1 Short Answer(3 parts) - 40 points

No explanation is required for your answers for full credit. Keep any explanations of your answers to 2 sentences maximum.

a. Consider the inductive definition of a language MYSTERY:

- $\mathtt{0} \in$ MYSTERY
- If $w \in$ MYSTERY, then $w\mathtt{1} \in$ MYSTERY
- If $w \in$ MYSTERY, then $\mathtt{0}w \in$ MYSTERY

Give a regular expression for this language.

Solution: $\mathtt{0}^*\mathtt{01}^*$ ■

b. Consider the following context free grammar:

$$S \to AB \mid B$$
$$A \to \epsilon \mid \mathtt{a}A$$
$$B \to \mathtt{b}B\mathtt{c} \mid \mathtt{bc}$$

Show the sequence of rules that can be used to derive $\mathtt{aabbcc}$

Solution:

$$\begin{aligned}
S &\rightsquigarrow AB && (S \to AB)\\
&\rightsquigarrow \mathtt{a}A\mathtt{b}B\mathtt{c} && (A \to \mathtt{a}A,\ B \to \mathtt{b}B\mathtt{c})\\
&\rightsquigarrow \mathtt{aa}A\mathtt{bbcc} && (A \to \mathtt{a}A,\ B \to \mathtt{bc})\\
&\rightsquigarrow \mathtt{aa}\epsilon\mathtt{bbcc} = \mathtt{aabbcc} && (A \to \epsilon)
\end{aligned}$$

■

c. Suppose L_{R} is a regular language, L_{NR} is a non-regular language, and we want to examine a new language L.

Which of the following claims are necessarily true? For each claim that is not necessarily true, give a counter example. A counter example must specify L, L_{NR}, and L_{R}.

i. If $L = L_{\text{NR}} \cap L_{\text{R}}$, then L is regular.

Solution: *False*. Counter example: any selection L_{NR} that is a subset of the chosen L_{R} would work. For instance,

$$\begin{aligned} L_{\text{NR}} &:= \{\mathtt{0}^n\mathtt{1}^n \mid n \geq 0\} \\ L_{\text{R}} &:= \mathtt{0}^*\mathtt{1}^* \\ \Rightarrow L &= L_{\text{NR}} \cap L_{\text{R}} \\ &= \{\mathtt{0}^n\mathtt{1}^n \mid n \geq 0\} \\ &= L_{\text{NR}} \end{aligned}$$

■

ii. If $L_{\text{NR}} = L \cap L_{\text{R}}$, then L is non-regular.

Solution: *True*. Regular languages are closed under intersections. If the statement is false, then there exists two regular languages whose intersection would be non-regular, which violates this rule. ■

iii. If $L_{\text{R}} = L_{\text{NR}} \cap L$, then L is non-regular.

Solution: *False*. Counter example: any selection L_{R} that is a subset of the chosen L_{NR} would work. For instance,

$$\begin{aligned} L_{\text{NR}} &:= \{\mathtt{0}^n\mathtt{1}^n \mid n \geq 0\} \\ L_{\text{R}} &:= \emptyset \\ L_{\text{R}} &= L_{\text{NR}} \cap \emptyset \\ \Rightarrow L &= \emptyset \end{aligned}$$

■

2 Language Transformation - 20 points

Let $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ and let L be an arbitrary regular language over Σ.

Define the operation $\textsc{TwoIsWild}(L)$ as follows:

$$\textsc{TwoIsWild}(L) = \{abx \mid a \in \Sigma, b \in \Sigma, acx \in L \text{ for some } c \in L\}.$$

To summarize in words, every string (of length 2 or longer) in L is also in $\textsc{TwoIsWild}(L)$. Additionally, you can take any string w from L, change the 2nd character to anything you want, and the resulting string will be in $\textsc{TwoIsWild}(L)$.

Show that $\textsc{TwoIsWild}(L)$ is regular by constructing an NFA. You may assume a DFA for L exists as $(Q, \Sigma, \delta, s, A)$.

Solution: Let $M = ((Q, \Sigma, \delta, s, A)$ be a NFA that accepts L. We construct a new NFA $M' = (Q', \Sigma', \delta', s', A')$ that accepts $\textsc{TwoIsWild}(L)$ as follows.

We need to track how many elements have read in to know when we can allow a "wild" element. To do this we use 3 markers to indicate zero, one, and more than one elements. The transition from zero elements to one element is done according to the original NFA we just switch the marker to indicate we have read one element. The states able to be transitioned to from one element to two elements follows the original NFA, however the transitions can be done with any input to accomodate the TwoIsWild property. We also switch the marker to indicate we have read more than one element. When we have more than one elements the transitions once again follow the original NFA just with the more than one element marker. The accepting states are the original NFA accepting states with the more than one element marker.

$\Sigma' = \Sigma = \{\mathtt{0}, \mathtt{1}\}$

$Q' = Q \times \{\text{zero,one,more}\}$ (the three markers)

$s' = (s, \text{zero})$

$\delta'((s, \text{zero}), a) = \{(\delta(s, a), \text{one})\}$

$\delta'((q, \text{one}), a) = \{(\delta(q, b), \text{more})\}$ for all $b \in \Sigma$

$\delta'((q, \text{more}), a) = \{(\delta(q, a), \text{more})\}$

$A' = A \times \{\text{more}\}$

■

3 Language classification I (2 parts) - 20 points

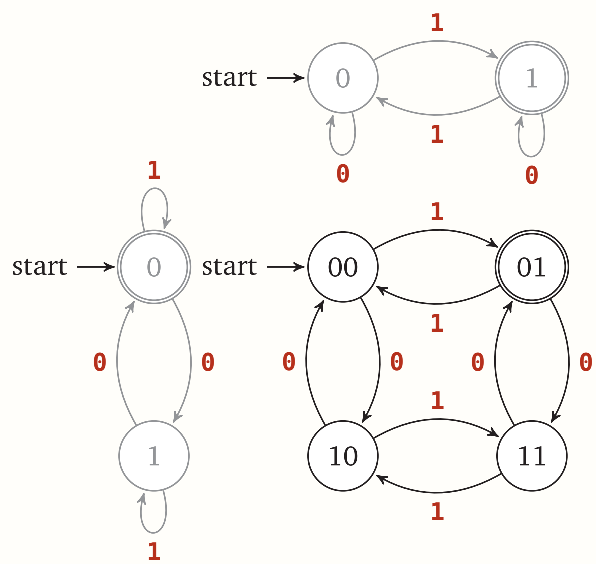
Let $\Sigma = \{0, 1\}$ and

$$L_3 = \{w \mid \text{ w contains a } \textit{even} \text{ number of } 0\text{'s and } \textit{odd} \text{ number of } 1\text{'s}\}.$$

1. Is L_3 regular? Indicate whether or not by circling one of the choices below. Either way, prove it.

Solution: regular not regular

The following DFA can accept L_3. We use a standard product construction of two DFAs, one accepting strings with an *even* number of 0's , and the other accepting strings with an *odd* number of 1's. The product DFA has four states, each labeled with a pair of integers. The two factor DFAs (in gray) is left and above for reference.



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2. Is L_3 context-free? Indicate whether or not by circling one of the choices below. Either way, prove it.

Solution: context-free not context-free

The context-free grammar for L_3 is as follows.

$S \rightarrow 0B \mid 1A$	*even* number of 0's and *even* number of 1's
$A \rightarrow 0C \mid 1S \mid \varepsilon$	*even* number of 0's and *odd* number of 1's
$B \rightarrow 0S \mid 1C$	*odd* number of 0's and *even* number of 1's
$C \rightarrow 0A \mid 1B$	*odd* number of 0's and *odd* number of 1's

■

Solution: Alternatively, you could just say that since the language is regular, it must be context-free. ■

4 Language classification II (2 parts) - 20 points

Let $\Sigma = \{\mathtt{0}, \mathtt{1}\}$ and

$$L_4 = \{w \mid \text{ w contains a } \textit{equal} \text{ number of } \mathtt{0}\text{'s and } \mathtt{1}\text{'s}\}.$$

1. Is L_4 regular? Indicate whether or not by circling one of the choices below. Either way, prove it.

Solution: regular not regular

Let F be the language $\mathtt{0}^*$.

Let x and y be arbitrary strings in F.

Then $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some non-negative integers $i \neq j$.

Let $z = \mathtt{1}^i$.

Then $xz = \mathtt{0}^i\mathtt{1}^i \in L_4$.

Then $yz = \mathtt{0}^j\mathtt{1}^i \notin L_4$, as $i \neq j$, so the number of $|\mathtt{0}| \neq |\mathtt{1}|$.

Thus F is a fooling set of L_4.

Because F is infinite, L_4 cannot be regular.

■

2. Is L_4 context-free? Indicate whether or not by circling one of the choices below. Either way, prove it.

Solution: context-free not context-free

The following CFG describes L_4.

$$S \rightarrow SS \mid 1S0 \mid 0S1 \mid \varepsilon$$

The is very similar to HW1 problem 2b, but we add the “SS" rule to account for strings that have zeros and ones on the same side. However, there are a few correct solutions to this. ■