Prove that each of the following languages is not regular.

1. \( \{0^{2n}1^n \mid n \geq 0 \} \)

**Solution (verbose):** Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Let \( z = 0^i1^i \).
Then \( xz = 0^{2i}1^i \in L \).
And \( yz = 0^{i+j}1^i \notin L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (concise):** For all non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( 0^i1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{i+j}1^i \notin L \). Thus, the language \( 0^* \) is an infinite fooling set for \( L \).

**Solution (concise, different fooling set):** For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{2j}1^i \notin L \). Thus, the language \( (00)^* \) is an infinite fooling set for \( L \).
2. \{\theta^m \theta^n \mid m \neq 2n\}

Solution (verbose): Let \( F \) be the language \( \theta^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = \theta^i \) and \( y = \theta^j \) for some non-negative integers \( i \neq j \).

Let \( z = \theta^i \theta^j \).

Then \( xz = \theta^{2i} \theta^j \notin L \).

And \( yz = \theta^{i+j} \theta^j \in L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution (concise, different fooling set): For all non-negative integers \( i \neq j \), the strings \( \theta^{2i} \) and \( \theta^{2j} \) are distinguished by the suffix \( \theta^j \), because \( \theta^{2i} \theta^j \notin L \) but \( \theta^{2j} \theta^j \in L \).

Thus, the language \( (\theta^0)^* \) is an infinite fooling set for \( L \).

3. \{\theta^{2n} \mid n \geq 0\}

Solution (verbose): Let \( F = L = \{\theta^{2n} \mid n \geq 0\} \).

Let \( x \) and \( y \) be arbitrary elements of \( F \).

Then \( x = \theta^{2i} \) and \( y = \theta^{2j} \) for some non-negative integers \( x \) and \( y \).

Let \( z = \theta^{2j} \).

Then \( xz = \theta^{2i} \theta^{2j} = \theta^{2i+1} \in L \).

And \( yz = \theta^{2j} \theta^{2i} = \theta^{2i+2j} \notin L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution (concise): For any non-negative integers \( i \neq j \), the strings \( \theta^{2i} \) and \( \theta^{2j} \) are distinguished by the suffix \( \theta^j \), because \( \theta^{2i} \theta^j = \theta^{2i+1} \in L \) but \( \theta^{2j} \theta^{2j} = \theta^{2j+2j} \notin L \).

Thus \( L \) itself is an infinite fooling set for \( L \).
4. Strings over \( \{0, 1\} \) where the number of 0s is exactly twice the number of 1s.

**Solution (verbose):** Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i 1^i \).

Then \( xz = 0^{2i} 1^i \in L \).

And \( yz = 0^{i+j} 1^i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (concise, different fooling set):** For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i} 1^i \in L \) but \( 0^{2j} 1^i \notin L \).

Thus, the language \((00)^*\) is an infinite fooling set for \( L \).
5. Strings of properly nested parentheses ( ), brackets [ ], and braces { }. For example, the string ( [ ] ) { } is in this language, but the string ( [ ] ) is not, because the left and right delimiters don't match.

**Solution (verbose):** Let \( F \) be the language \( \{ \} \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = (i) \) and \( y = (j) \) for some non-negative integers \( i \neq j \).

Let \( z = )^i \).

Then \( xz = )^i i \in L \).

And \( yz = )^i i \notin L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (concise):** For any non-negative integers \( i \neq j \), the strings \( (i) \) and \( (j) \) are distinguished by the suffix \( )^i \), because \( (i)^i i \in L \) but \( (i)^i j \notin L \). Thus, the language \( \{ \} \) is an infinite fooling set.

**Solution (closure properties):** If \( L \) were regular, then the language \( L \cap \{ \}^* = \{ (i)^n \mid n \geq 0 \} \) would be regular. The language \( \{ (i)^n \mid n \geq 0 \} \) is the same as \( \{ 0^n 1^n \mid n \geq 0 \} \) modulo changing the symbol names and is not regular from lecture. Thus \( L \) is not regular.
6. \( w \), such that \(|w| = \lceil k \sqrt{k} \rceil\), for some natural number \( k \).

Hint: since this one is more difficult, we'll even give you a fooling set that works: try \( F = \{ \theta^m | m \geq 1 \} \). We'll also provide a bound that can help: the difference between consecutive strings in the language, \( \lceil (k + 1)^{1.5} \rceil - \lceil k^{1.5} \rceil \), is bounded above and below as follows

\[
1.5 \sqrt{k} - 1 \leq \lceil (k + 1)^{1.5} \rceil - \lceil k^{1.5} \rceil \leq 1.5 \sqrt{k} + 3
\]

All that's left is you need to carefully prove that \( F \) is a fooling set for \( L \).

**Solution:** HW Problem.
7. Strings of the form $w_1 \# w_2 \# \cdots \# w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in 
\{0, 1\}*, and some pair of substrings $w_i$ and $w_j$ are equal.

**Solution (verbose):** Let $F$ be the language $0^*$. 
Let $x$ and $y$ be arbitrary strings in $F$. Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$. Let $z = \#0^i$. Then $xz = 0^i\#0^i \in L$. And $yz = 0^j\#0^i \notin L$, because $i \neq j$. Thus, $F$ is a fooling set for $L$. Because $F$ is infinite, $L$ cannot be regular.

**Solution (concise):** For any non-negative integers $i \neq j$, the strings $0^i$ and $0^j$ are distinguished by the suffix $\#0^i$, because $0^i\#0^i \in L$ but $0^j\#0^i \notin L$. Thus, the language $0^*$ is an infinite fooling set.
Work on these later:

7. \( \{ \theta^n \mid n \geq 0 \} \)

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( L \).

Without loss of generality, \( x = \theta^{2i+1} \) and \( y = \theta^{2j+1} \) for some \( i > j \geq 0 \).

Let \( z = \theta^i \).

Then \( xz = \theta^{i+2i+1} = \theta^{(i+1)^2} \in L \).

On the other hand, \( yz = \theta^{i^2+2j+1} \notin L \), because \( i^2 < i^2 + 2j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■

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**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( \theta^* \).

Without loss of generality, \( x = \theta^i \) and \( y = \theta^j \) for some \( i > j \geq 0 \).

Let \( z = \theta^{i^2+i+1} \).

Then \( xz = \theta^{i^2+2i+1} = \theta^{(i+1)^2} \in L \).

On the other hand, \( yz = \theta^{i^2+j+1} \notin L \), because \( i^2 < i^2 + i + j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( \theta^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■

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**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( \theta\theta\theta\theta^* \).

Without loss of generality, \( x = \theta^i \) and \( y = \theta^j \) for some \( i > j \geq 3 \).

Let \( z = \theta^{i^2-i} \).

Then \( xz = \theta^{i^2} \in L \).

On the other hand, \( yz = \theta^{i^2-i+j} \notin L \), because

\[
(i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.
\]

(The first inequalities requires \( i \geq 2 \), and the second \( j \geq 1 \).)

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( \theta\theta\theta\theta^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■
8. \{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\}

**Solution:** We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L\), for any integer \(k \geq 2\). The argument is somewhat simpler if we further restrict \(k\) to be even.

Let \(F = 1(00)^*1\), and let \(x\) and \(y\) be arbitrary strings in \(F\).
Then \(x = 10^{2i-2}1\) and \(y = 10^{2j-2}1\), for some positive integers \(i \neq j\).

Without loss of generality, assume \(i < j\). (Otherwise, swap \(x\) and \(y\).)

Let \(z = 0^{2i}1\).

Then \(xz = 10^{2i-2}10^{2i}1\) is the binary representation of \(2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2\), and therefore \(xz \in L\).

On the other hand, \(yz = 10^{2j-2}10^{2j}1\) is the binary representation of \(2^{2i+2j} + 2^{2i+1} + 1\). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \(2^{2i+2j} + 2^{2i+1} + 1\) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \(yz \notin L\).

We conclude that \(F\) is a fooling set for \(L\). Because \(F\) is infinite, \(L\) cannot be regular. ■