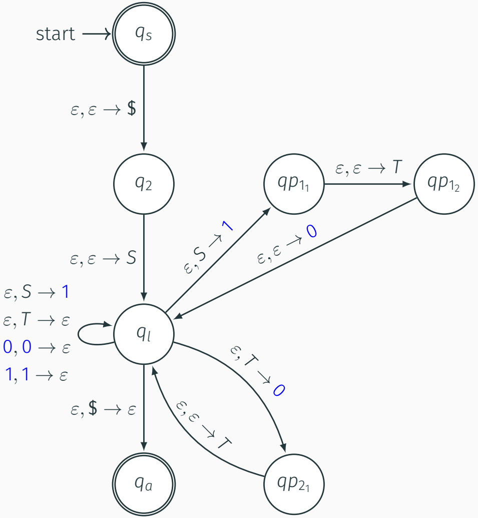
What is the context-free grammar of the following push-down automata:



ECE-374-B: Lecture 7 - Context-sensitive and decidable languages

Instructor: Nickvash Kani

September 14, 2023

University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

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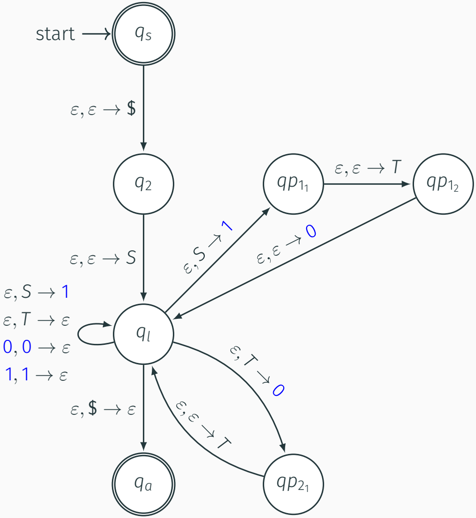
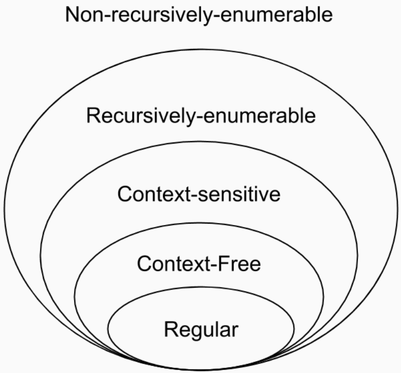


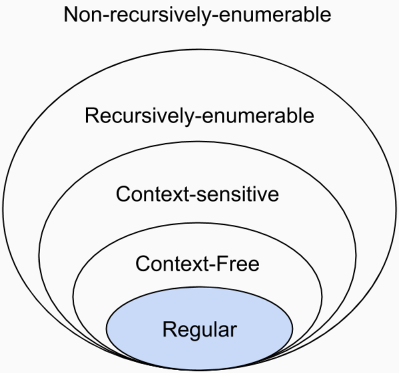
Larger world of languages!

Chomsky Hierarchy



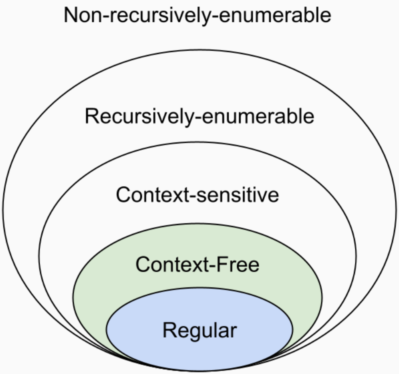
Remember our hierarchy of languages

Chomsky Hierarchy



You've mastered regular expressions.

Chomsky Hierarchy



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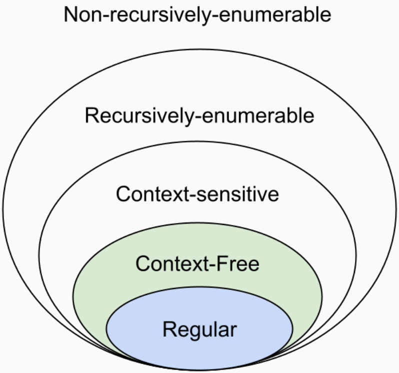
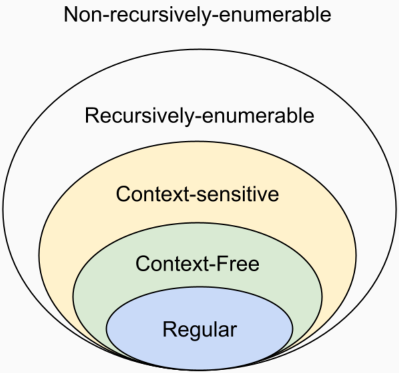


Now what about the next level up?

Chomsky Hierarchy



On to the next one.....

Context-Sensitive Languages

Example

The language $L = \{a^n b^n c^n | n \geq 1\}$ is not a context free language.

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- $V = \{S, A, B\}$
- $T = \{a, b, c\}$
- $P = \left\{ \begin{array}{c} S \rightarrow abc|aAbc, \\ Ab \rightarrow bA, \\ Ac \rightarrow Bbcc \\ bB \rightarrow Bb \\ aB \rightarrow aa|aaA \end{array} \right\}$

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$$S \rightsquigarrow aAbc \rightsquigarrow abAc \rightsquigarrow abBbcc \rightsquigarrow aBbbcc \rightsquigarrow aaAbbcc \rightsquigarrow aabAbcc \rightsquigarrow aabbAcc \rightsquigarrow aabbBbccc \rightsquigarrow aabBbbccc \rightsquigarrow aaBbbbccc \rightsquigarrow aaabbbccc$$

Context Sensitive Grammar (CSG) Definition

Definition
A CSG is a quadruple $G = (V, T, P, S)$

- V is a finite set of non-terminal symbols
- T is a finite set of terminal symbols (alphabet)
- P is a finite set of productions, each of the form
  $\alpha \rightarrow \beta$
  where α and β are strings in $(V \cup T)^*$.
- $S \in V$ is a start symbol

$$G = \Big(\text{ Variables, } \quad \text{Terminals, } \quad \text{Productions, } \quad \text{Start var } \Big)$$

Example formally...

$$L = \{a^n b^n c^n | n \geq 1\}$$

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$$G = \left(\{S, A, B\}, \quad \{a, b, c\}, \quad \left\{ \begin{array}{c} S \to abc|aAbc, \\ Ab \to bA, \\ Ac \to Bbcc \\ bB \to Bb \\ aB \to aa|aaA \end{array} \right\} \quad S \right)$$

Other examples of context-sensitive languages

$$L_{Cross} = \{a^m b^n c^m d^n | m, n \geq 1\} \tag{1}$$

Turing Machines

“Most General” computer?

- DFAs are simple model of computation.
- Accept only the regular languages.
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages: $\{L \mid L \subseteq \{0,1\}^*\}$ is ~~countably infinite~~ / uncountably infinite

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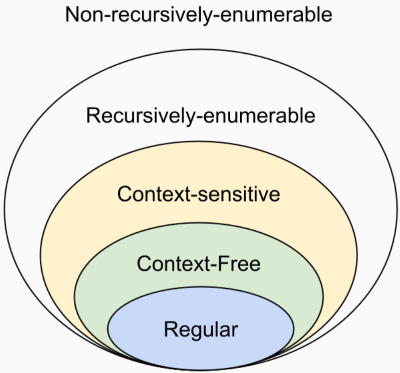
- Set of all programs: $\{P \mid P$ is a finite length computer program$\}$: is countably infinite / ~~uncountably infinite~~.
- **Conclusion:** There are languages for which there are no programs.

Chomsky Hierarchy



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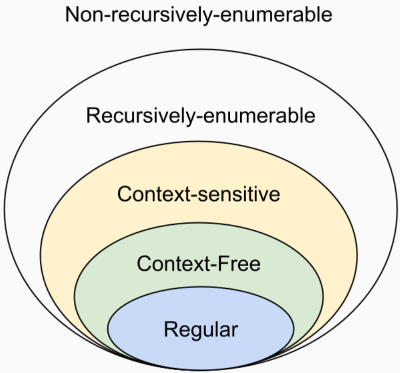
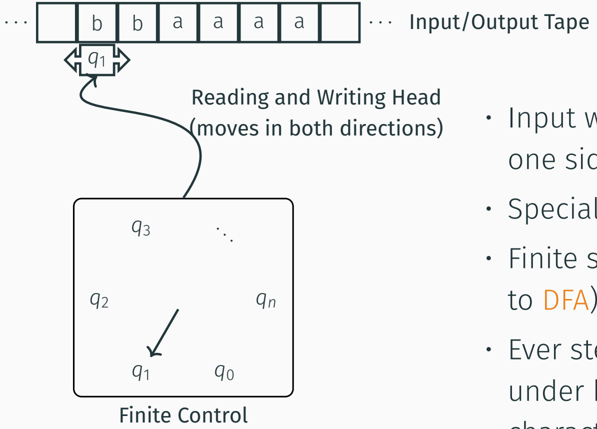


Onto our final class of languages - recursively enumerable (aka Turing-recognizable) languages.

What is a Turing machine

Turing machine



- Input written on (infinite) one sided tape.
- Special blank characters.
- Finite state control (similar to DFA).
- Ever step: Read character under head, write character out, move the head right or left (or stay).

High level goals

- Church-Turing thesis: TMs are the most general computing devices. So far no counter example.
- Every TM can be represented as a string.
- Existence of Universal Turing Machine which is the model/inspiration for stored program computing. UTM can simulate any TM
- Implications for what can be computed and what cannot be computed

Examples of Turing

- binary increment

Turing machine: Formal definition

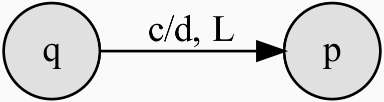
A Turing machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$

- Q: finite set of states.
- Σ: finite input alphabet.
- Γ: finite tape alphabet.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}$: Transition function.
- $q_0 \in Q$ is the initial state.
- $q_{\mathrm{acc}} \in Q$ is the accepting/final state.
- $q_{\mathrm{rej}} \in Q$ is the rejecting state.
- $\sqcup$ or $\boxed{?}$: Special blank symbol on the tape.

Turing machine: Transition function

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}$$

As such, the transition



$$\delta(q, c) = (p, d, \mathsf{L})$$

- q: current state.
- c: character under tape head.
- p: new state.
- d: character to write under tape head
- L: Move tape head left.

Can also be written as

$$c \rightarrow d, L \qquad (2)$$

Turing machine: Transition function

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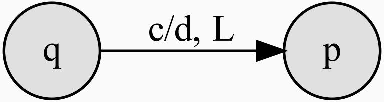

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Missing transitions lead to hell state. “Blue screen of death.” “Machine crashes.”

Some examples of Turing machines

turingmachine.io

- equal strings TM
- palindrome TM

Languages defined by a Turing machine

Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages

$$L = \{L(M) \mid M \text{ some Turing machine}\}.$$

- Recursive / decidable languages

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}.$$

Recursive vs. Recursively Enumerable

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- Fundamental questions:
  - What languages are RE?
  - Which are recursive?
  - What is the difference?
  - What makes a language decidable?

What is Decidable?

Decidable vs recursively-enumerable

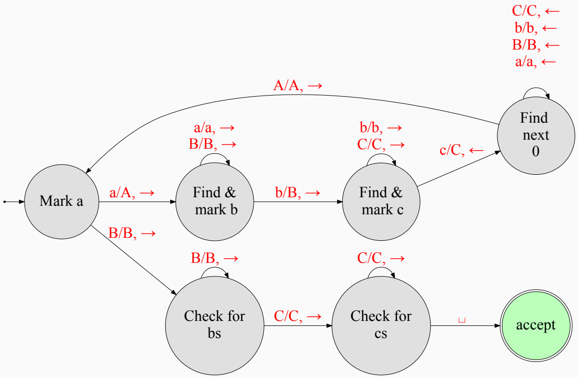
A semi-decidable problem (equivalent of recursively enumerable) could be:

- **Decidable** - equivalent of recursive (TM always accepts or rejects).
- **Undecidable** - Problem is not recursive (doesn't always halt on negative)

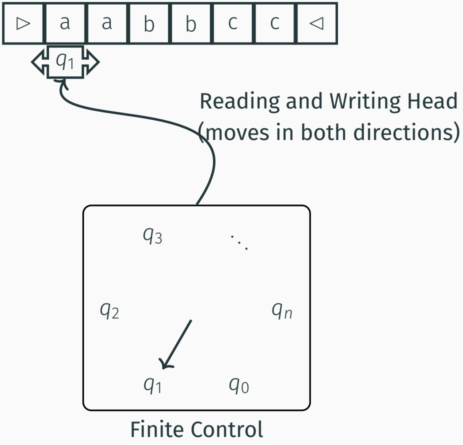
There are undecidable problem that are not semi-decidable (recursively enumerable).

Infinite Tapes? Do we need them?

Let's look at the TM that recognizes $L = \{a^n b^n c^n | n \geq 0\}$:



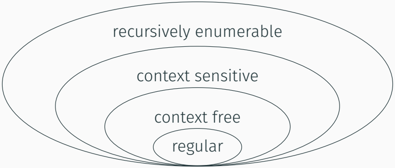
Linear Bounded Automata



- (Nondeterministic) Linear bounded automata can recognize all context sensitive languages.
- Machine can non-deterministically apply all production rule to input in reverse and see if we end up with the start token.

Well that was a journey....

Zooming out



Grammar	Languages	Production Rules	Automation	Examples
Type-0	Turing machine	$\gamma \rightarrow \alpha$ (no constraints)	Turing machine	$L = \{w \| w$ is a TM whihc halts$\}$
Type-1	Context-sensitive	$\alpha A \beta \rightarrow \alpha \gamma \beta$	Linear bounded Non-deterministic Turing machine	$L = \{a^n b^n c^n \| n > 0\}$
Type-2	Context-free	$A \rightarrow \alpha$	Non-deterministic Push-down automata	$L = \{a^n b^n \| n > 0\}$ 1
Type-3	Regular	$A \rightarrow aB$	Finite State Machine	$L = \{a^n \| n > 0\}$

Meaning of symbols:

- a = terminal
- A, B = variables
- α, β, γ = string of $\{a \cup A\}^*$
- α, β = maybe empty —– γ = never empty