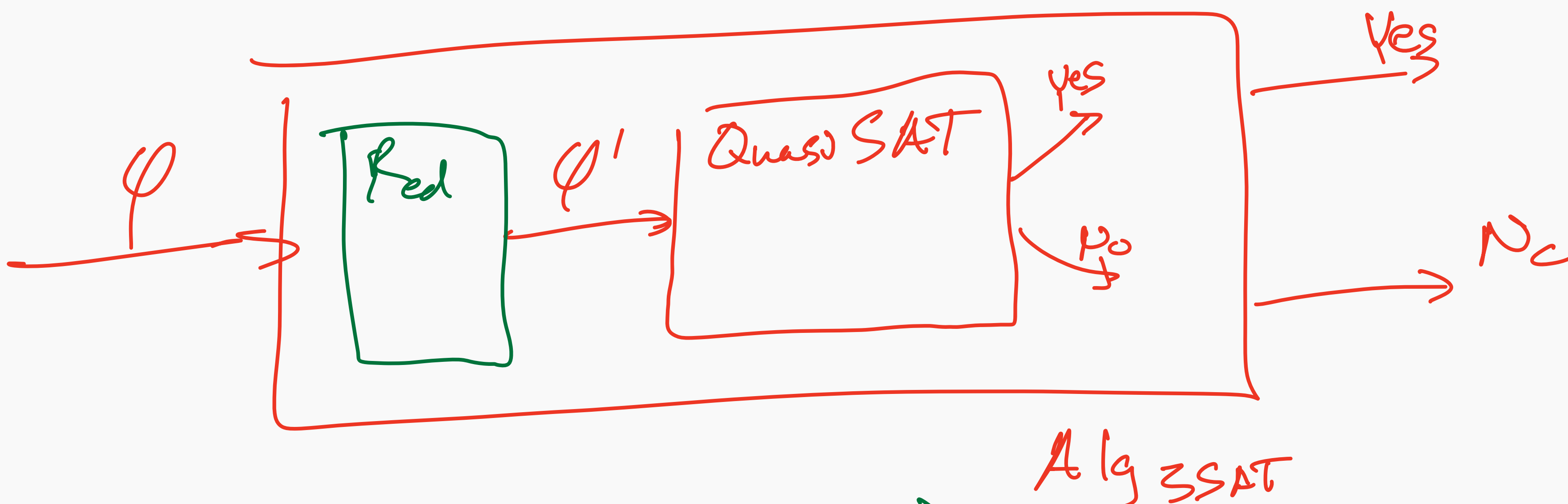
**Prove quasiSAT is in NP**

Certificate: truth assignment

Certifier:

A quasi-satisfying assignment for a 3CNF boolean formula $\Phi$ is an assignment of truth values to the variables such that at most one clause in $\Phi$ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

**Prove quasiSAT is NP-hard** $3SAT \leq Quasi\ SAT$




$$\varphi' = \varphi \wedge (x \vee x \vee x) \wedge (\bar{x} \wedge \bar{x} \wedge \bar{x})$$

$= (x \vee y \vee z) \wedge (\bar{x} \vee y \vee z) \wedge (\ldots$

# Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard

# Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard

3SAT: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment.

# Good luck on the exam