

Pre-lecture brain teaser

Find the regular expression for the language containing all binary strings that do not contain the subsequence 111000

ECE-374-B: Lecture 3 - NFAs

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University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

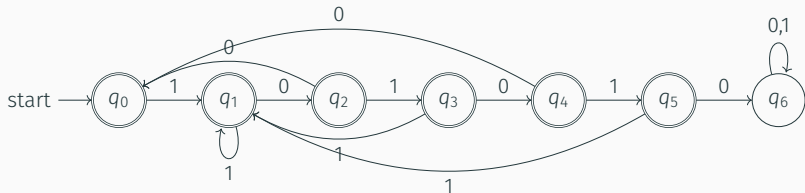
Find the regular expression for the language containing all binary strings that **do not** contain the subsequence 111000

Pre-lecture brain teaser II

Find the regular expression for the language containing all binary strings that **do not** contain the substring 101010

Pre-lecture brain teaser II

Find the regular expression for the language containing all binary strings that **do not** contain the substring **101010**

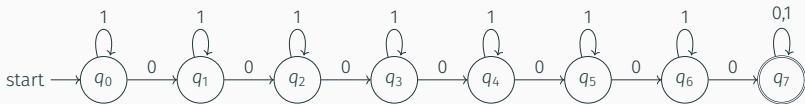


Pre-lecture brain teaser III

Find the regular expression for the language contains all binary strings whose $\#_0(w) \% 7 = 0$ (number of 0's divisible by 7).

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Pre-lecture brain teaser III

Show that the following string(w) is a member of the language that:

- does not contain the subsequence 111000 or
- does not contain the substring 101010 or
- or has a number of 0's divisible by 7

Pre-lecture brain teaser III

Show that the following string(w) is a member of the language that:

- does not contain the subsequence 111000 or
- does not contain the substring 101010 or
- or has a number of 0's divisible by 7

$w =$ 1001110110111001
1000010111110010
0101010011001111
1001001011111100

You have 30 seconds.

Pre-lecture brain teaser III

Show that the following string(w) is a member of the language that:

- does not contain the subsequence **111000** or
- does not contain the substring **101010** or
- or has a number of 0's divisible by 7

$w =$ 1001110110111001
1000010111110010
0101010011001111
1001001011111100

You have 30 seconds. Pray, choose a strategy and hope you get **lucky**.

Tangential Thought

Does luck allow us to solve unsolvable problems?

Tangential Thought

Does luck allow us to solve unsolvable problems? New example: Consider two machines: M_1 and M_2

- M_1 is a classic deterministic machine.
- M_2 is a “lucky” machine that will always make the right choice.

Lucky machine programs

Problem: Find shortest path from a to b

Program on M_1 (Dijkstra's algorithm):

```
Initialize for each node  $v$ ,  $\text{Dist}(s, v) = d'(s, v) = \infty$   
Initialize  $X = \emptyset$ ,  $d'(s, s) = 0$   
for  $i = 1$  to  $|V|$  do  
    Let  $v$  be node realizing  $d'(s, v) = \min_{u \in V - X} d'(s, u)$   
     $\text{Dist}(s, v) = d'(s, v)$   
     $X = X \cup \{v\}$   
    Update  $d'(s, u)$  for each  $u$  in  $V - X$  as follows:  
         $d'(s, u) = \min(d'(s, u), \text{Dist}(s, v) + \ell(v, u))$ 
```

Lucky machine programs

Problem: Find shortest path from a to b

Program on M_2 (Blind luck):

```
Initialize  $path = []$   
 $path += a$   
While( $not\ at\ b$ )  
    take an outgoing edge  $(u, v)$  from current node  $u$  to  $v$   
     $current = v$   
     $path += v$   
return  $path$ 
```

Tangential Thought

Does luck allow us to solve unsolvable problems?

Consider two machines: M_1 and M_2

- M_1 is a classic deterministic machine.
- M_2 is a “lucky” machine that will always make the right choice.

Question:

Tangential Thought

Does luck allow us to solve unsolvable problems?

Consider two machines: M_1 and M_2

- M_1 is a classic deterministic machine.
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Question: Are there problems which M_2 can solve that M_1 cannot.

Tangential Thought

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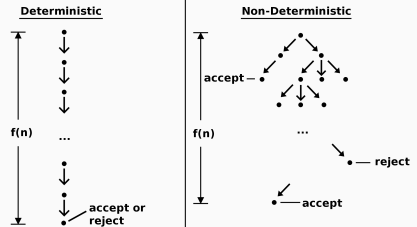
The notion was first posed by **Robert W. Floyd** in 1967.

Non-determinism in computing

In computer science, a nondeterministic machine is a theoretical device that can have more than one output for the same input.

A machine that is capable of taking multiple states concurrently. Whenever it reaches a choice, it takes both paths.

If there is a path for the string to be accepted by the machine, then the string is part of the language.



Non-determinism in media

Placeholder slide for youtube.

Why non-determinism?

- Non-determinism adds power to the model; richer programming language and hence (much) easier to “design” programs
- Fundamental in **theory** to prove many theorems
- Very important in **practice** directly and indirectly
- Many deep connections to various fields in Computer Science and Mathematics

Many interpretations of non-determinism. Hard to understand at the outset. Get used to it and then you will appreciate it slowly.

Non-deterministic finite automata (NFA) Introduction

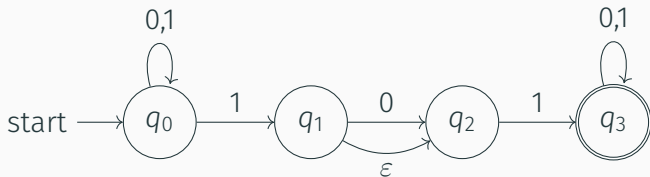
Non-deterministic Finite State Automata by example

When you come to a fork in the road, take it.

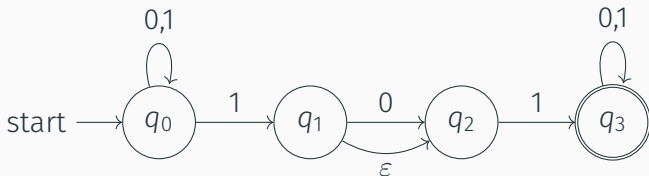
Non-deterministic Finite State Automata by example

When you come to a fork in the road, take it.

Today we'll talk about automata whose logic **is not** deterministic.

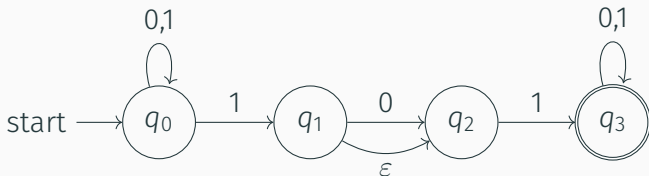


NFA acceptance: Informal



Informal definition: An NFA N **accepts a string** w iff some accepting state is reached by N from the start state on input w .

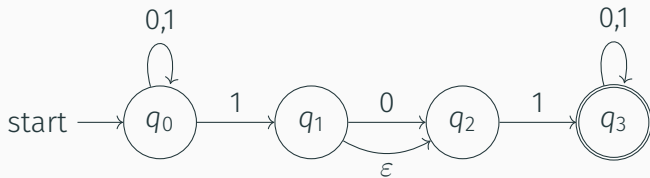
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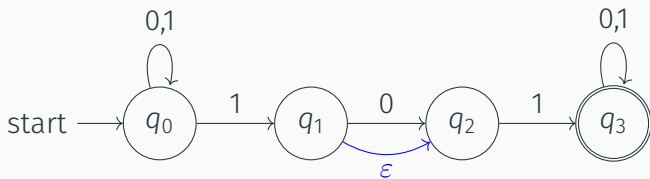
The **language accepted** (or recognized) by a NFA N is denoted by $L(N)$ and defined as: $L(N) = \{w \mid N \text{ accepts } w\}$.

NFA acceptance: Example



- Is **010110** accepted?

NFA acceptance: Wait! what about the ϵ ?!

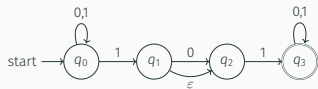


NFA acceptance: Example

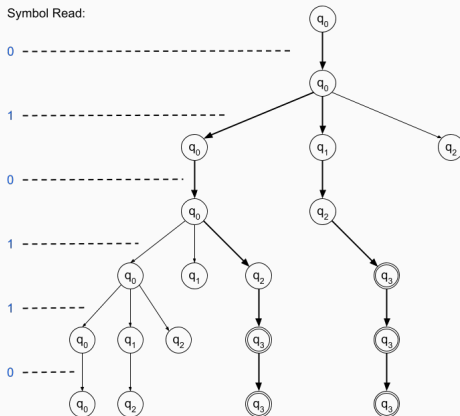


Is **010110** accepted?

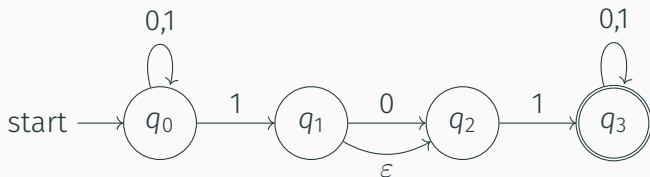
NFA acceptance: Example



Is **010110** accepted?

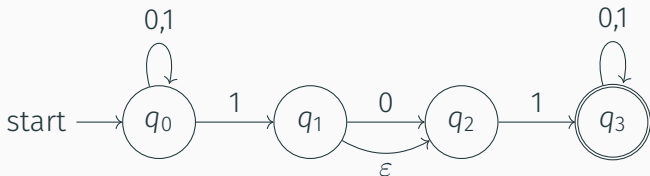


NFA acceptance: Example



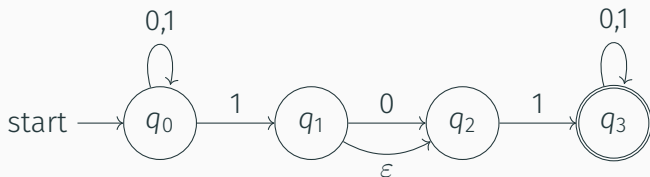
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NFA acceptance: Example



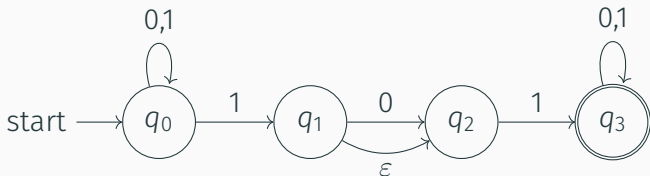
- Is **010110** accepted?
- Is **010** accepted?

NFA acceptance: Example



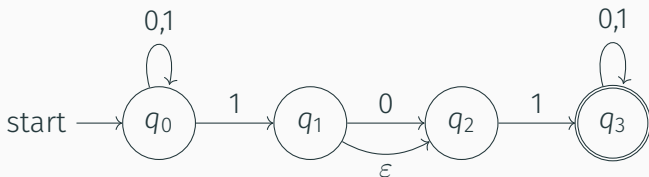
- Is **010110** accepted?
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- Is **101** accepted?

NFA acceptance: Example



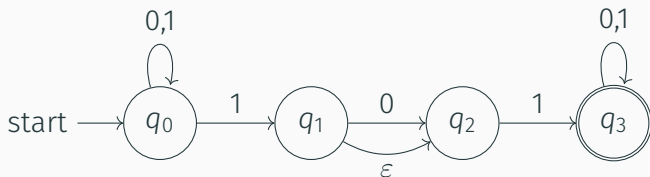
- Is **010110** accepted?
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- Is **101** accepted?
- Is **10011** accepted?

NFA acceptance: Example



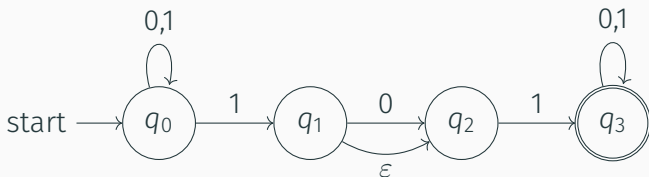
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- Is **10011** accepted?
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NFA acceptance: Example



- Is **010110** accepted?
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NFA acceptance: Example



- Is **010110** accepted?
- Is **010** accepted?
- Is **101** accepted?
- Is **10011** accepted?
- What is the language accepted by N ?

Comment: Unlike DFAs, it is easier in NFAs to show that a string is accepted than to show that a string is **not** accepted.

Formal definition of NFA

Formal Tuple Notation

Definition

A **non-deterministic finite automata (NFA)** $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

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- $\delta : Q \times \Sigma \cup \{\varepsilon\} \rightarrow \mathcal{P}(Q)$ is the **transition function** (here $\mathcal{P}(Q)$ is the power set of Q),

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$\mathcal{P}(Q)$?

Reminder: Power set

Q : a set. Power set of Q is: $\mathcal{P}(Q) = 2^Q = \{X \mid X \subseteq Q\}$ is set of all subsets of Q .

Example

$$Q = \{1, 2, 3, 4\}$$

$$\mathcal{P}(Q) = \left\{ \begin{array}{c} \{1, 2, 3, 4\}, \\ \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \\ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ \{1\}, \{2\}, \{3\}, \{4\}, \\ \{\} \end{array} \right\}$$

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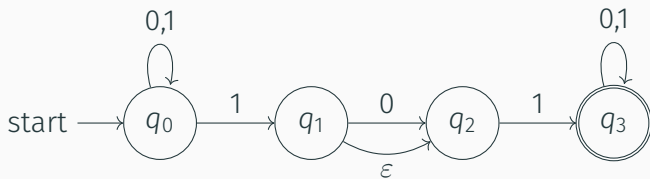
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- $s \in Q$ is the **start state**,
- $A \subseteq Q$ is the set of **accepting/final** states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\varepsilon\}$ is a subset of Q — a set of states.

Example



• $Q =$

• $\Sigma =$

• $\delta =$

• $S =$

• $A =$

Extending the transition function to strings

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- NFA $N = (Q, \Sigma, \delta, s, A)$

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Extending the transition function to strings

- NFA $N = (Q, \Sigma, \delta, s, A)$
- $\delta(q, a)$: **set of states** that N can go to from q on reading $a \in \Sigma \cup \{\epsilon\}$.
- Want transition function $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$

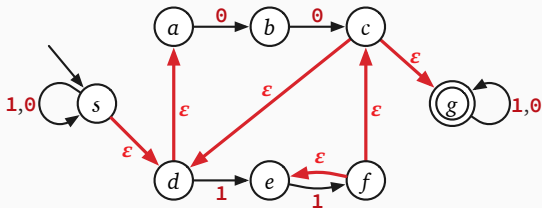
Extending the transition function to strings

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- Want transition function $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$
- $\delta^*(q, w)$: set of states reachable on input w starting in state q .

Extending the transition function to strings

Definition

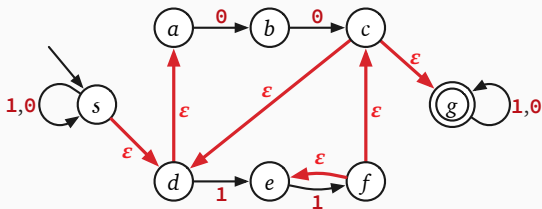
For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that q can reach using only ϵ -transitions.



Extending the transition function to strings

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For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that q can reach using only ϵ -transitions.



Definition

For $X \subseteq Q$: $\epsilon\text{reach}(X) = \bigcup_{x \in X} \epsilon\text{reach}(x)$.

Extending the transition function to strings

$\epsilon\text{reach}(q)$: set of all states that q can reach using only ϵ -transitions.

Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$:

- if $w = \epsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$

Extending the transition function to strings

$\epsilon\text{reach}(q)$: set of all states that q can reach using only ϵ -transitions.

Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$:

- if $w = \epsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$:

$$\delta^*(q, a) = \epsilon\text{reach} \left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a) \right)$$

Extending the transition function to strings

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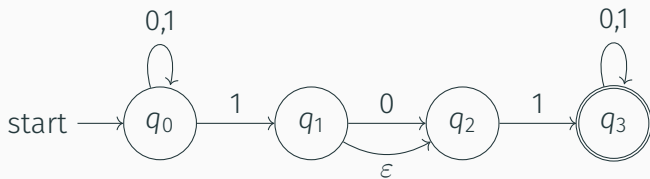
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- if $w = ax$:

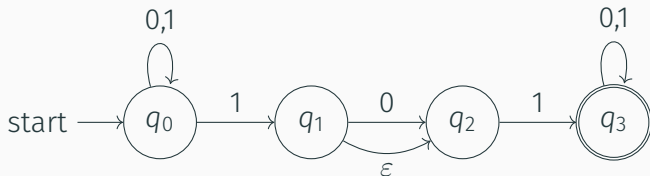
$$\delta^*(q, w) = \epsilon\text{reach} \left(\bigcup_{p \in \epsilon\text{reach}(q)} \left(\bigcup_{r \in \delta^*(p, a)} \delta^*(r, x) \right) \right)$$

Example of extended transition function



Find $\delta^*(q_0, 11)$:

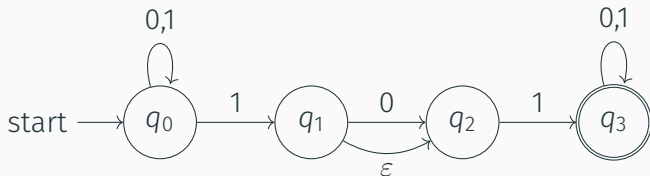
Example of extended transition function



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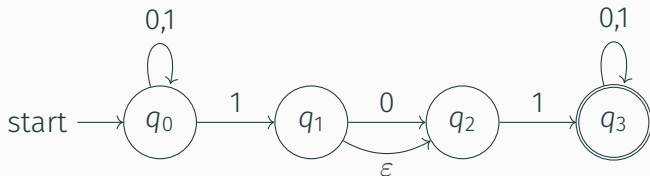
Example of extended transition function



We know $w = 11 = ax$ so $a = 1$ and $x = 1$

$$\delta^*(q_0, 11) = \epsilon\text{reach} \left(\bigcup_{p \in \epsilon\text{reach}(q_0)} \left(\bigcup_{r \in \delta^*(p, 1)} \delta^*(r, 1) \right) \right)$$

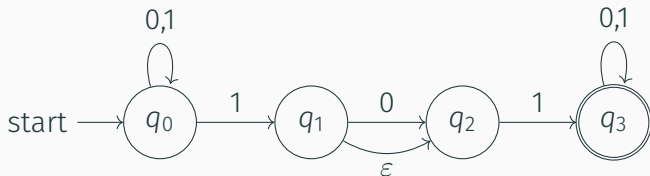
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$$\epsilon\text{reach}(q_0) = \{q_0\}$$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{p \in \{q_0\}} \left(\bigcup_{r \in \delta^*(p, 1)} \delta^*(r, 1)\right)\right)$$

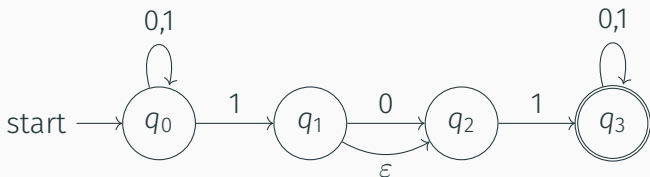
Example of extended transition function



Simplify:

$$\delta^*(q_0, 11) = \epsilon\text{reach} \left(\bigcup_{r \in \delta^*({q_0}, 1)} \delta^*(r, 1) \right)$$

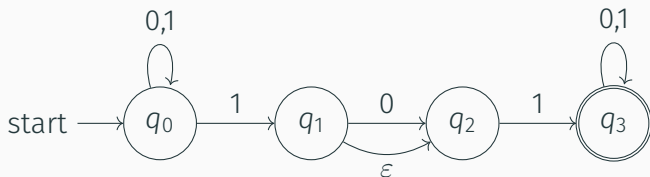
Example of extended transition function



Need $\delta^*(q_0, 1) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a)\right) = \epsilon\text{reach}(\delta(q_0, 1))$:
 $= \epsilon\text{reach}(\{q_0, q_1\}) = \{q_0, q_1, q_2\}$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{r \in \delta^*(\{q_0\}, 1)} \delta^*(r, 1)\right)$$

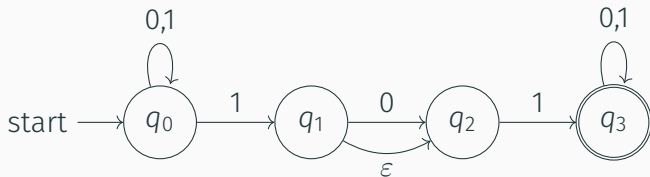
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Example of extended transition function



Simplify

$$\delta^*(q_0, 11) = \epsilon\text{reach}(\delta^*(q_0, 1) \cup \delta^*(q_1, 1) \cup \delta^*(q_2, 1))$$

Transition for strings: $w = ax$

$$\delta^*(q, w) = \epsilon\text{reach} \left(\bigcup_{p \in \epsilon\text{reach}(q)} \left(\bigcup_{r \in \delta^*(p, a)} \delta^*(r, x) \right) \right)$$

- $R = \epsilon\text{reach}(q) \implies$

$$\delta^*(q, w) = \epsilon\text{reach} \left(\bigcup_{p \in R} \bigcup_{r \in \delta^*(p, a)} \delta^*(r, x) \right)$$

- $N = \bigcup_{p \in R} \delta^*(p, a)$: All the states reachable from q with the letter a .

- $\delta^*(q, w) = \epsilon\text{reach} \left(\bigcup_{r \in N} \delta^*(r, x) \right)$

Formal definition of language accepted by **N**

Definition

A string w is accepted by NFA N if $\delta_N^*(s, w) \cap A \neq \emptyset$.

Definition

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

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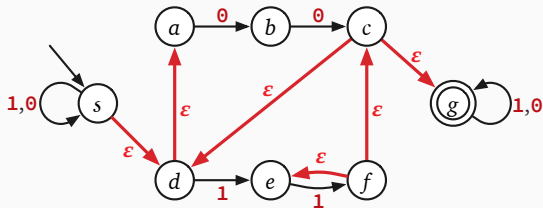
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$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

Important: Formal definition of the language of NFA above uses δ^* and not δ . As such, one does not need to include ϵ -transitions closure when specifying δ , since δ^* takes care of that.

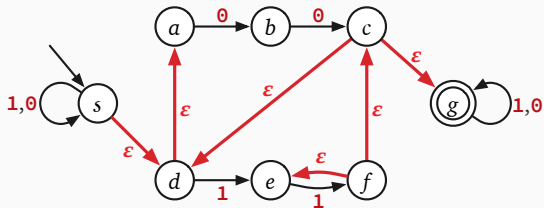
Example



What is:

$$\cdot \delta^*(s, \epsilon) =$$

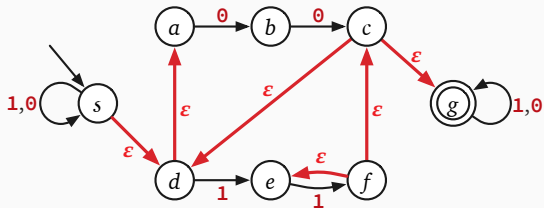
Example



What is:

- $\delta^*(s, \epsilon) =$
- $\delta^*(s, 0) =$

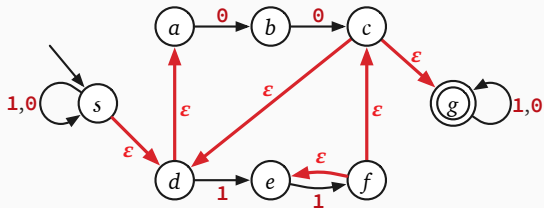
Example



What is:

- $\delta^*(s, \epsilon) =$
- $\delta^*(s, 0) =$
- $\delta^*(b, 0) =$

Example



What is:

- $\delta^*(s, \epsilon) =$
- $\delta^*(s, 0) =$
- $\delta^*(b, 0) =$
- $\delta^*(b, 00) =$

Constructing generalized NFAs

DFAs and NFAs

- Every DFA is a NFA so NFAs are at least as powerful as DFAs.
- NFAs prove ability to “guess and verify” which simplifies design and reduces number of states
- Easy proofs of some closure properties

Example

$L = \{\text{bitstrings that have a } 1 \text{ three positions from the end}\}$

A simple transformation

Theorem

For every NFA N there is another NFA N' such that $L(N) = L(N')$ and such that N' has the following two properties:

- N' has single final state f that has no outgoing transitions*
- The start state s of N is different from f*

A simple transformation

Theorem

For every NFA N there is another NFA N' such that $L(N) = L(N')$ and such that N' has the following two properties:

- N' has single final state f that has no outgoing transitions*
- The start state s of N is different from f*

Why couldn't we say this for DFA's?

A simple transformation

Hint: Consider the $L = 0^* + 1^*$.

Closure Properties of NFAs

Closure properties of NFAs

Are the class of languages accepted by NFAs closed under the following operations?

- union
- intersection
- concatenation
- Kleene star
- complement

Closure under union

Theorem

For any two NFAs N_1 and N_2 there is a NFA N such that

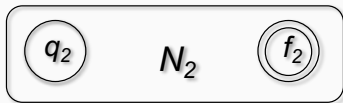
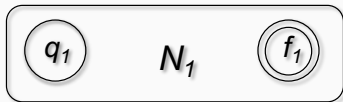
$$L(N) = L(N_1) \cup L(N_2).$$

Closure under union

Theorem

For any two NFAs N_1 and N_2 there is a NFA N such that

$$L(N) = L(N_1) \cup L(N_2).$$



Closure under concatenation

Theorem

For any two NFAs N_1 and N_2 there is a NFA N such that

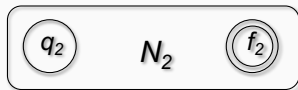
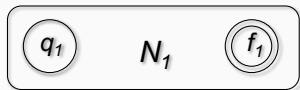
$$L(N) = L(N_1) \cdot L(N_2).$$

Closure under concatenation

Theorem

For any two NFAs N_1 and N_2 there is a NFA N such that

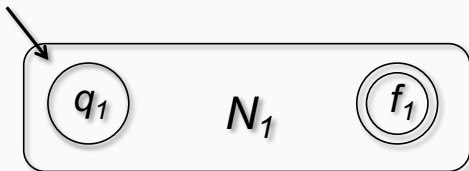
$$L(N) = L(N_1) \cdot L(N_2).$$



Closure under Kleene star

Theorem

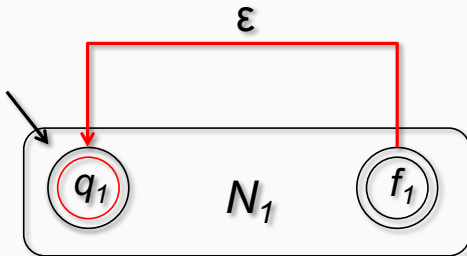
For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.



Closure under Kleene star

Theorem

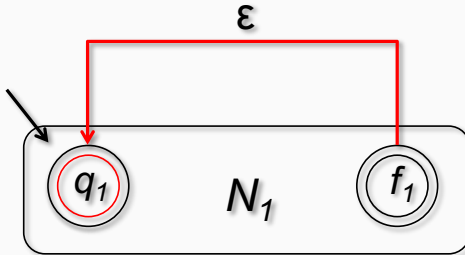
For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.



Closure under Kleene star

Theorem

For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.

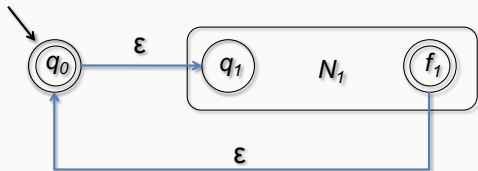


Does not work! Why?

Closure under Kleene star

Theorem

For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.



Transformations

All these examples are examples of language *transformations*.

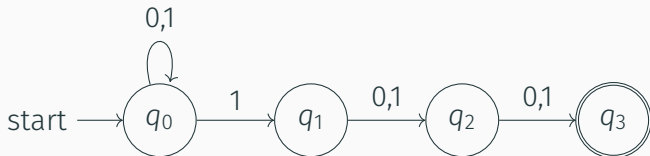
A language transformation is one where you take one class or languages, perform some operation and get a new language **that belongs to that same class (closure)**.

Tomorrow's lab will go over more examples of language transformations.

Last thought

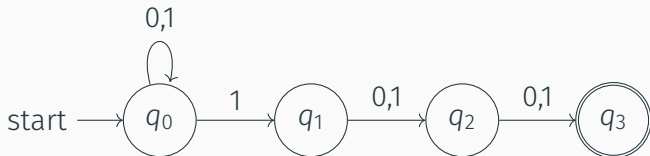
Equivalence

Do all NFAs have a corresponding DFA?



Equivalence

Do all NFAs have a corresponding DFA?



Yes but it likely won't be pretty.

