Recall fooling sets and distinguishability. Two strings $x, y \in \Sigma^*$ are suffix distinguishable with respect to a given language $L$ if there is a string $z$ such that exactly one of $xz$ and $yz$ is in $L$. This means that any DFA that accepts $L$ must necessarily take $x$ and $y$ to different states from its start state. A set of strings $F$ is a fooling set for $L$ if any pair of strings $x, y \in F, x \neq y$ are distinguishable. This means that any DFA for $L$ requires at least $|F|$ states. To prove non-regularity of a language $L$ you need to find an infinite fooling set $F$ for $L$. Given a language $L$ try to find a constant size fooling set first and then prove that one of size $n$ exists for any given $n$ which is basically the same as finding an infinite fooling set.

Note that another method to prove non-regularity is via reductions. Suppose you want to prove that $L$ is non-regular. You can do regularity preserving operations on $L$ to obtain a language $L'$ which you already know is non-regular. Then $L$ must not have been regular. For instance if $\bar{L}$ is not regular then $L$ is also not regular. You will see an example in Problem 4 below.

Prove that each of the following languages is not regular.

1. $\{0^{2n}1^n \mid n \geq 0\}$

2. $\{0^m1^n \mid m \neq 2n\}$

3. $\{0^{2n} \mid n \geq 0\}$

4. Strings over $\{0, 1\}$ where the number of $0$s is exactly twice the number of $1$s.
   - Describe an infinite fooling set for the language.
   - Use closure properties. What is language if you intersect the given language with $0^*1^*$?

5. Strings of properly nested parentheses $()$, brackets $[]$, and braces ${}$. For example, the string $([]){}$ is in this language, but the string $(())$ is not, because the left and right delimiters don’t match.
   - Describe an infinite fooling set for the language.
   - Use closure properties.

6. $w$, such that $|w| = \lceil k\sqrt{k} \rceil$, for some natural number $k$.

   Hint: since this one is more difficult, we’ll even give you a fooling set that works: try $F = \{0^m \mid m \geq 1\}$. We’ll also provide a bound that can help: the difference between consecutive strings in the language, $[(k + 1)^{1.5}] - [k^{1.5}]$, is bounded above and below as follows
   
   $$1.5\sqrt{k} - 1 \leq [(k + 1)^{1.5}] - [k^{1.5}] \leq 1.5\sqrt{k} + 3$$

   All that’s left is you need to carefully prove that $F$ is a fooling set for $L$.

7. Strings of the form $w_1\#w_2\# \cdots \#w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in $\{0, 1\}^*$, and some pair of substrings $w_i$ and $w_j$ are equal.
Work on these later:

7. \( \{0^n^2 \mid n \geq 0\} \)

8. \( \{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\} \)