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Formulate a **language** that describes the above problem.

ECE-374-B: Lecture 1 - Regular Languages

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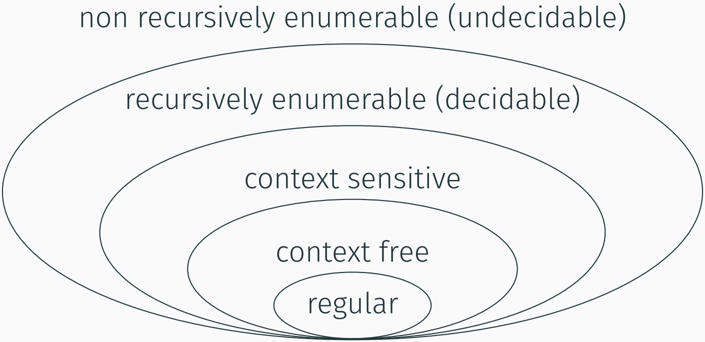
This is an example of a regular language which we'll be discussing today.

Chomsky Hierarchy



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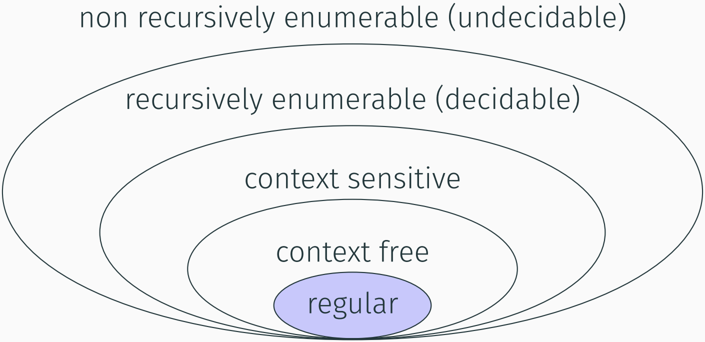

Regular Languages

Theorem (Kleene's Theorem)

A language is regular if and only if it can be obtained from finite languages by applying the three operations:

- *Union*
- *Concatenation*
- *Repetition*

a finite number of times.

Regular Languages

A class of simple but useful languages.

The set of regular languages over some alphabet Σ is defined inductively.

Base Case

- $\emptyset$ is a regular language.
- $\{\epsilon\}$ is a regular language.
- $\{a\}$ is a regular language for each $a \in \Sigma$. Interpreting a as string of length 1.

Inductive step:

We can build up languages using a few basic operations:

- If L_1, L_2 are regular then $L_1 \cup L_2$ is regular.
- If L_1, L_2 are regular then $L_1 L_2$ is regular.
- If L is regular, then $L^* = \cup_{n \geq 0} L^n$ is regular.
  The $\cdot^*$ operator name is Kleene star.
- If L is regular, then so is $\overline{L} = \Sigma^* \setminus L$.

Regular languages are closed under operations of union, concatenation and Kleene star.

Some simple regular languages

Lemma
If w is a string then $L = \{w\}$ is regular.

Example: $\{aba\}$ or $\{abbabbab\}$. Why?

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Lemma
Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

Regular Languages

Have basic operations to build regular languages.

Important: Any language generated by a finite sequence of such operations is regular.

Lemma
Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

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Note:Kleene star (repetition) is a **single** operation!

Regular Languages - Example

Example: The language $L_{01} = 0^i 1^j |$ for all $i, j \geq 0$ is regular:

Rapid-fire questions - regular languages

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4. $L_4 = \{ w \in \{0, 1\}^* \mid w \text{ has at most } 2\ 1\text{s} \}$. L_4 is regular. T/F?

Regular Expressions

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star names after him [1].

Inductive Definition

A regular expression $\mathbf{r}$ over an alphabet Σ is one of the following:

Base cases:

- $\emptyset$ denotes the language $\emptyset$
- ϵ denotes the language $\{\epsilon\}$.
- a denote the language $\{a\}$.

Inductive cases: If $\mathbf{r_1}$ and $\mathbf{r_2}$ are regular expressions denoting languages R_1 and R_2 respectively then,

- $(\mathbf{r_1} + \mathbf{r_2})$ denotes the language $R_1 \cup R_2$
- $(\mathbf{r_1} \bullet \mathbf{r_2}) = r_1 \bullet r_2 = (\mathbf{r_1}\mathbf{r_2})$ denotes the language R_1R_2
- $(\mathbf{r_1})^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages	Regular Expressions
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	$\mathbf{a}$ denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r_1} + \mathbf{r_2}$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r_1} \bullet \mathbf{r_2}$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

Notation and Parenthesis

- For a regular expression **r**, $L(\mathbf{r})$ is the language denoted by **r**. Multiple regular expressions can denote the same language!
  Example: $(0 + 1)$ and $(1 + 0)$ denotes same language $\{0, 1\}$

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- Superscript $+$. For convenience, define $\mathbf{r}^+ = \mathbf{r}\mathbf{r}^*$. Hence if $L(\mathbf{r}) = R$ then $L(\mathbf{r}^+) = R^+$.
- Other notation: $r + s$, $r \cup s$, $r|s$ all denote union. rs is sometimes written as $r \bullet s$.

Some examples of regular expressions

Creating regular expressions

1. All strings that end in 1011?

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1. All strings that end in 1011?
2. All strings except 11?
3. All strings that do not contain 000 as a subsequence?
4. All strings that do not contain the substring 10?

Interpreting regular expressions

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2. $(0 + 1)^*001(0 + 1)^*$:
3. $0^* + (0^*10^*10^*10^*)^*$:
4. $(\epsilon + 1)(01)^*(\epsilon + 0)$:

Tying everything together

Consider the problem of a n-input AND function. The input (x) is a string n-digits long with an input alphabet $\Sigma_i = \{0, 1\}$ and has an output (y) which is the logical AND of all the elements of x. We knwo the language used to describe it is:

$$L_{AND_N} = \begin{Bmatrix} 0 \cdot |0, & 1 \cdot |1, & & \\ 0 \cdot 0 \cdot |0, & 0 \cdot 1 \cdot |0, & 1 \cdot 0 \cdot |0, & 1 \cdot 1 \cdot |1 \\ \vdots & \vdots & \vdots & \vdots \\ (0 \cdot)^n|0, & (0 \cdot)^{n-1}1|0, & \ldots & (1 \cdot)^n|1 \ldots \end{Bmatrix}$$

Formulate the regular expression which describes the above language:

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Formulate the regular expression which describes the above language: $\Sigma = \{0, 1, `\cdot', `|'\}$

$$r_{AND_N} = \underbrace{(\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* \text{“}0\cdot\text{”} (\text{“}0\cdot\text{”} + \text{“}1\cdot\text{”})^* \text{“}|0\text{”}}_{\text{all output 0 instances}} + \overbrace{(\text{“}1\cdot\text{”})^* \text{“}|1\text{”}}^{\text{all output 1 instances}}$$

Regular expressions in programming

One last expression....

Bit strings with odd number of 0s and 1s

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The regular expression is

$$(00+11)^*(01+10)$$
$$\Big(00+11+(01+10)(00+11)^*(01+10)\Big)^*$$

Bit strings with odd number of 0s and 1s

The regular expression is

$$(00 + 11)^*(01 + 10) \Big(00 + 11 + (01 + 10)(00 + 11)^*(01 + 10)\Big)^*$$

(Solved using techniques to be presented in the following lectures...)