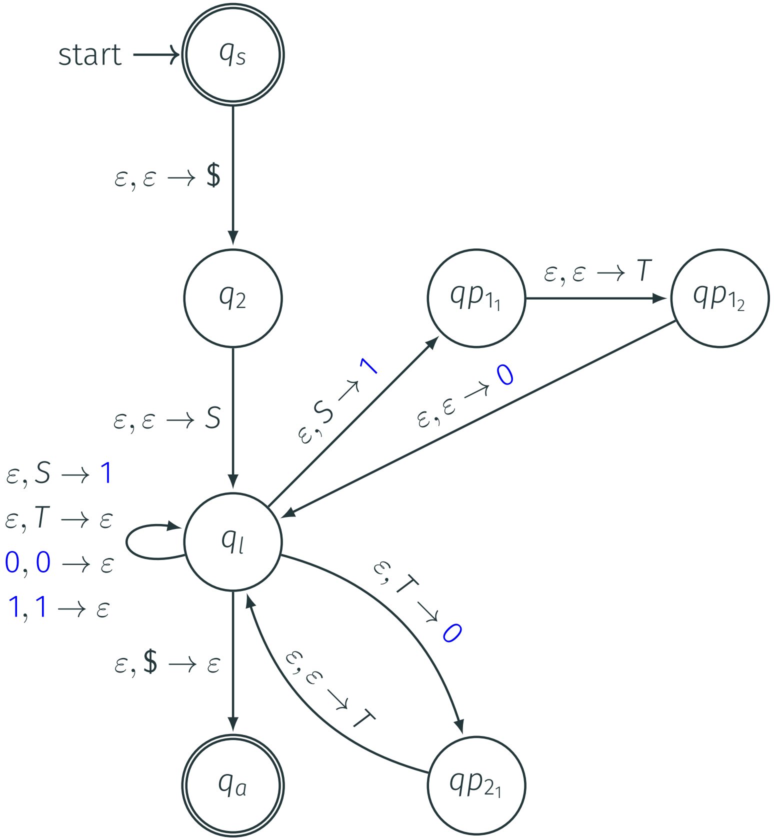
Pre-lecture brain teaser

What is the context-free grammar of the following push-down automata:



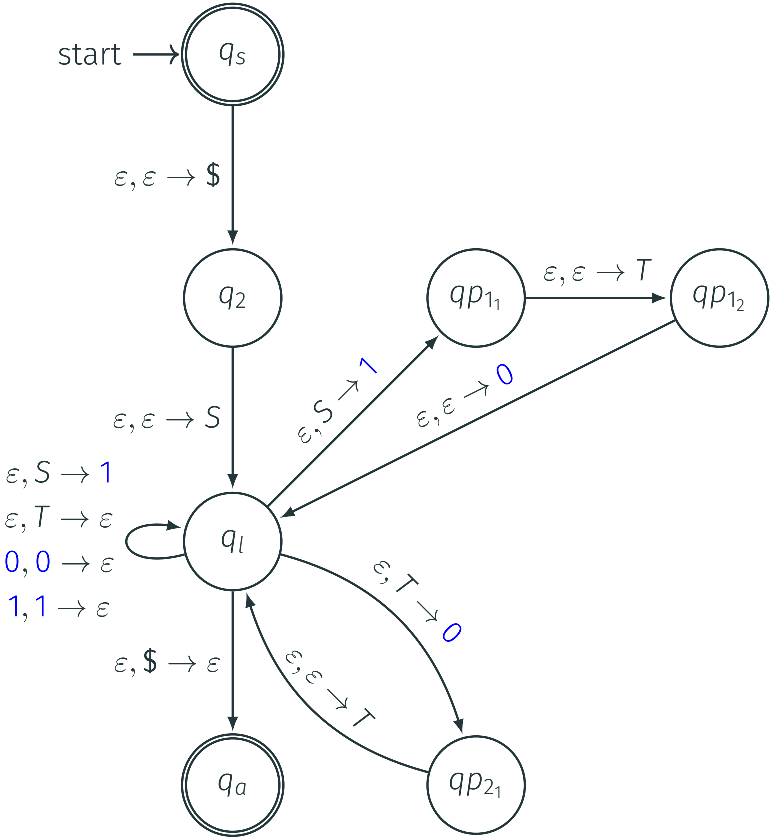
ECE-374-B: Lecture 7 - Context-sensitive and decidable languages

Instructor: Nickvash Kani

September 18, 2025

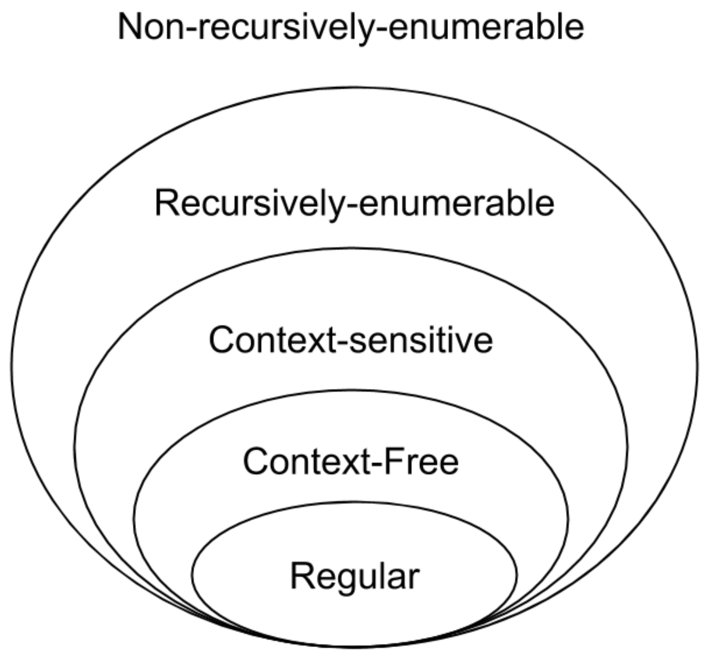
University of Illinois Urbana-Champaign

What is the context-free grammar of the following push-down automata:



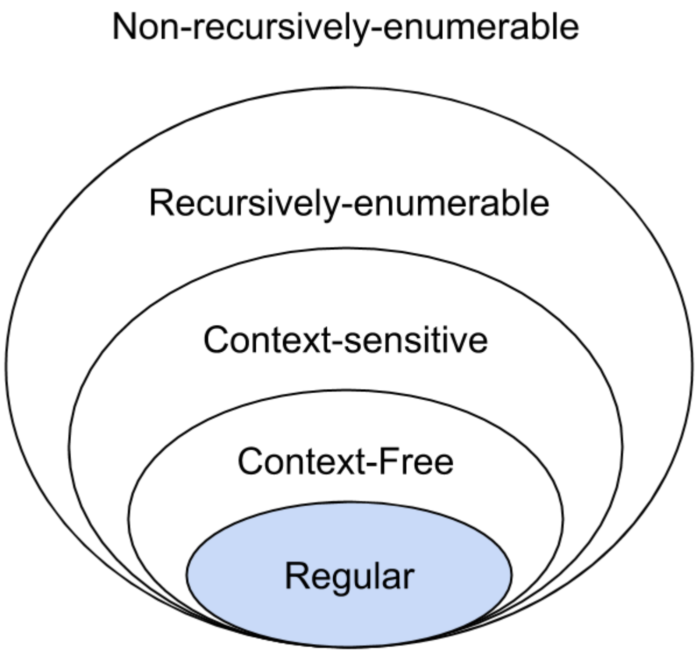
Larger world of languages!

Chomsky Hierarchy



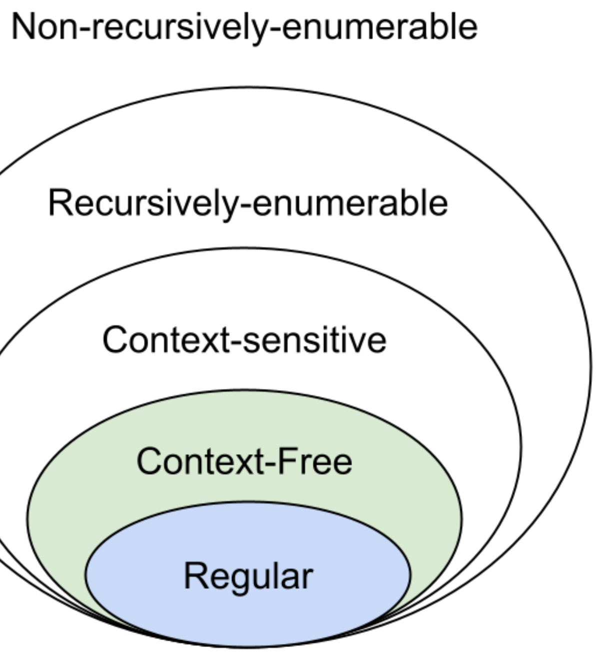
Remember our hierarchy of languages

Chomsky Hierarchy

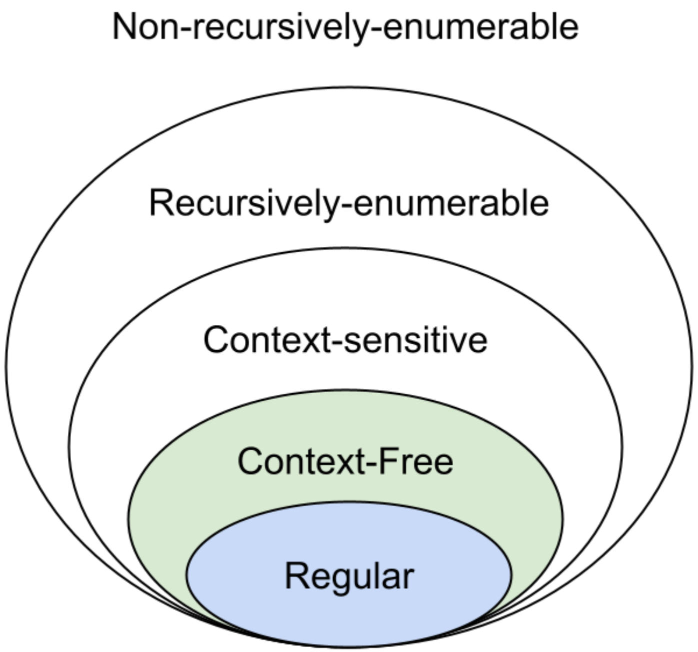


You've mastered regular expressions.

Chomsky Hierarchy

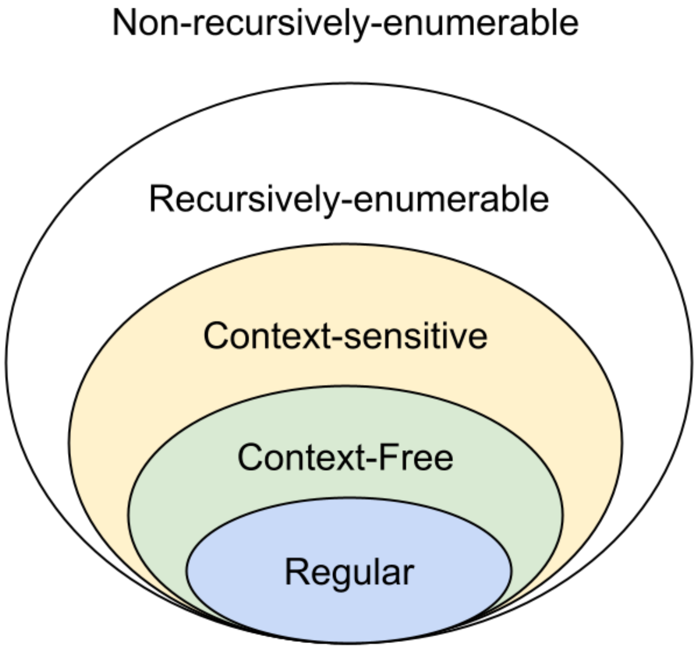


Chomsky Hierarchy



Now what about the next level up?

Chomsky Hierarchy



On to the next one.....

Context-Sensitive Languages

Example

The language $L = \{a^n b^n c^n | n \geq 1\}$ is not a context free language.

$0^n 1^n \to$ CF Languages

Reg Grammar

$V \to TV$

Context Free

$V \to TVT$

Context Sensitive

$TVT \to TVT$

Example

The language $L = \{a^n b^n c^n | n \geq 1\}$ is not a context free language. *but it is a context-sensitive language!*

- $V = \{S, A, B\}$
- $T = \{a, b, c\}$
- $P = \left\{ \begin{array}{c} S \to abc | aAbc, \\ Ab \to bA, \\ Ac \to Bbcc \\ bB \to Bb \\ aB \to aa | aaA \end{array} \right\}$

abc

aabbcc

$S \to aAbc \to abAc \to abBbcc \to aBbbcc \to aabbcc$

Example

The language $L = \{a^n b^n c^n | n \geq 1\}$ is not a context free language. *but it is a context-sensitive language!*

- $V = \{S, A, B\}$
- $T = \{a, b, c\}$
- $P = \left\{ \begin{array}{c} S \rightarrow abc|aAbc, \\ Ab \rightarrow bA, \\ Ac \rightarrow Bbcc \\ bB \rightarrow Bb \\ aB \rightarrow aa|aaA \end{array} \right\}$

$$S \rightsquigarrow aAbc \rightsquigarrow abAc \rightsquigarrow abBbcc \rightsquigarrow aBbbcc \rightsquigarrow aaAbbcc \rightsquigarrow aabAbcc$$
$$\rightsquigarrow aabbAcc \rightsquigarrow aabbBbccc \rightsquigarrow aabBbbccc \rightsquigarrow aaBbbbccc$$

Context Sensitive Grammar (CSG) Definition

Definition
A CSG is a quadruple $G = (V, T, P, S)$

- V is a finite set of non-terminal symbols variables
- T is a finite set of terminal symbols (alphabet)
- P is a finite set of productions, each of the form
  $\alpha \rightarrow \beta$
  where α and β are strings in $(V \cup T)^*$.
- $S \in V$ is a start symbol

$$G = \Big(\quad \text{Variables}, \quad \text{Terminals}, \quad \text{Productions}, \quad \text{Start var} \quad \Big)$$

Example formally...

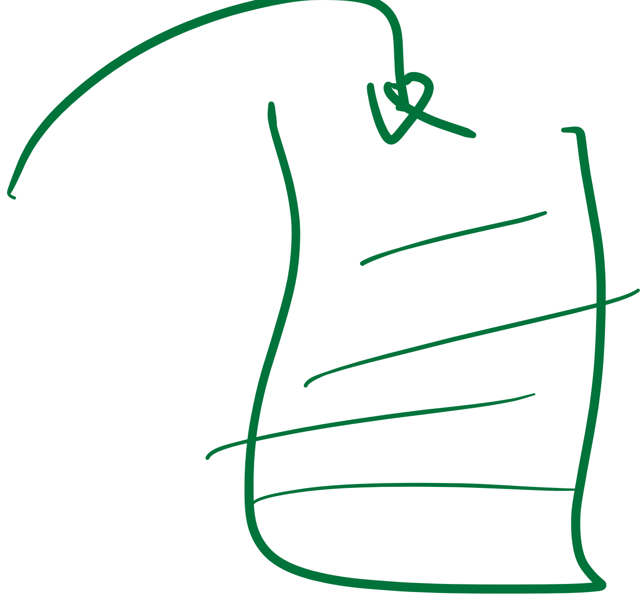
$L = \{a^n b^n c^n | n \geq 1\}$

- $V = \{S, A, B\}$
- $T = \{a, b, c\}$
- $P = \left\{ \begin{array}{c} S \rightarrow abc|aAbc, \\ Ab \rightarrow bA, \\ Ac \rightarrow Bbcc \\ bB \rightarrow Bb \\ aB \rightarrow aa|aaA \end{array} \right\}$

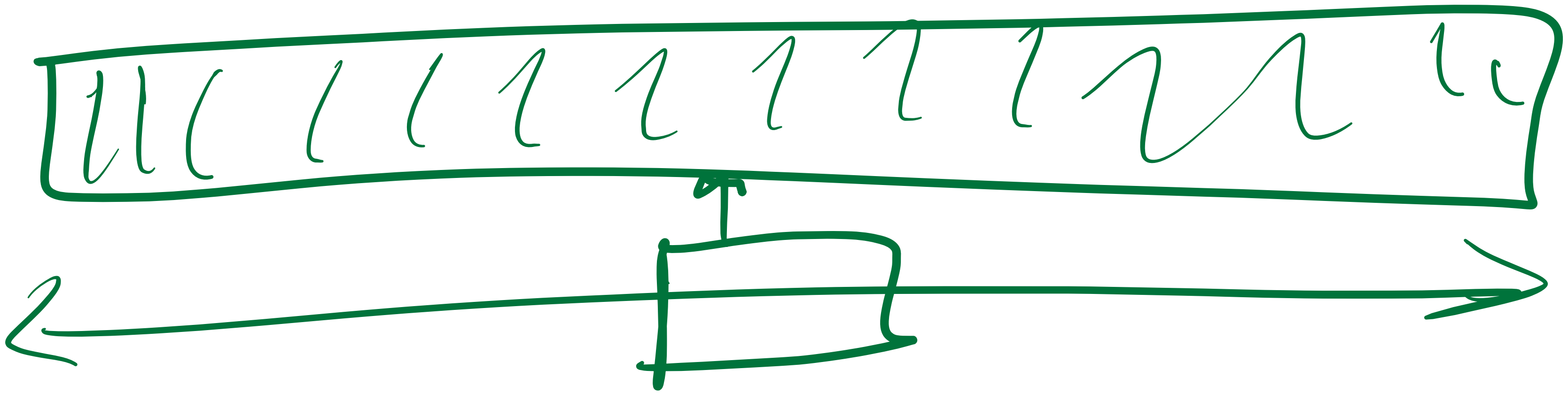
$$G = \left(\{S, A, B\}, \quad \{a, b, c\}, \quad \left\{ \begin{array}{c} S \rightarrow abc|aAbc, \\ Ab \rightarrow bA, \\ Ac \rightarrow Bbcc \\ bB \rightarrow Bb \\ \end{array} \right\} \quad S \right)$$

Other examples of context-sensitive languages

$$L_{Cross} = \{a^m b^n c^m d^n | m, n \geq 1\} \quad (1)$$



Turing Machines



“Most General” computer?

- DFAs are simple model of computation.
- Accept only the regular languages.
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages: $\{L \mid L \subseteq \{0,1\}^*\}$ is ~~countably infinite~~ / uncountably infinite

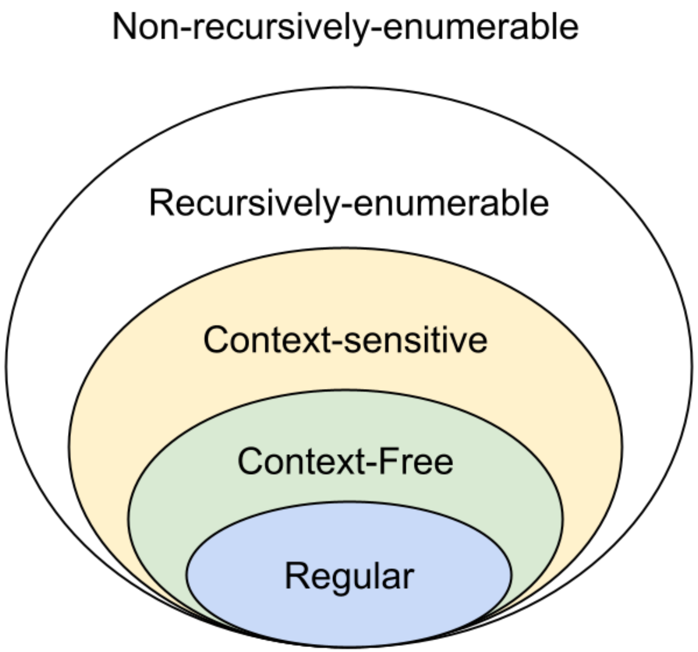
“Most General” computer?

- DFAs are simple model of computation.
- Accept only the regular languages.
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages: $\{L \mid L \subseteq \{0,1\}^*\}$ is ~~countably infinite~~ / uncountably infinite
- Set of all programs: $\{P \mid P$ is a finite length computer program$\}$: is countably infinite / ~~uncountably infinite.~~

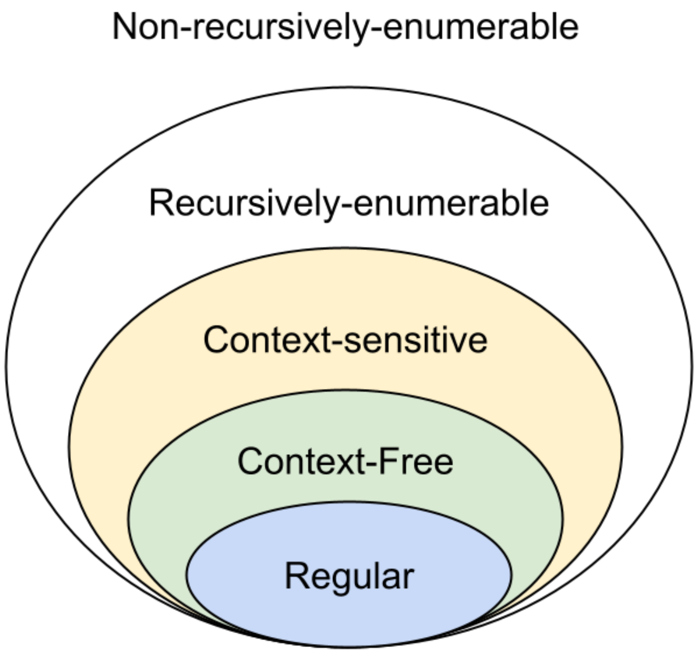
“Most General” computer?

- DFAs are simple model of computation.
- Accept only the regular languages.
- Is there a kind of computer that can accept any language, or compute any function?
- Recall counting argument. Set of all languages: $\{L \mid L \subseteq \{0,1\}^*\}$ is ~~countably infinite~~ / uncountably infinite
- Set of all programs: $\{P \mid P \text{ is a finite length computer program}\}$: is countably infinite / ~~uncountably infinite.~~
- **Conclusion:** There are languages for which there are no programs.

Chomsky Hierarchy



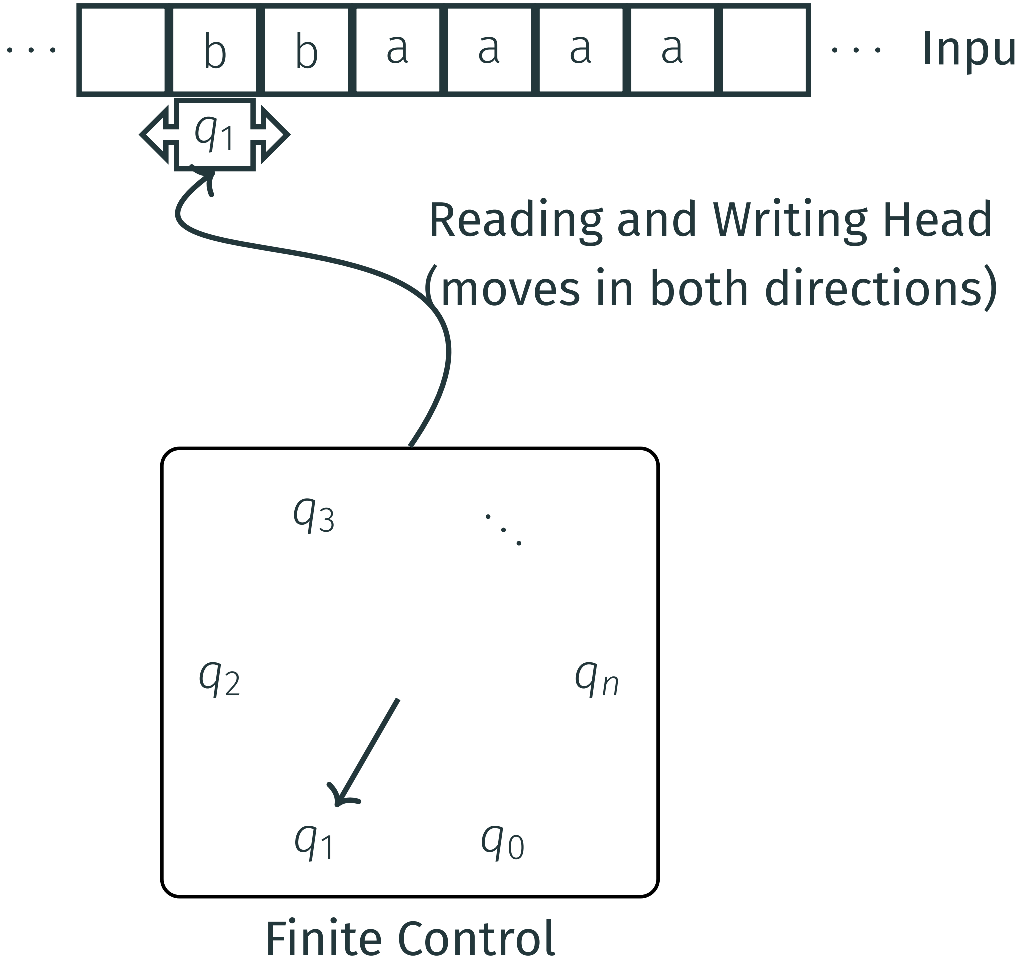
Chomsky Hierarchy



Onto our final class of languages - recursively enumerable (aka Turing-recognizable)

What is a Turing machine

Turing machine



- Input written on (infinite) one sided tape.
- Special blank characters.
- Finite state control (similar to DFA).
- Ever step: Read character under head, write character out, move the head right or left (or stay).

High level goals

- Church-Turing thesis: TMs are the most general computing devices. So far no counter example.
- Every TM can be represented as a string.
- Existence of Universal Turing Machine which is the model/inspiration for stored program computing. UTM can simulate any TM
- Implications for what can be computed and what cannot be computed

Decidability

Examples of Turing

turingmachine.io

- binary increment

A Turing machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}})$

- Q: finite set of states.
- Σ: finite input alphabet.
- Γ: finite tape alphabet.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}$: Transition function.
- $q_0 \in Q$ is the initial state.
- $q_{\mathrm{acc}} \in Q$ is the accepting/final state.
- $q_{\mathrm{rej}} \in Q$ is the rejecting state.
- $\sqcup$ or ?: Special blank symbol on the tape.

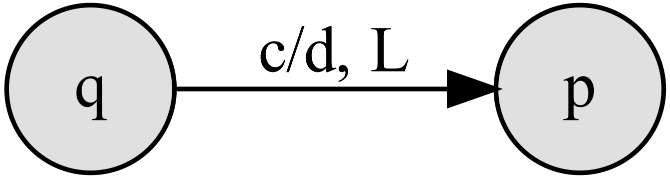
Turing machine: Transition function

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}$$

As such, the transition

$$\delta(q, c) = (p, d, \mathsf{L})$$

- q: current state.
- c: character under tape head.
- p: new state.
- d: character to write under tape head
- L: Move tape head left.



Can also be written as

$$c \to d, L \tag{2}$$

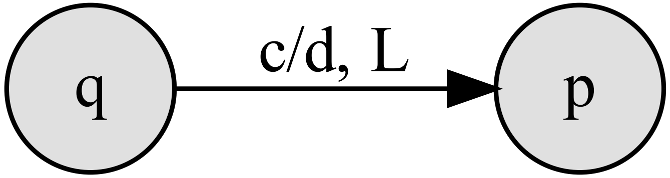
Turing machine: Transition function

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}$$

As such, the transition

$$\delta(q, c) = (p, d, \mathsf{L})$$

- q: current state.
- c: character under tape head.
- p: new state.
- d: character to write under tape head
- L: Move tape head left.



Missing transitions lead to hell state.
"Blue screen of death."
"Machine crashes."

Some examples of Turing machines

turingmachine.io

- equal strings TM
- palindrome TM

Languages defined by a Turing machine

Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages

$$L = \{L(M) \mid M \text{ some Turing machine}\}\,.$$

- Recursive / decidable languages

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}\,.$$

Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages (bad)

$$L = \{L(M) \mid M \text{ some Turing machine}\}.$$

- Recursive / decidable languages (good)

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}.$$

Recursive vs. Recursively Enumerable

- Recursively enumerable (aka RE) languages (bad)

$$L = \{L(M) \mid M \text{ some Turing machine}\}\,.$$

- Recursive / decidable languages (good)

$$L = \{L(M) \mid M \text{ some Turing machine that halts on all inputs}\}\,.$$

- Fundamental questions:
  - What languages are RE?
  - Which are recursive?
  - What is the difference?
  - What makes a language decidable?

} Last week of course

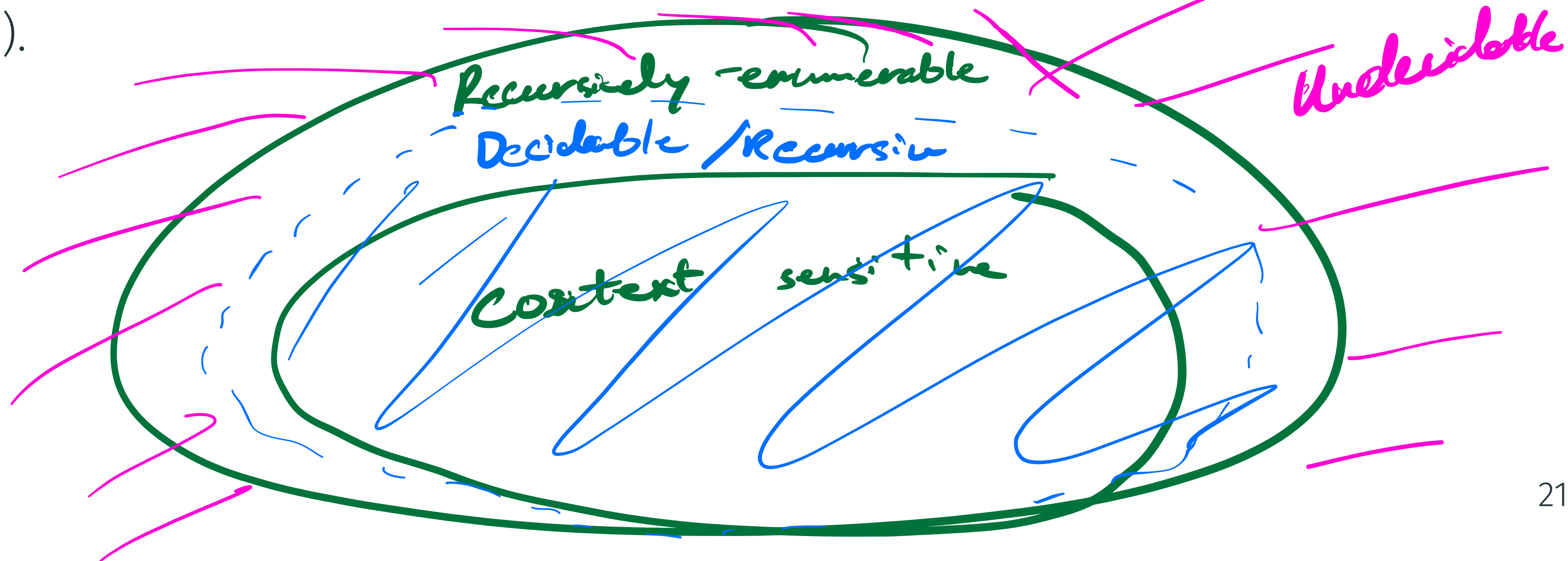
What is Decidable?

Decidable vs recursively-enumerable

A semi-decidable problem (equivalent of recursively enumerable) could be:

- **Decidable** - equivalent of recursive (TM always accepts or rejects).
- **Undecidable** - Problem is not recursive (doesn't always halt on negative)

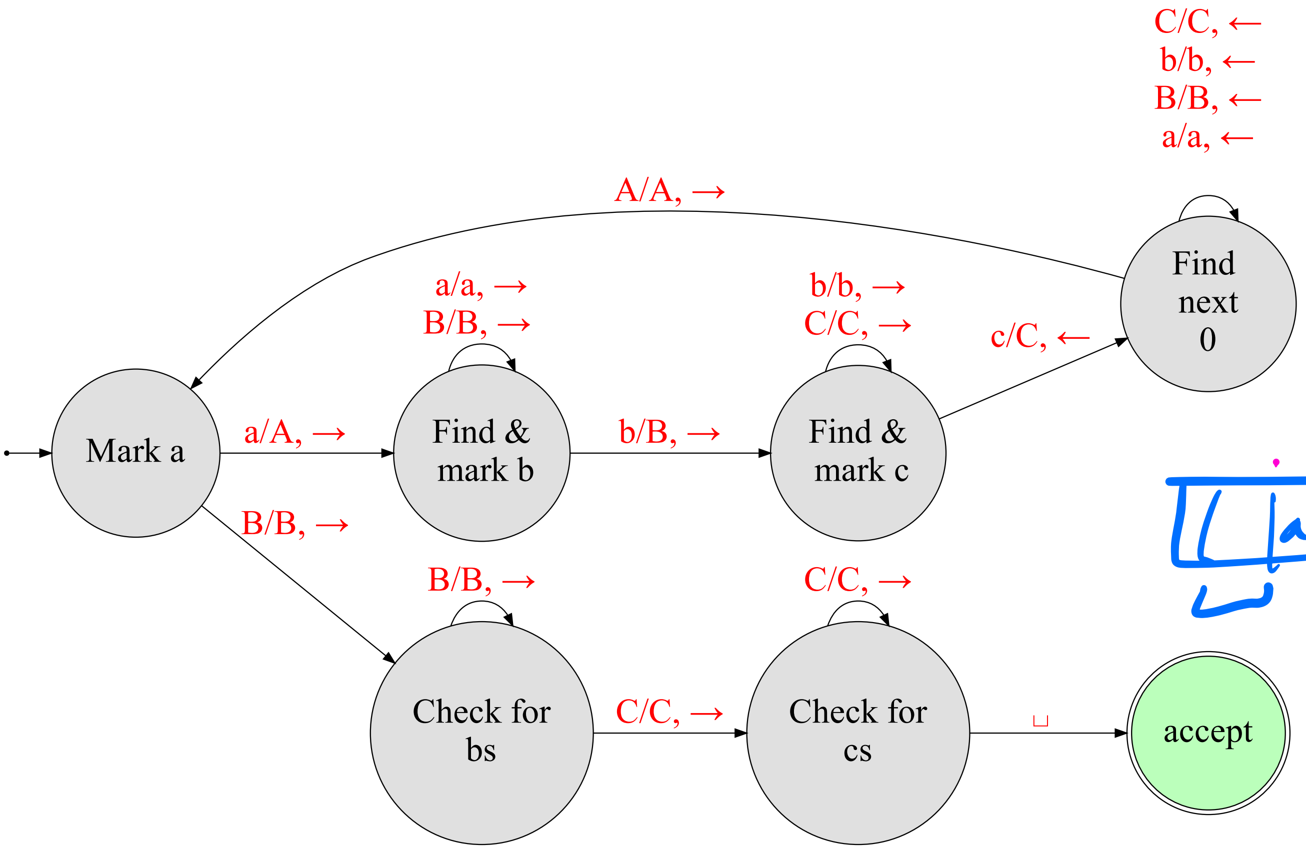
There are undecidable problem that are not semi-decidable (recursively enumerable).

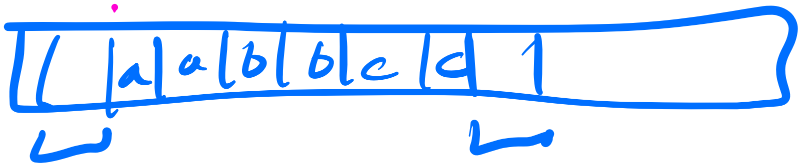


Infinite Tapes? Do we need them?

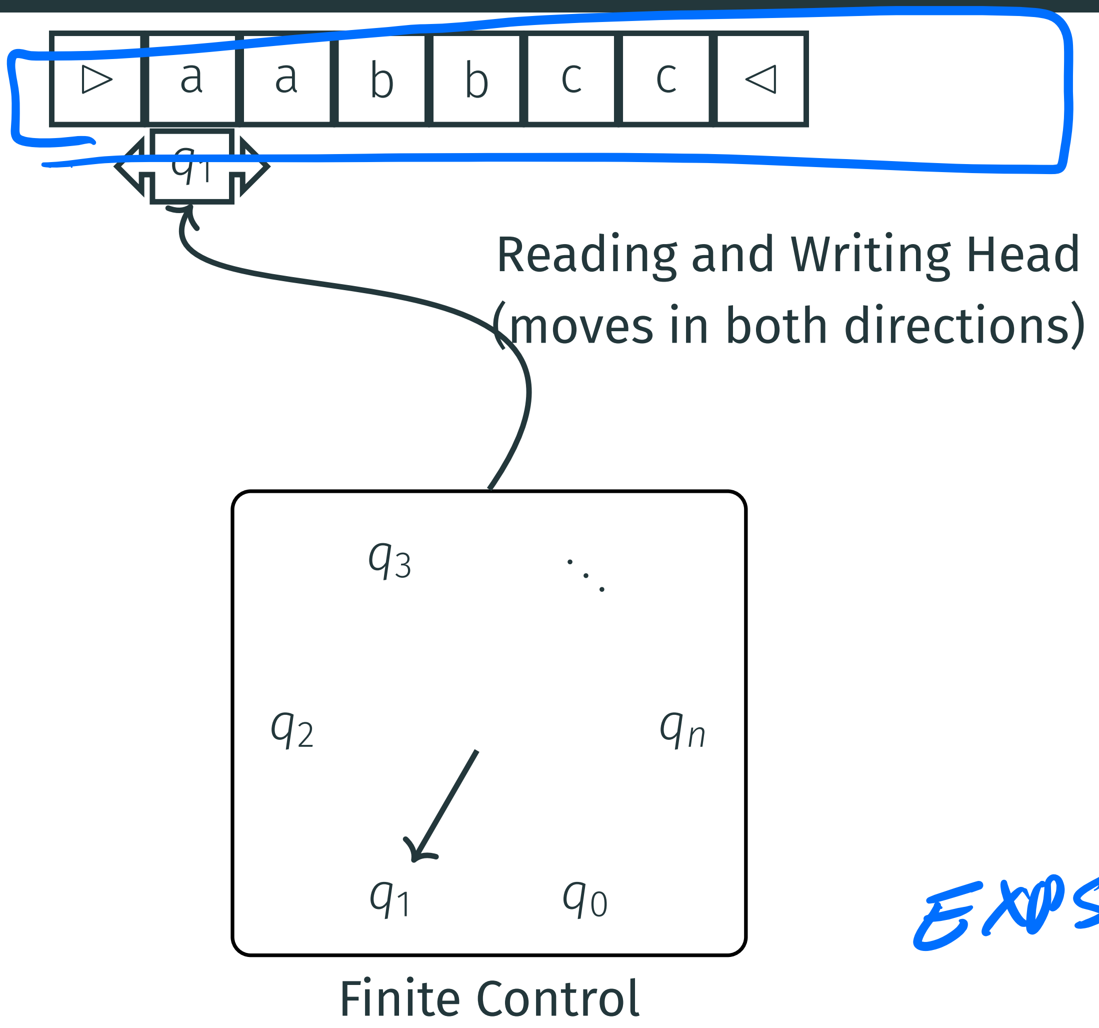
$a^n b^n c^n$

Let's look at the TM that recognizes $L = \{a^n b^n c^n | n \geq 0\}$:





Linear Bounded Automata

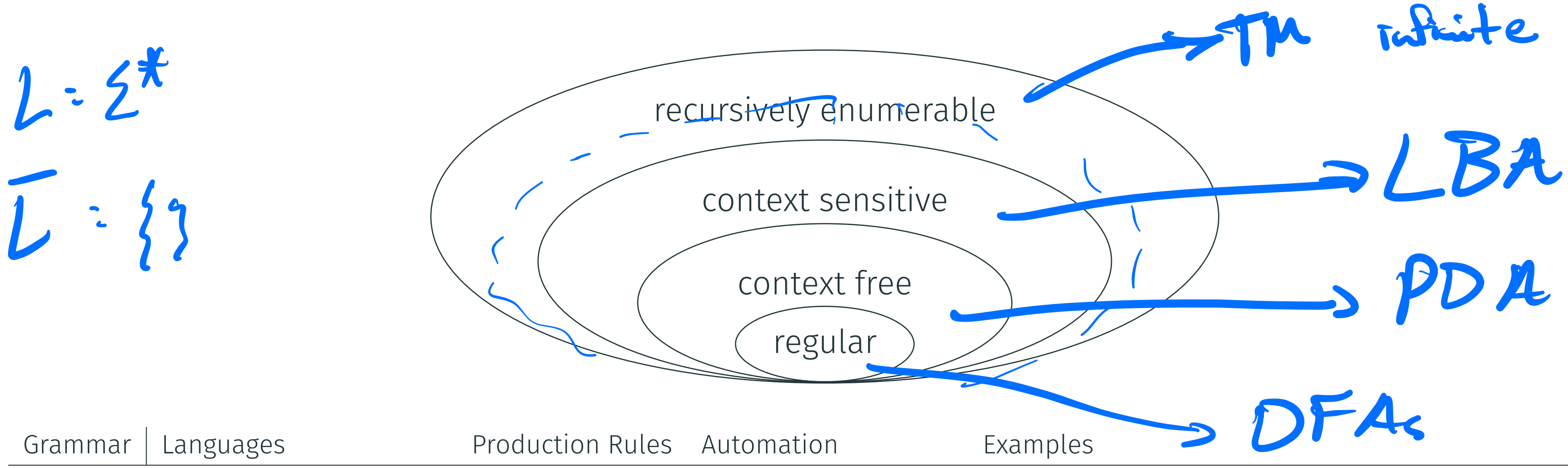


- (Nondeterministic) Linear bounded automata can recognize all context sensitive languages.
- Machine can non-deterministically apply all production rule to input in reverse and see if we end up with the start token.

EXPSPACE complete

Well that was a journey….

Zooming out



Grammar	Languages	Production Rules	Automation	Examples
Type-0	Recursively enumerable	$\gamma \to \alpha$ (no constraints)	Turing machine	$L = \{\langle M, w \rangle \| M$ is a TM which halts on $w\}$
Type-1	Context-sensitive	$\alpha A \beta \to \alpha \gamma \beta$	Linear bounded Non-deterministic Turing machine	$L = \{a^n b^n c^n \| n > 0\}$
Type-2	Context-free	$A \to \alpha$	Non-deterministic Push-down automata	$L = \{a^n b^n \| n > 0\}$
Type-3	Regular	$A \to aB$	Finite State Machine	$L = \{a^n \| n > 0\}$

1

Meaning of symbols: • a = terminal • A, B = variables • α, β, γ = string of $\{a \cup A\}^*$ • α, β = maybe empty –– γ = never empty