Prove that each of the following problems is NP-hard.

1. Given an undirected graph \( G \), does \( G \) contain a simple path that visits all but 374 vertices?

**Solution:** We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph \( G \), let \( H \) be the graph obtained from \( G \) by adding 374 isolated vertices. Call a path in \( H \) almost-Hamiltonian if it visits all but 374 vertices. I claim that \( G \) contains a Hamiltonian path if and only if \( H \) contains an almost-Hamiltonian path.

\[ \Rightarrow \text{Suppose } G \text{ has a Hamiltonian path } P. \text{ Then } P \text{ is an almost-Hamiltonian path in } H, \text{ because it misses only the 374 isolated vertices.} \]

\[ \Leftarrow \text{Suppose } H \text{ has an almost-Hamiltonian path } P. \text{ This path must miss all 374 isolated vertices in } H, \text{ and therefore must visit every vertex in } G. \text{ Every edge in } H, \text{ and therefore every edge in } P, \text{ is also an edge in } G. \text{ We conclude that } P \text{ is a Hamiltonian path in } G. \]

Given \( G \), we can easily build \( H \) in polynomial time by brute force.

2. Given an undirected graph \( G \), does \( G \) have a spanning tree in which every node has degree at most 374?

**Solution:** We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph \( G \), let \( H \) be the graph obtained by attaching a fan of 372 edges to every vertex of \( G \). Call a spanning tree of \( H \) almost-Hamiltonian if it has maximum degree 374. I claim that \( G \) contains a Hamiltonian path if and only if \( H \) contains an almost-Hamiltonian spanning tree.

\[ \Rightarrow \text{Suppose } G \text{ has a Hamiltonian path } P. \text{ Let } T \text{ be the spanning tree of } H \text{ obtained by adding every fan edge in } H \text{ to } P. \text{ Every vertex } v \text{ of } H \text{ is either a leaf of } T \text{ or a vertex of } P. \text{ If } v \in P, \text{ then } \deg_P(v) \leq 2, \text{ and therefore } \deg_H(v) = \deg_P(v) + 372 \leq 374. \text{ We conclude that } H \text{ is an almost-Hamiltonian spanning tree.} \]

\[ \Leftarrow \text{Suppose } H \text{ has an almost-Hamiltonian spanning tree } T. \text{ The leaves of } T \text{ are precisely the vertices of } H \text{ with degree 1; these are also precisely the vertices of } H \text{ that are not vertices of } G. \text{ Let } P \text{ be the subtree of } T \text{ obtained by deleting every leaf of } T. \text{ Observe that } P \text{ is a spanning tree of } G, \text{ and for every vertex } v \in P, \text{ we have } \deg_P(v) = \deg_T(v) - 372 \leq 2. \text{ We conclude that } P \text{ is a Hamiltonian path in } G. \]

Given \( G \), we can easily build \( H \) in polynomial time by brute force.
3. Given an undirected graph \( G \), does \( G \) have a spanning tree with at most 374 leaves?

**Solution:** We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem.\(^a\) Given an arbitrary graph \( G \), let \( H \) be the graph obtained from \( G \) by adding the following vertices and edges:

- First we add a vertex \( z \) with edges to every other vertex in \( G \).
- Then we add 373 vertices \( \ell_1, \ldots, \ell_{373} \), each with edges to \( z \) and nothing else.

Call a spanning tree of \( H \) **almost-Hamiltonian** if it has at most 374 leaves. I claim that \( G \) contains a Hamiltonian path if and only if \( H \) contains an almost-Hamiltonian spanning tree.

\[ \Rightarrow \] Suppose \( G \) has a Hamiltonian path \( P \). Suppose \( P \) starts at vertex \( s \) and ends at vertex \( t \). Let \( T \) be subgraph of \( H \) obtained by adding the edge \((t, z)\) and all edges \((z, \ell_i)\) to \( P \). Then \( T \) is a spanning tree of \( H \) with exactly 374 leaves, namely \( s \) and all 373 new vertices \( \ell_i \).

\[ \Leftarrow \] Suppose \( H \) has an almost-Hamiltonian spanning tree \( T \). Every node \( \ell_i \) is a leaf of \( T \), so \( T \) must consist of the 373 edges \( z\ell_i \) and a simple path from \( z \) to some vertex \( s \) of \( G \). Let \( t \) be the only neighbor of \( z \) in \( T \) that is not a leaf \( \ell_i \), and let \( P \) be the unique path in \( T \) from \( s \) to \( t \). This path visits every vertex of \( G \); in other words, \( P \) is a Hamiltonian path in \( G \).

Given \( G \), we can easily build \( H \) in polynomial time by brute force. \( \blacksquare \)

\(^a\)Are you noticing a pattern here?
4. Recall that a 5-coloring of a graph $G$ is a function that assigns each vertex of $G$ a “color” from the set \{0, 1, 2, 3, 4\}, such that for any edge $uv$, vertices $u$ and $v$ are assigned different “colors”. A 5-coloring is careful if the colors assigned to adjacent vertices are not only distinct, but differ by more than $1$ (mod $5$). Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

**Solution:** We prove that careful 5-coloring is NP-hard by reduction from the standard 5Color problem.

Given a graph $G$, we construct a new graph $H$ by replacing each edge in $G$ with a path of length three. I claim that $H$ has a careful 5-coloring if and only if $G$ has a (not necessarily careful) 5-coloring.

$\Leftarrow$ Suppose $G$ has a 5-coloring. Consider a single edge $uv$ in $G$, and suppose $\text{color}(u) = a$ and $\text{color}(v) = b$. We color the path from $u$ to $v$ in $H$ as follows:

- If $b = (a + 1) \mod 5$, use colors $(a, (a + 2) \mod 5, (a - 1) \mod 5, b)$.
- If $b = (a - 1) \mod 5$, use colors $(a, (a - 2) \mod 5, (a + 1) \mod 5, b)$.
- Otherwise, use colors $(a, b, a, b)$.

In particular, every vertex in $G$ retains its color in $H$. The resulting 5-coloring of $H$ is careful.

$\Rightarrow$ On the other hand, suppose $H$ has a careful 5-coloring. Consider a path $(u, x, y, v)$ in $H$ corresponding to an arbitrary edge $uv$ in $G$. There are exactly eight careful colorings of this path with $\text{color}(u) = 0$, namely: $(0, 2, 0, 2), (0, 2, 0, 3), (0, 2, 4, 1), (0, 2, 4, 2), (0, 3, 0, 3), (0, 3, 0, 2), (0, 3, 1, 3), (0, 3, 1, 4)$. It follows immediately that $\text{color}(u) \neq \text{color}(v)$. Thus, if we color each vertex of $G$ with its color in $H$, we obtain a valid 5-coloring of $G$.

Given $G$, we can clearly construct $H$ in polynomial time. ■
5. Prove that the following problem is NP-hard: Given an undirected graph $G$, find any integer $k > 374$ such that $G$ has a proper coloring with $k$ colors but $G$ does not have a proper coloring with $k - 374$ colors.

**Solution:** Let $G'$ be the union of 374 copies of $G$, with additional edges between every vertex of each copy and every vertex in every other copy. Given $G$, we can easily build $G'$ in polynomial time by brute force. Let $\chi(G)$ and $\chi(G')$ denote the minimum number of colors in any proper coloring of $G$, and define $\chi(G')$ similarly.

$$\implies \text{Fix any coloring of } G \text{ with } \chi(G) \text{ colors. We can obtain a proper coloring of } G' \text{ with } 374 \cdot \chi(G) \text{ colors, by using a distinct set of } \chi(G) \text{ colors in each copy of } G. \text{ Thus, } \chi(G') \leq 374 \cdot \chi(G).$$

$$\Leftarrow \text{Now fix any coloring of } G' \text{ with } \chi(G') \text{ colors. Each copy of } G \text{ in } G' \text{ must use its own distinct set of colors, so at least one copy of } G \text{ uses at most } \lfloor \chi(G')/374 \rfloor \text{ colors. Thus, } \chi(G) \leq \lfloor \chi(G')/374 \rfloor.$$ 

These two observations immediately imply that $\chi(G') = 374 \cdot \chi(G)$. It follows that if $k$ is an integer such that $k - 374 < \chi(G') \leq k$, then $\chi(G) = \chi(G')/374 = \lfloor k/374 \rfloor$. Thus, if we could compute such an integer $k$ in polynomial time, we could compute $\chi(G)$ in polynomial time. But computing $\chi(G)$ is NP-hard! ■
6. A **bicoloring** of an undirected graph assigns each vertex a set of two colors. There are two types of bicoloring: In a weak bicoloring, the endpoints of each edge must use different sets of colors; however, these two sets may share one color. In a strong bicoloring, the endpoints of each edge must use distinct sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

(a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

**Solution:** It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3COLOR problem.

Let $G$ be an arbitrary undirected graph. I claim that $G$ has a proper 3-coloring if and only if $G$ has a weak bicoloring with 3 colors.

• Suppose $G$ has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of $G$ using only the colors cyan, magenta, and yellow by recoloring each red vertex with \{magenta, yellow\}, recoloring each blue vertex with \{magenta, cyan\}, and recoloring each green vertex with \{yellow, cyan\}.

• Suppose $G$ has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of $G$ by defining red = \{magenta, yellow\}, defining blue = \{magenta, cyan\}, and defining green = \{yellow, cyan\}.

More generally, for any integer $k$ and any graph $G$, every weak $k$-bicoloring of $G$ is also a proper $\binom{k}{2}$-coloring of $G$, and vice versa.
(b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

**Solution:** It suffices to prove that deciding whether a graph has a strong bicoloring with six colors is NP-hard, using the following reduction from the standard 3Color problem.

Let \( G \) be an arbitrary undirected graph. We build a new graph \( H \) from \( G \) as follows:

- For every vertex \( v \) in \( G \), the graph \( H \) contains three vertices \( v_1, v_2, \) and \( v_3 \) and three edges \( v_1v_2, v_2v_3, \) and \( v_3v_1 \).
- For every edge \( uv \) in \( G \), the graph \( H \) contains three edges \( u_1v_1, u_2v_2, \) and \( u_3v_3 \).

I claim that \( G \) has a proper 3-coloring if and only if \( H \) has a strong bicoloring with six colors. Without loss of generality, we can assume that \( G \) (and therefore \( H \)) is connected; otherwise, consider each component independently.

\[ \Rightarrow \] Suppose \( G \) has a proper 3-coloring with colors red, green, and blue. Then we define a strong bicoloring of \( H \) with colors 1, 2, 3, 4, 5, 6 as follows:

- For every red vertex \( v \) in \( G \), let \( \text{color}(v_1) = \{1, 2\} \) and \( \text{color}(v_3) = \{5, 6\} \).
- For every blue vertex \( v \) in \( G \), let \( \text{color}(v_1) = \{3, 4\} \) and \( \text{color}(v_3) = \{1, 2\} \).
- For every green vertex \( v \) in \( G \), let \( \text{color}(v_1) = \{5, 6\} \) and \( \text{color}(v_3) = \{3, 4\} \).

Exhaustive case analysis confirms that every pair of adjacent vertices of \( H \) has disjoint color sets.

\[ \Rightarrow \] Suppose \( H \) has a strong bicoloring with six colors. Fix an arbitrary vertex \( v \) in \( G \), and without loss of generality, suppose \( \text{color}(v_1) = \{1, 2\} \) and \( \text{color}(v_3) = \{5, 6\} \). Exhaustive case analysis implies that for any edge \( uv \), each vertex \( u_i \) must be colored either \( \{1, 2\} \) or \( \{3, 4\} \) or \( \{5, 6\} \). It follows by induction that every vertex in \( H \) must be colored either \( \{1, 2\} \) or \( \{3, 4\} \) or \( \{5, 6\} \).

Now for each vertex \( w \) in \( G \), color \( w \) red if \( \text{color}(w_1) = \{1, 2\} \), blue if \( \text{color}(w_1) = \{3, 4\} \), and green if \( \text{color}(w_1) = \{5, 6\} \). This assignment of colors is a proper 3-coloring of \( G \).

Given \( G \), we can build \( H \) in polynomial time by brute force. \[ \blacksquare \]

I believe that deciding whether a graph has a strong bicoloring with five colors is also NP-hard, but I don't have a proof yet. A graph has a strong bicoloring with four colors if and only if it is bipartite, and a strong bicoloring with two or three colors if and only if it has no edges.