Prove that each of the following problems is NP-hard.

1. Given an undirected graph G, does G contain a simple path that visits all but 374 vertices?

Solution: We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph G, let H be the graph obtained from G by adding 374 isolated vertices. Call a path in H ***almost-Hamiltonian*** if it visits all but 374 vertices. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian path.

- $\Rightarrow$ Suppose G has a Hamiltonian path P. Then P is an almost-Hamiltonian path in H, because it misses only the 374 isolated vertices.
- $\Leftarrow$ Suppose H has an almost-Hamiltonian path P. This path must miss all 374 isolated vertices in H, and therefore must visit every vertex in G. Every edge in H, and therefore every edge in P, is also na edge in G. We conclude that P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force. ■

2. Given an undirected graph G, does G have a spanning tree in which every node has degree at most 374?

Solution: We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph G, let H be the graph obtained by attaching a fan of 372 edges to every vertex of G. Call a spanning tree of H ***almost-Hamiltonian*** if it has maximum degree 374. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian spanning tree.

- $\Rightarrow$ Suppose G has a Hamiltonian path P. Let T be the spanning tree of H obtained by adding every fan edge in H to P. Every vertex v of H is either a leaf of T or a vertex of P. If $v \in P$, then $\deg_P(v) \leq 2$, and therefore $\deg_H(v) = \deg_P(v) + 372 \leq 374$. We conclude that H is an almost-Hamiltonian spanning tree.
- $\Leftarrow$ Suppose H has an almost-Hamiltonian spanning tree T. The leaves of T are precisely the vertices of H with degree 1; these are also precisely the vertices of H that are not vertices of G. Let P be the subtree of T obtained by deleting every leaf of T. Observe that P is a spanning tree of G, and for every vertex $v \in P$, we have $\deg_P(v) = \deg_T(v) - 372 \leq 2$. We conclude that P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force. ■

3. Given an undirected graph G, does G have a spanning tree with at most 374 leaves?

Solution: We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem.[a] Given an arbitrary graph G, let H be the graph obtained from G by adding the following vertices and edges:

- First we add a vertex z with edges to every other vertex in G.
- Then we add 373 vertices $\ell_1, \ldots, \ell_{373}$, each with edges to z and nothing else.

Call a spanning tree of H ***almost-Hamiltonian*** if it has at most 374 leaves. I claim that G contains a Hamiltonian path if and only if H contains an almost-Hamiltonian spanning tree.

$\Rightarrow$ Suppose G has a Hamiltonian path P. Suppose P starts at vertex s and ends at vertex t. Let T be subgraph of H obtained by adding the edge (t, z) and all edges (z, ℓ_i) to P. Then T is a spanning tree of H with exactly 374 leaves, namely s and all 373 new vertices ℓ_i.

$\Leftarrow$ Suppose H has an almost-Hamiltonian spanning tree T. Every node ℓ_i is a leaf of T, so T must consist of the 373 edges $z\ell_i$ and a simple path from z to some vertex s of G. Let t be the only neighbor of z in T that is not a leaf ℓ_i, and let P be the unique path in T from s to t. This path visits every vertex of G; in other words, P is a Hamiltonian path in G.

Given G, we can easily build H in polynomial time by brute force. ■

[a] Are you noticing a pattern here?

4. Recall that a 5-coloring of a graph G is a function that assigns each vertex of G a "color" from the set $\{0, 1, 2, 3, 4\}$, such that for any edge uv, vertices u and v are assigned different "colors". A 5-coloring is ***careful*** if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

Solution: We prove that careful 5-coloring is NP-hard by reduction from the standard 5Color problem.

Given a graph G, we construct a new graph H by replacing each edge in G with a path of length three. I claim that H has a careful 5-coloring if and only if G has a (not necessarily careful) 5-coloring.

$\Longleftarrow$ Suppose G has a 5-coloring. Consider a single edge uv in G, and suppose $color(u) = a$ and $color(v) = b$. We color the path from u to v in H as follows:

- If $b = (a+1) \bmod 5$, use colors $(a,\ (a+2) \bmod 5,\ (a-1) \pmod 5,\ b)$.
- If $b = (a-1) \bmod 5$, use colors $(a,\ (a-2) \bmod 5,\ (a+1) \pmod 5,\ b)$.
- Otherwise, use colors $(a,\ b,\ a,\ b)$.

In particular, every vertex in G retains its color in H. The resulting 5-coloring of H is careful.

$\Longrightarrow$ On the other hand, suppose H has a careful 5-coloring. Consider a path (u, x, y, v) in H corresponding to an arbitrary edge uv in G. There are exactly eight careful colorings of this path with $color(u) = 0$, namely: $(0, 2, 0, 2)$, $(0, 2, 0, 3)$, $(0, 2, 4, 1)$, $(0, 2, 4, 2)$, $(0, 3, 0, 3)$, $(0, 3, 0, 2)$, $(0, 3, 1, 3)$, $(0, 3, 1, 4)$. It follows immediately that $color(u) \neq color(v)$. Thus, if we color each vertex of G with its color in H, we obtain a valid 5-coloring of G.

Given G, we can clearly construct H in polynomial time. ■

5. Prove that the following problem is NP-hard: Given an undirected graph G, find *any* integer $k > 374$ such that G has a proper coloring with k colors but G does not have a proper coloring with $k - 374$ colors.

Solution: Let G' be the union of 374 copies of G, with additional edges between *every* vertex of each copy and *every* vertex in *every* other copy. Given G, we can easily build G' in polynomial time by brute force. Let $\chi(G)$ and $\chi(G')$ denote the minimum number of colors in any proper coloring of G, and define $\chi(G')$ similarly.

$\Longrightarrow$ Fix any coloring of G with $\chi(G)$ colors. We can obtain a proper coloring of G' with $374 \cdot \chi(G)$ colors, by using a distinct set of $\chi(G)$ colors in each copy of G. Thus, $\chi(G') \leq 374 \cdot \chi(G)$.

$\Longleftarrow$ Now fix any coloring of G' with $\chi(G')$ colors. Each copy of G in G' must use its own distinct set of colors, so at least one copy of G uses at most $\lfloor \chi(G')/374 \rfloor$ colors. Thus, $\chi(G) \leq \lfloor \chi(G')/374 \rfloor$.

These two observations immediately imply that $\chi(G') = 374 \cdot \chi(G)$. It follows that if k is an integer such that $k - 374 < \chi(G') \leq k$, then $\chi(G) = \chi(G')/374 = \lceil k/374 \rceil$. Thus, if we could compute such an integer k in polyomial time, we could compute $\chi(G)$ in polynomial time. But computing $\chi(G)$ is NP-hard! ■

6. A ***bicoloring*** of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

   (a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

Solution: It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3Color problem.

Let G be an arbitrary undirected graph. I claim that G has a proper 3-coloring if and only if G has a weak bicoloring with 3 colors.

- Suppose G has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of G using only the colors cyan, magenta, and yellow by recoloring each red vertex with $\{\text{magenta}, \text{yellow}\}$, recoloring each blue vertex with $\{\text{magenta}, \text{cyan}\}$, and recoloring each green vertex with $\{\text{yellow}, \text{cyan}\}$.
- Suppose G has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of G by defining red $= \{\text{magenta}, \text{yellow}\}$, defining blue $= \{\text{magenta}, \text{cyan}\}$, and defining green $= \{\text{yellow}, \text{cyan}\}$.

More generally, for any integer k and any graph G, every weak k-bicoloring of G is also a proper $\binom{k}{2}$-coloring of G, and vice versa. ■

(b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

Solution: It suffices to prove that deciding whether a graph has a strong bicoloring with six colors is NP-hard, using the following reduction from the standard 3Color problem.

Let G be an arbitrary undirected graph. We build a new graph H from G as follows:

- For every vertex v in G, the graph H contains three vertices v_1, v_2, and v_3 and three edges v_1v_2, v_2v_3, and v_3v_1.
- For every edge uv in G, the graph H contains three edges u_1v_1, u_2v_2, and u_3v_3.

I claim that G has a proper 3-coloring if and only if H has a strong bicoloring with six colors. Without loss of generality, we can assume that G (and therefore H) is connected; otherwise, consider each component independently.

⇒ Suppose G has a proper 3-coloring with colors red, green, and blue. Then we define a strong bicoloring of H with colors $1, 2, 3, 4, 5, 6$ as follows:

- For every red vertex v in G, let $\mathit{color}(v_1) = \{1, 2\}$ and $\mathit{color}(v_1) = \{3, 4\}$ and $\mathit{color}(v_3) = \{5, 6\}$.
- For every blue vertex v in G, let $\mathit{color}(v_1) = \{3, 4\}$ and $\mathit{color}(v_1) = \{5, 6\}$ and $\mathit{color}(v_3) = \{1, 2\}$.
- For every green vertex v in G, let $\mathit{color}(v_1) = \{5, 6\}$ and $\mathit{color}(v_1) = \{1, 2\}$ and $\mathit{color}(v_3) = \{3, 4\}$.

Exhaustive case analysis confirms that every pair of adjacent vertices of H has disjoint color sets.

- Suppose H has a strong bicoloring with six colors. Fix an arbitrary vertex v in G, and without loss of generality, suppose $\mathit{color}(v_1) = \{1, 2\}$ and $\mathit{color}(v_1) = \{3, 4\}$ and $\mathit{color}(v_3) = \{5, 6\}$. Exhaustive case analysis implies that for any edge uv, each vertex u_i must be colored either $\{1, 2\}$ or $\{3, 4\}$ or $\{5, 6\}$. It follows by induction that *every* vertex in H must be colored either $\{1, 2\}$ or $\{3, 4\}$ or $\{5, 6\}$.

  Now for each vertex w in G, color w red if $\mathit{color}(w_1) = \{1, 2\}$, blue if $\mathit{color}(w_1) = \{3, 4\}$, and green if $\mathit{color}(w_1) = \{5, 6\}$. This assignment of colors is a proper 3-coloring of G.

Given G, we can build H in polynomial time by brute force. ■

I believe that deciding whether a graph has a strong bicoloring with five *colors is also NP-hard, but I don't have a proof yet. A graph has a strong bicoloring with four colors if and only if it is bipartite, and a strong bicoloring with two or three colors if and only if it has no edges.*