Pre-lecture brain teaser

We talked a lot about languages representing problems. Consider the problem of adding two numbers. What language class does it belong to?

ECE-374-B: Lecture 9 - Recursion, Sorting and Recurrences

Instructor: Abhishek Kumar Umrawal

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University of Illinois at Urbana-Champaign

About your instructor – Basic info

- Name: Abhishek Kumar Umrawal
- Webpage: ece.illinois.edu/about/directory/faculty/aumrawal
- Email: aumrawal@illinois.edu
- Office: ECEB 3054
- Office hours: Thursdays, 3 p.m. to 4 p.m., ECEB 4036

About your instructor – Education

- Purdue University, Ph.D. in Industrial Engineering

  Dissertation: Machine Learning Algorithms for Influence Maximization on Social Networks
- Purdue University, MS in Economics
- Indian Institute of Technology (IIT) Kanpur, MS in Statistics

About your instructor – Prior teaching experience

- University of Maryland, Visiting Lecturer of Computer Science and Electrical Engineering

About your instructor – Research interests

Core areas:

1. Combinatorial optimization
2. Approximation algorithms
3. Statistical learning theory
4. Reinforcement learning (RL)
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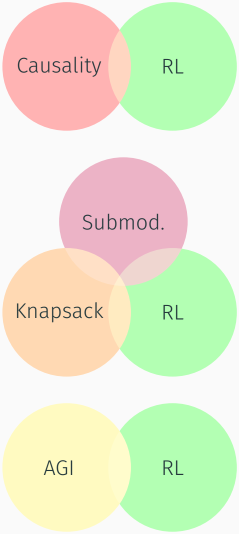
Intersections:



Working with me on research

If you are interested in working with me then please send me an email with subject line ‘Expressing interest in working with you on research’ with a brief description of your interests and skills with no attachments.

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Preferred (but not required) skills:

- Mathematical thinking
- Probability and statistics
- Python programming – graphs, object-oriented programming, recursion, etc.
- Algorithms (you're doing it this semester!)

You may fill out this form to provide further information.

Pre-lecture brain teaser

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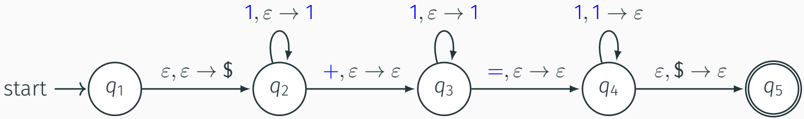
Pre-lecture brain teaser

Let's say we are adding two unary numbers.

$$3 + 4 = 7 \to 111 + 1111 = 1111111 \tag{1}$$

Seems like we can make a PDA that considers

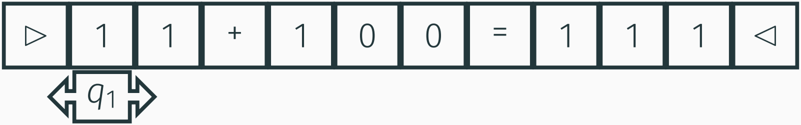


Pre-lecture brain teaser

What if we wanted add two binary numbers?

$$3 + 4 = 7 \rightarrow 11 + 100 = 111 \tag{2}$$

At least context-sensitive b/c we can build a finite Turing machine which takes in the encoding

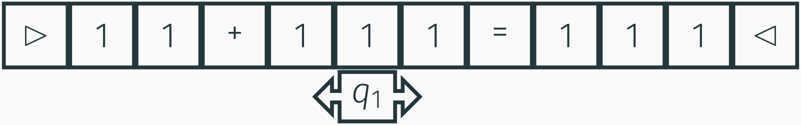


Pre-lecture brain teaser

What if we wanted add two binary numbers?

$$3 + 4 = 7 \to 11 + 100 = 111 \tag{3}$$

Computes value on left hand side



Pre-lecture brain teaser

What if we wanted add two binary numbers?

$$3 + 4 = 7 \rightarrow 11 + 100 = 111 \tag{4}$$

And compares it to the value on the right..

New Course Section: Introductory algorithms

Brief intro to the RAM model

Algorithms and Computing

- Algorithm solves a specific *problem*.
- Steps/instructions of an algorithm are *simple/primitive* and can be executed mechanically.
- Algorithm has a *finite description; same description* for all instances of the problem
- Algorithm implicitly may have *state/memory*

A computer is a device that

- *implements* the primitive instructions
- allows for an *automated* implementation of the entire algorithm by keeping track of state

Models of Computation vs Computers

- Model of Computation: an *idealized mathematical construct* that describes the primitive instructions and other details
- Computer: an actual *physical device* that implements a very specific model of computation

In this course: design algorithms in a high-level model of computation.

Question: What model of computation will we use to design algorithms?

Models of Computation vs Computers

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In this course: design algorithms in a high-level model of computation.

Question: What model of computation will we use to design algorithms?

The standard programming model that you are used to in programming languages such as Java/C++. We have already seen the Turing Machine model.

Unit-Cost RAM Model

Informal description:

- Basic data type is an integer number
- Numbers in input fit in a *word*
- Arithmetic/comparison operations on words take constant time
- Arrays allow random access (constant time to access $A[i]$)
- Pointer based data structures via storing addresses in a word

Example

Sorting: input is an array of n numbers

- input size is n (ignore the bits in each number),
- comparing two numbers takes $O(1)$ time,
- random access to array elements,
- addition of indices takes constant time,
- basic arithmetic operations take constant time,
- reading/writing one word from/to memory takes constant time.

We will usually do not allow (or be careful about allowing):

- bitwise operations (and, or, xor, shift, etc).
- floor function.
- limit word size (usually assume unbounded word size).

What is an algorithmic problem?

What is an algorithmic problem?

An algorithmic problem is simply to compute a function $f : \Sigma^* \to \Sigma^*$ over strings of a finite alphabet.

Algorithm $\mathcal{A}$ solves f if for all **input strings** w, $\mathcal{A}$ outputs $f(w)$.

Types of Problems

We will broadly see three types of problems.

- Decision Problem: Is the input a YES or NO input?
  Example: Given graph G, nodes s, t, is there a path from s to t in G?
  Example: Given a CFG grammar G and string w, is $w \in L(G)$?
- Search Problem: Find a *solution* if input is a YES input.
  Example: Given graph G, nodes s, t, find an s-t path.
- Optimization Problem: Find a *best* solution among all solutions for the input.
  Example: Given graph G, nodes s, t, find a shortest s-t path.

Analysis of Algorithms

Given a problem P and an algorithm $\mathcal{A}$ for P we want to know:

- Does $\mathcal{A}$ **correctly** solve problem P?
- What is the **asymptotic worst-case running time** of $\mathcal{A}$?
- What is the **asymptotic worst-case space** used by $\mathcal{A}$.

Asymptotic running-time analysis: $\mathcal{A}$ runs in $O(f(n))$ time if:

"for all n and for all inputs I of size n, $\mathcal{A}$ on input I terminates after $O(f(n))$ primitive steps."

Algorithmic Techniques

- Reduction to known problem/algorithm
- Recursion, divide-and-conquer, dynamic programming
- Graph algorithms to use as basic reductions
- Greedy

Some advanced techniques not covered in this class:

- Combinatorial optimization
- Linear and Convex Programming, more generally continuous optimization method
- Advanced data structure
- Randomization
- Many specialized areas

Reductions

Reduction

Reducing problem A to problem B:

- Algorithm for A uses algorithm for B as a *black box*

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Q: How do you hunt a blue elephant?
A: With a blue elephant gun.

Q: How do you hunt a red elephant?
A: Hold his trunk shut until it turns blue, and then shoot it with the blue elephant gun.

Q: How do you shoot a white elephant?
A: Embarrass it till it becomes red. Now use your algorithm for hunting red elephants.

UNIQUENESS: Distinct Elements Problem

Problem Given an array A of n integers, are there any *duplicates* in A?

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Naive algorithm:

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DistinctElements(A[1..n])
    for i = 1 to n − 1 do
        for j = i + 1 to n do
            if (A[i] = A[j])
                return YES
    return NO
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Running time:

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```

Running time: $O(n^2)$

Reduction to Sorting

```
DistinctElements(A[1..n])
    Sort A
    for i = 1 to n − 1 do
        if (A[i] = A[i + 1]) then
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Running time: $O(n)$ plus time to sort an array of n numbers

Important point: algorithm uses sorting as a *black box*

Advantage of naive algorithm: works for objects that cannot be "sorted". Can also consider hashing but outside scope of current course.

Two sides of Reductions

Suppose problem A reduces to problem B

- Positive direction: Algorithm for B implies an algorithm for A
- Negative direction: Suppose there is no “efficient” algorithm for A then it implies no efficient algorithm for B (technical condition for reduction time necessary for this)

Two sides of Reductions

Suppose problem A reduces to problem B

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Example: Distinct Elements reduces to Sorting in $O(n)$ time

- An $O(n \log n)$ time algorithm for Sorting implies an $O(n \log n)$ time algorithm for Distinct Elements problem.
- If there is *no* $o(n \log n)$ time algorithm for Distinct Elements problem then there is *no* $o(n \log n)$ time algorithm for Sorting.

Recursion as self reductions

Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction

- reduce problem to a *smaller* instance of *itself*
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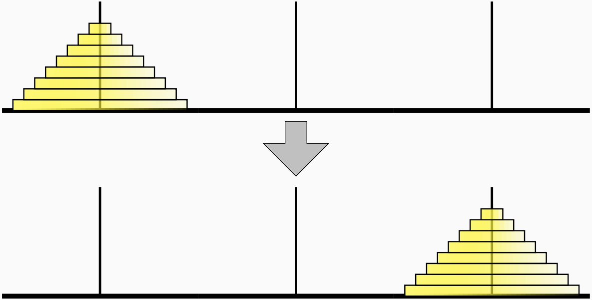
- Problem instance of size n is reduced to *one or more* instances of size $n-1$ or less.
- For termination, problem instances of small size are solved by some other method as *base cases*

Recursion

- Recursion is a very powerful and fundamental technique
- Basis for several other methods
    - Divide and conquer
    - Dynamic programming
    - Enumeration and branch and bound etc
    - Some classes of greedy algorithms
- Makes proof of correctness easy (via induction)
- Recurrences arise in analysis

Tower of Hanoi



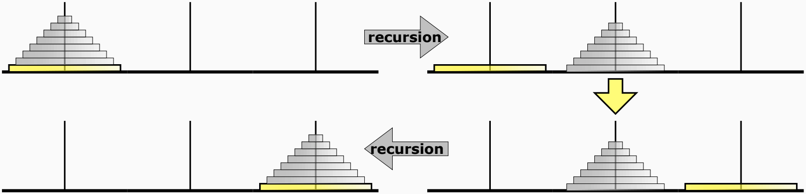
The Tower of Hanoi puzzle

Move stack of n disks from peg 1 to peg 2, one disk at a time.

Rule: cannot put a larger disk on a smaller disk.

Question: what is a strategy and how many moves does it take?

Tower of Hanoi via Recursion



The Tower of Hanoi algorithm; ignore everything but the bottom disk

Recursive Algorithm

```
Hanoi(n, src, dest, tmp):
    if (n > 0) then
        Hanoi(n − 1, src, tmp, dest)
        Move disk n from src to dest
        Hanoi(n − 1, tmp, dest, src)
```

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$T(n)$: time to move n disks via recursive strategy

$$T(n) = 2T(n-1) + 1 \qquad n > 1 \qquad \text{and } T(1) = 1$$

Analysis

$$
\begin{aligned}
T(n) &= 2T(n-1)+1 \\
&= 2^2T(n-2)+2+1 \\
&= \ldots \\
&= 2^iT(n-i)+2^{i-1}+2^{i-2}+\ldots+1 \\
&= \ldots \\
&= 2^{n-1}T(1)+2^{n-2}+\ldots+1 \\
&= 2^{n-1}+2^{n-2}+\ldots+1 \\
&= (2^n-1)/(2-1)=2^n-1
\end{aligned}
$$

Merge Sort

Sorting

Input Given an array of n elements

Goal Rearrange them in ascending order

MergeSort

1. Input: Array $A[1 \ldots n]$

$$ALGORITHMS$$

MergeSort

1. Input: Array $A[1 \dots n]$

$$A\ L\ G\ O\ R\ I\ T\ H\ M\ S$$

2. Divide into subarrays $A[1 \dots m]$ and $A[m+1 \dots n]$, where $m = \lfloor n/2 \rfloor$

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4. Merge the sorted arrays

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Merging Sorted Arrays

- Use a new array C to store the merged array
- Scan A and B from left-to-right, storing elements in C in order

$$\color{orange}{A}\ G\ L\ O\ R \qquad \color{orange}{H}\ I\ M\ S\ T$$

$$A$$

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$$A\ G\ H\ I\ L\ M\ O\ R\ S\ T$$

- Merge two arrays using only constantly more extra space (in-place merge sort): doable but complicated and typically impractical.

Formal Code

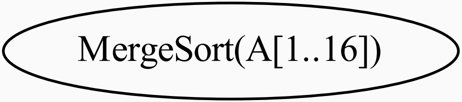
```
MergeSort(A[1..n]):
  if n > 1
      m ← ⌊n/2⌋
      MergeSort(A[1..m])
      MergeSort(A[m+1..n])
      Merge(A[1..n], m)
```

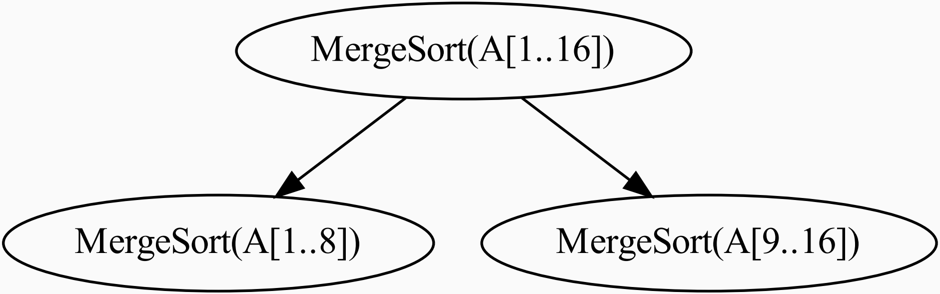
```
Merge(A[1..n], m):
  i ← 1;  j ← m+1
  for k ← 1 to n
      if j > n
          B[k] ← A[i];  i ← i+1
      else if i > m
          B[k] ← A[j];  j ← j+1
      else if A[i] < A[j]
          B[k] ← A[i];  i ← i+1
      else
          B[k] ← A[j];  j ← j+1

  for k ← 1 to n
      A[k] ← B[k]
```

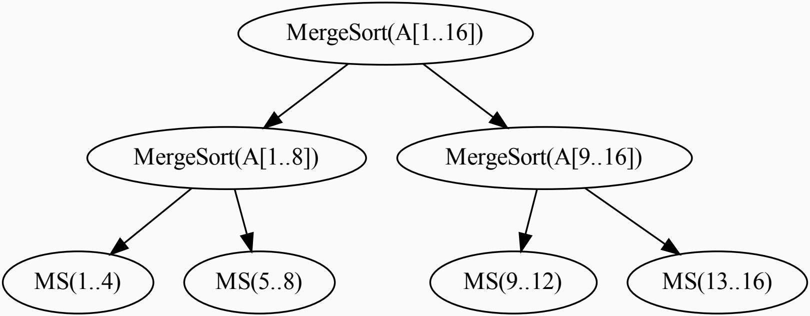
Running time analysis of merge-sort: Recursion tree method

Recursion tree




MergeSort(A[1..16])
MergeSort(A[1..8])
MergeSort(A[9..16])

Recursion tree



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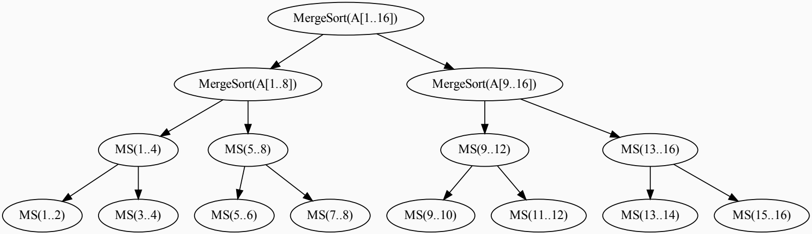

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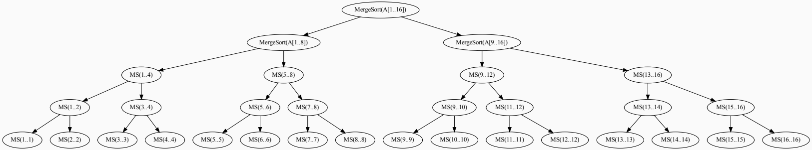
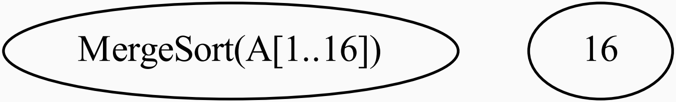


Recursion tree: subproblem sizes



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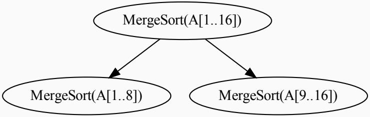

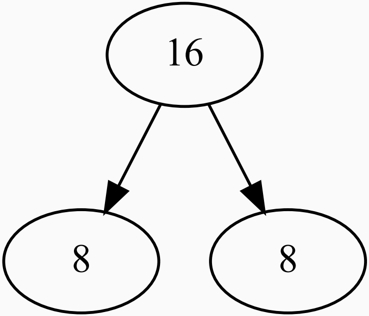

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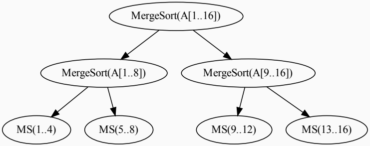

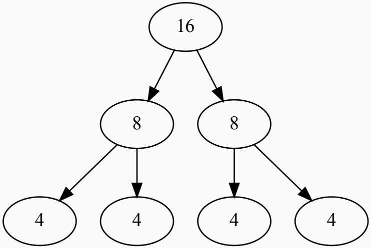

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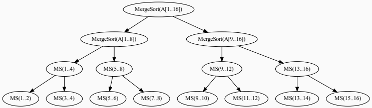

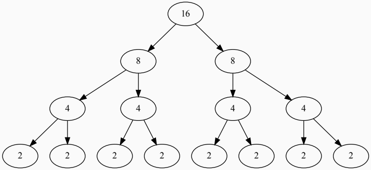

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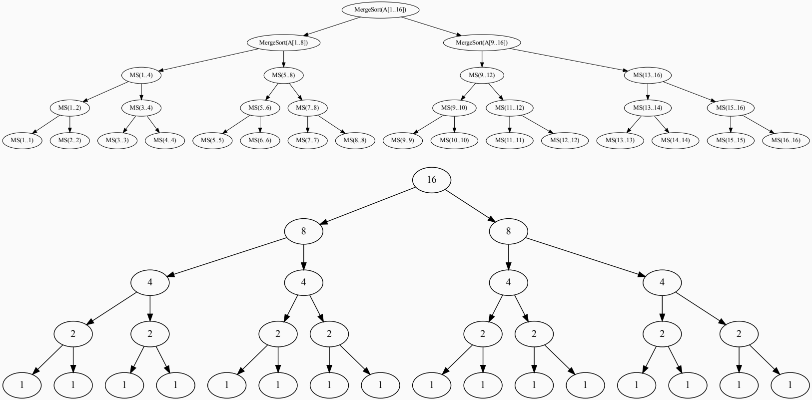
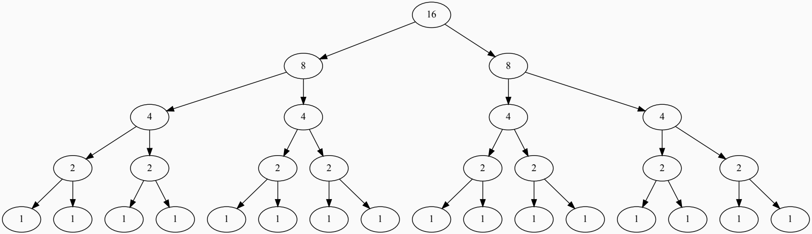


Recursion tree: Total work?



Running Time

$T(n)$: time for merge sort to sort an n element array

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$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

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What do we want as a solution to the recurrence?

Almost always only an *asymptotically* tight bound. That is we want to know $f(n)$ such that $T(n) = \Theta(f(n))$.

- $T(n) = O(f(n))$ - upper bound
- $T(n) = \Omega(f(n))$ - lower bound

Solving Recurrences: Some Techniques

- Know some basic math: geometric series, logarithms, exponentials, elementary calculus
- Expand the recurrence and spot a pattern and use simple math
- Recursion tree method — imagine the computation as a tree
- Guess and verify — useful for proving upper and lower bounds even if not tight bounds

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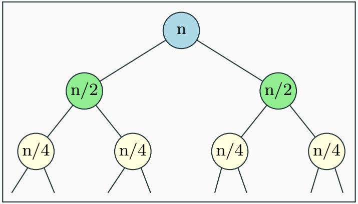
Albert Einstein: "Everything should be made as simple as possible, but not simpler."

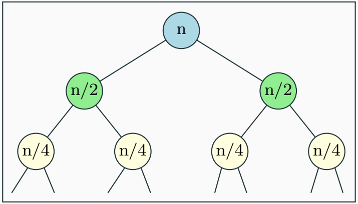
Know where to be loose in analysis and where to be tight. Comes with practice, practice, practice!

Recursion Trees : MergeSort: n is a power of 2



- Unroll the recurrence.
  $T(n) = 2T(n/2) + cn$

Recursion Trees : MergeSort: n is a power of 2



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.

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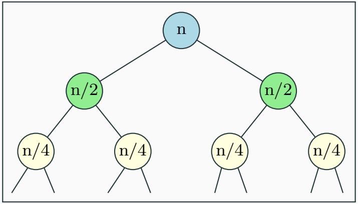

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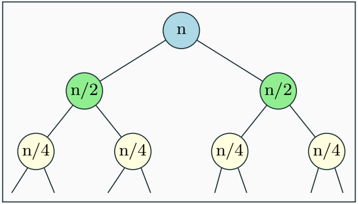

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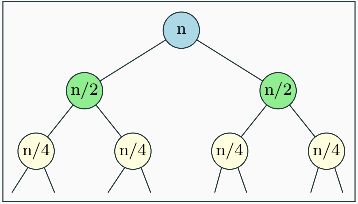

- Unroll the recurrence.
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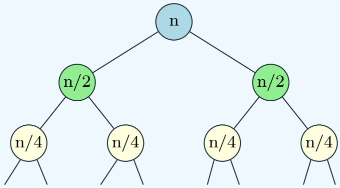
- Sum over all levels. The number of levels is $\log n$. So total is $cn \log n = O(n \log n)$.

Recursion Trees



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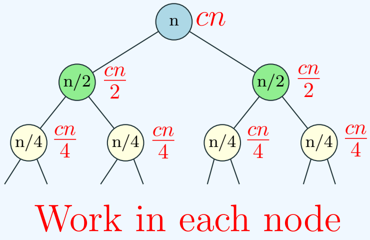

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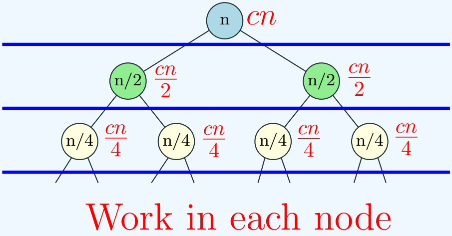

Recursion Trees

$$\log n \left\{ \begin{array}{cl} cn & = cn \\ \frac{cn}{2} \quad + \quad \frac{cn}{2} & = cn \\ \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} & = cn \\ \vdots & \vdots \\ & = cn \end{array} \right.$$

Recursion Trees

$$\log n \left\{ \begin{array}{c} cn \\ \hline \frac{cn}{2} + \frac{cn}{2} \\ \hline \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} + \frac{cn}{4} \\ \hline \vdots \\ \hline \end{array} \right. \begin{array}{l} = cn \\ \quad + \\ = cn \\ \quad + \\ = cn \\ \quad \vdots \\ = cn \end{array}$$

$$= cn \log n = O(n \log n)$$

Merge Sort Variant

Question: Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say k arrays of size n/k each?

Binary Search

Binary Search in Sorted Arrays

Input Sorted array A of n numbers and number x

Goal Is x in A?

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```
BinarySearch (A[a..b], x):
        if (b − a < 0) return NO
        mid = A[⌊(a + b)/2⌋]
        if (x = mid) return YES
        if (x < mid)
            return BinarySearch (A[a..⌊(a + b)/2⌋ − 1], x)
        else
            return BinarySearch (A[⌊(a + b)/2⌋ + 1..b],x)
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```

Analysis: $T(n) = T(\lfloor n/2 \rfloor) + O(1)$. $T(n) = O(\log n)$.

Observation: After k steps, size of array left is $n/2^k$