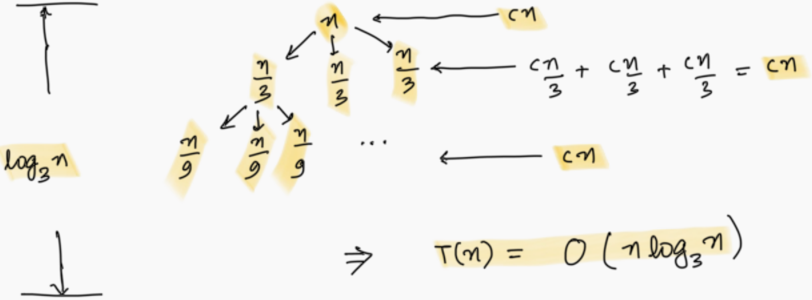
Pre-lecture brain teaser

Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say k arrays of size n/k each?

Recall: $k=2$: $T(n) = 2T\left(\frac{n}{2}\right) + cn \Rightarrow T(n) = O(n\log_2 n)$

$k=3$: $T(n) = 3T\left(\frac{n}{3}\right) + cn$



$\Rightarrow T(n) = O(n\log_3 n)$

In general: $T(n) = k\,T\left(\frac{n}{k}\right) + cn$

$$\Rightarrow \quad T(n) = O(n \log_k n)$$

$$\boxed{\log_a x}\ \underbrace{\log_b a}_{\text{constant}} = \log_b x$$

$$O(n \log_2 n) = O(n \log_k n \, \log_2 k)$$

$$= O(n \log_k n)$$

ECE-374-B: Lecture 10 - Divide and Conquer Algorithms

Instructor: Abhishek Kumar Umrawal

February 22, 2023

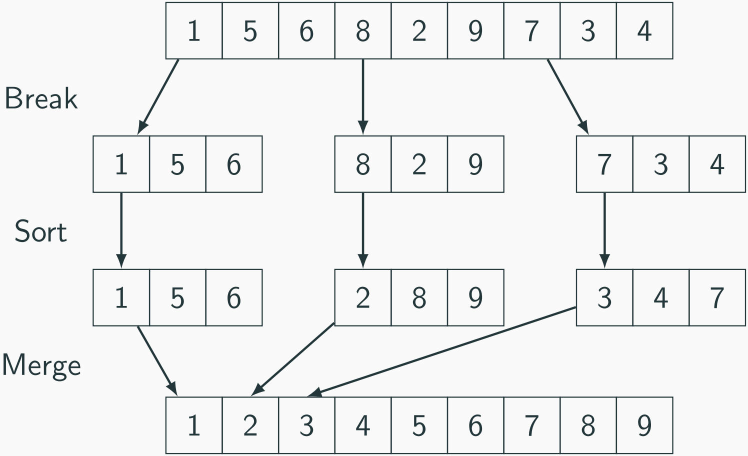
University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say k arrays of size n/k each?

Pre-lecture brain teaser

Simpler case: Break into 3 lists:



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What does the recurrence for $k = 3$ look like?

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$$T(n) = 3T(\frac{n}{3}) + cn$$

What is the solution to this recurrence?

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$$T(n) = 3T(\frac{n}{3}) + cn = O(nlogn)$$

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What is the solution to this recurrence?

$$T(n) = kT(\frac{n}{k}) + cn = O(nlogn)$$

So why don't we use smaller lists?

Learning Objectives

Learning Objectives

At the end of the lecture, you should be able to understand

- the idea of divide and conquer and how recursion forms a basis of it,
- the quicksort algorithm and its runtime analysis,
- the selection problem, quickselect algorithm and its runtime analysis, and
- the multiplication of numbers problem, a simple divide and conquer algorithm, and Karatsuba's algorithm, and runtime analysis of these algorithms.

Quick Sort

Quick Sort

Quick Sort [Hoare]

Tony Hoare (1959-60) A British Mathematician, Turing Awardee, FRS, FREng.

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.

3. Recursively sort the subarrays, and concatenate them.

Quick Sort

Quick Sort [Hoare]

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Quick Sort

Quick Sort [Hoare]

1. Pick a pivot element from array ← $O(1)$
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself. Linear scan of array does it. Time is $\underline{O(n)}$
3. Recursively sort the subarrays, and concatenate them. ?

Quick Sort

Quick Sort [Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself. Linear scan of array does it. Time is $O(n)$
3. Recursively sort the subarrays, and concatenate them.

Quick Sort: Example

- array: 16, 12, 14, 20, 5, 3, 18, 19, 1
- pivot: 16

See visualizer:

hackerearth.com/practice/algorithms/sorting/quick-sort/visualize

Time Analysis

- Let k be the rank of the chosen pivot. Then,

✓ $T(n) = T(k-1) + T(n-k) + O(n)$

$k=1$: $T(n) = T(1-1) + T(n-1) + O(n)$

$= T(0) + T(n-1) + O(n) = T(n-1) + O(n)$

$k=n$: $T(n) = T(n-1) + T(n-n) + O(n)$

$= T(0) + T(n-1) + O(n) = T(n-1) + O(n)$

$k = \underline{1}$ or $\underline{n}$: $T(n) = T(n-1) + O(n)$

$\Rightarrow$ $T(n) = O(n \cdot n) = \underline{O(n^2)}$

$n-0 \cdots O(n)$

$\downarrow$

$n-1 \cdots O(n-1)$

$\downarrow$

$n-2$ $\vdots$ n

$\downarrow$

$\vdots$

$n-(n-1)$

Time Analysis

- Let k be the rank of the chosen pivot. Then,
  $T(n) = T(k-1) + T(n-k) + O(n)$
- If $k = \lceil n/2 \rceil$ then median
  $T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n)$.
  Then, $T(n) = O(n \log n)$.

$$T(n) \leq 2T\left(\frac{n}{2}\right) + O(n)$$

$$\Rightarrow \quad T(n) = O(n \log_2 n)$$

Time Analysis

- Let k be the rank of the chosen pivot. Then, $T(n) = T(k-1) + T(n-k) + O(n)$
- If $k = \lceil n/2 \rceil$ then $T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n)$. Then, $T(n) = O(n \log n)$.

Time Analysis

- Let k be the rank of the chosen pivot. Then, $T(n) = T(k-1) + T(n-k) + O(n)$
- If $k = \lceil n/2 \rceil$ then $T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n)$. Then, $\boxed{T(n) = O(n \log n).}$
- Typically, pivot is the first or last element of array. Then,

$$T(n) = \underbrace{\max_{1 \leq k \leq n}}_{} \overbrace{\left(T(k-1) + T(n-k) + O(n)\right)}^{}$$

In the worst case $T(n) = T(n-1) + O(n)$, which means $\ast$ $\boxed{T(n) = O(n^2)}$. Happens if array is already sorted and pivot is always first element.

Selecting in Unsorted Lists

The Selection Problem

Big problem with QuickSort is that the pivot might not be the median.

How long would it take us to find the median of an unsorted list?

The Selection Problem

Big problem with QuickSort is that the pivot might not be the median.

How long would it take us to find the median of an unsorted list?

Sort, then $A[n/2]$. **Is this the optimal way?**

Rank of element in an array

A: an unsorted array of n integers

For $1 \leq j \leq n$, element of rank j is the j-th smallest element in A.

Unsorted array	16	14	34	20	12	5	3	19	11
Ranks	6	5	9	8	4	2	1	7	3

Sorted ~~Sort of~~ array	3	5	11	12	14	16	19	20	34

Problem - Selection

Input Unsorted array A of n integers **and** integer j

Goal Find the j-th smallest number in A (rank j number)

Median: $j = \lfloor (n+1)/2 \rfloor$

$A: \quad 9 \quad 5 \quad 1 \quad 7 \quad 2 \qquad n = 5 \quad \Rightarrow \quad \frac{n+1}{2} = 3$

Sorted: $\quad 1 \quad 2 \quad \textcircled{5} \quad 7 \quad 9$

$\uparrow$

Median!

Problem - Selection

Input Unsorted array A of n integers **and** integer j

Goal Find the j-th smallest number in A (rank j number)

Median: $j = \lfloor (n+1)/2 \rfloor$

Simplifying assumption for sake of notation: elements of A are distinct

Algorithm I

- Sort the elements in A
- Pick jth element in sorted order

Time taken $= O(n \log n)$

Algorithm I

- Sort the elements in A
- Pick jth element in sorted order

Time taken $= O(n \log n)$

Do we need to sort? Is there an $O(n)$ time algorithm?

Algorithm II

If j is small or $n - j$ is small then

- Find j smallest/largest elements in A in $O(jn)$ time. (How?)
- Time to find median is $O(n^2)$.

E.g. $j = 1$: We want the (1st) smallest element from A !

Claim: We can do that in $O(1n) = O(n)$ time!

$j = 2$: $O(2n)$ time!

$j = \lfloor \frac{n+1}{2} \rfloor$: $O(\lfloor \frac{n+1}{2} \rfloor n) = O(n^2)$

Quick select

QuickSelect

- Pick a pivot element a from A
- Partition A based on a.
  $A_{\text{less}} = \{x \in A \mid x \leq a\}$ and $A_{\text{greater}} = \{x \in A \mid x > a\}$
- $|A_{\text{less}}| = j$: return a
- $|A_{\text{less}}| > j$: recursively find jth smallest element in A_{less}
- $|A_{\text{less}}| < j$: recursively find kth smallest element in A_{greater}
  where $k = j - |A_{\text{less}}|$.

Example

16	14	34	20	12	5	3	19	11

Time Analysis

- Partitioning step: $O(n)$ time to scan A
- How do we choose pivot? Recursive running time?

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Time Analysis

- Partitioning step: $O(n)$ time to scan A
- How do we choose pivot? Recursive running time?

Suppose we always choose pivot to be $A[1]$.

Say A is sorted in increasing order and $j = n$. largest element

How long does this new algorithm take?

$O(n^2)$: Quick Select

Does this help with QuickSort?

Should we combine this with QuickSort

Does this help with QuickSort?

Should we combine this with QuickSort

Of course not! It takes $O(n^2)$ which is already the worse case of QuickSort. Need another method....

Does this help with QuickSort?

Looking at the quicksort recurrence again:

$$\bigstar\ \boxed{T(n) = \underline{T(k-1)} + \underline{T(n-k)} + \underline{O(n)}}$$

Does $\underline{k}$ *need* to be $\underline{n/2}$?

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What if $k = \frac{3}{5}n$?

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What if $k = \frac{3}{5}n$?

What if $k = \frac{7}{10}n$?

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Does k *need* to be $n/2$?

What if $k = \frac{3}{5}n$?

What if $k = \frac{7}{10}n$?

we only need to be able to find a rough median! .... How do we do that?

Median of Medians

Divide and Conquer Approach

Idea

- Break input A into many subarrays: $L_1, \ldots L_k$.
- Find median m_i in each subarray L_i.
- Find the median x of the medians $m_1, \ldots, m_k$.
- Intuition: The median x should be close to being a good median of all the numbers in A.
- Use x as pivot in previous algorithm.

$n \longrightarrow$ groups of 5 $\longrightarrow$ $\frac{n}{5}$ groups

↓

Calculate median of each group :

↳ $O(n)$

Example

11	7	3	42	174	310	1	92	87	12	19	15

Example

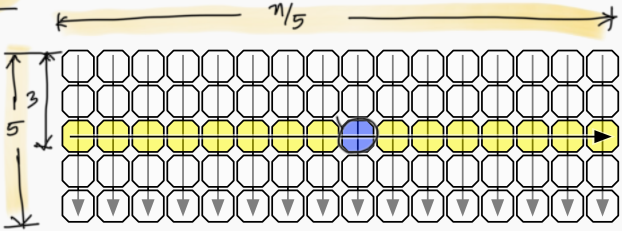
$\frac{1}{3} > \frac{3}{10} > \frac{2}{7}$

33% 30% 28%

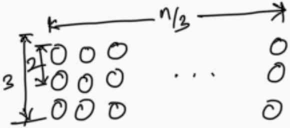
11	7	3	42	174	310	1	92	87	12	19	15

k = 7:

$$\frac{1}{2}\cdot\frac{n}{7}\cdot 4 = \frac{2}{7}n$$



k = 3



$$\underbrace{3\cdot\frac{1}{2}\left(\frac{n}{5}-1\right)+2}_{\approx \frac{3}{10}n} \le \text{MoM}$$

$$\approx \frac{3}{10}n \le \text{MoM}$$

$$\frac{1}{2}\cdot\frac{n}{3}\cdot 2 \le \text{MoM}$$

$$= \frac{1}{3}n$$

Choosing the pivot

- Partition array A into $\lceil n/5 \rceil$ lists of 5 items each.
  $L_1 = \{A[1], A[2], \ldots, A[5]\}$, $L_2 = \{A[6], \ldots, A[10]\}$, $\ldots$,
  $L_i = \{A[5i+1], \ldots, A[5i-4]\}$, $\ldots$,
  $L_{\lceil n/5 \rceil} = \{A[5\lceil n/5 \rceil - 4, \ldots, A[n]\}$.
- For each i find median b_i of L_i using brute-force in $O(1)$ time. Total $O(n)$ time
- Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
- Find median b of B

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- Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
- Find median b of B

Median of B is an *approximate* median of A. That is, if b is used a pivot to partition A, then $|A_{\text{less}}| \leq 7n/10$ *and* $|A_{\text{greater}}| \leq 7n/10$.

Algorithm for Selection

```
select(A, j):
    Form lists $L_1, L_2, \ldots, L_{\lceil n/5 \rceil}$ where $L_i = \{A[5i-4], \ldots, A[5i]\}$
    Find median $b_i$ of each $L_i$ using brute-force
    Find median $b$ of $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
    Partition $A$ into $A_{\texttt{less}}$ and $A_{\texttt{greater}}$ using $b$ as pivot
    if $(|A_{\texttt{less}}|) = j$ return $b$
    else if $(|A_{\texttt{less}}|) > j)$
        return select($A_{\texttt{less}}$, $j$)
    else
        return select($A_{\texttt{greater}}$, $j - |A_{\texttt{less}}|$)
```

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How do we find median of B?

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    Find median $b$ of $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
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    if ($|A_{\text{less}}|) = j$ return $b$
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        return select($A_{\text{less}}$, $j$)
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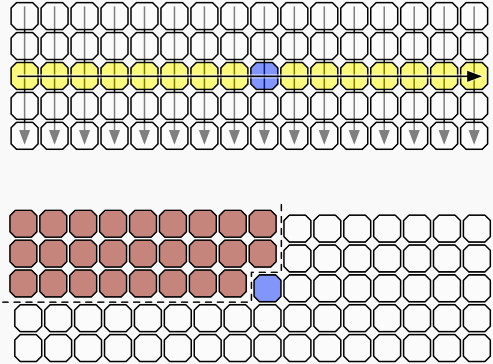
        return select($A_{\text{greater}}$, $j - |A_{\text{less}}|$)
```

How do we find median of B? Recursively!

Median of medians is a good median

Median of Medians: Proof of Lemma

There are at least $3n/10$ elements smaller than the median of medians b.



Median of Medians: Proof of Lemma

There are at least $3n/10$ elements smaller than the median of medians b.

At least half of the $\lfloor n/5 \rfloor$ groups have at least 3 elements smaller than b, except for the group containing b which has 2 elements smaller than b. Hence number of elements smaller than b is:

$$3\lfloor \frac{\lfloor n/5 \rfloor + 1}{2} \rfloor - 1 \geq 3n/10$$

Median of Medians: Proof of Lemma

There are at least $3n/10$ elements smaller than the median of medians b.

$|A_{\text{greater}}| \leq 7n/10$.

(R1Y)

Via symmetric argument,

$|A_{\text{less}}| \leq 7n/10$.

Running time of deterministic median selection

Running time of deterministic median selection

$$\underset{n}{A} \longrightarrow B \ (n/5)$$

$$T(n) \leq T(\lceil n/5 \rceil) + \max\{T(|A_{\text{less}}|), T(|A_{\text{greater}}|)\} + O(n)$$

Running time of deterministic median selection

$$T(n) \leq T(\lceil n/5 \rceil) + \underbrace{\max\{T(|A_{\text{less}}|), T(|A_{\text{greater}}|)\}}_{} + O(n)$$

From Lemma,

$$\boxed{T(n) \leq T(\lceil n/5 \rceil) + T(\underline{\lfloor 7n/10 \rfloor}) + O(n)}$$

and

$$T(n) = O(1) \qquad n < 10$$

Running time of deterministic median selection

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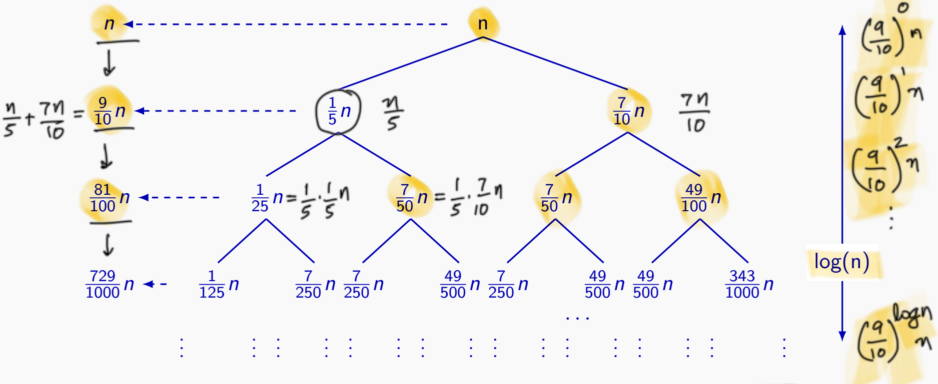
and

$$T(n) = O(1) \qquad n < 10$$

Exercise: show that $\boxed{T(n) = O(n)}$?

Recursion tree fill-in

If the workload is decreasing at every level, then total work is dominated by the root.



$$\boxed{T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 \rfloor) + \underline{O(n)} = O(n)}$$

$$\left(\frac{9}{10}\right)^0 n + \left(\frac{9}{10}\right)^1 n + \cdots + \left(\frac{9}{10}\right)^{\log n} \cdot n$$

$$\Big\downarrow \text{ Geometric series}$$

$$\vdots$$

$$O(n)$$

What about QuickSort?

How would we use the median of medians approach for quicksort?

What about QuickSort?

How would we use the median of medians approach for quicksort?

Just use MoM if find pivot!

- Original recurrence: $\boxed{T(n) = \overbrace{T(k-1)} + \overbrace{T(n-k)} + \overbrace{O(n)}}$
- With MoM: $T(n) = \underbrace{\underbrace{T(\frac{3}{10}n)} + \underbrace{T(\frac{7}{10}n)} + \underbrace{O(n)}}_{k} + \underbrace{O(n)}_{\text{MoM}}$

Solve!

$T(n) = \ldots$ (Exercise!)

Median of Medians Algorithm

Due to the following.

M. Blum, R. Floyd, D. Knuth, V. Pratt, R. Rivest, and R. Tarjan.

"Time bounds for selection".

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Favorite Knuth quote: He once warned a correspondent, “Beware of bugs in the above code; I have only proved it correct, not tried it.”

Takeaway Points

- Recursion tree method and guess and verify are the most reliable methods to analyze recursions in algorithms.
- Recursive algorithms naturally lead to recurrences.
- Some times one can look for certain type of recursive algorithms (reverse engineering) by understanding recurrences and their behavior.

Problem statement: Multiplying numbers + a slow algorithm

The Problem: Multiplying numbers

Given two large positive integer numbers b and c, with n digits, compute the number $b * c$.

Egyptian multiplication: 1850BC (3870 years ago?)

76	35

Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76

Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
$\underline{76}$	$\underline{34}$	

Egyptian multiplication: 1850BC (3870 years ago?)

$m \cdot n = 2m \cdot \frac{n}{2}$

	76	35	
	76	$34 + 1$	76
x2	76	34 /2	
	152	$\underline{17}$	

Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	34 + 1	76
76	34	
152	17	
152	16 + 1	152

Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	$34 + 1$	76
76	34	
152	17	
152	$16 + 1$	152
152	16	

Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
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76	34	
152	17	
152	$16+1$	152
152	16	
304	8	

Egyptian multiplication: 1850BC (3870 years ago?)

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152	16	
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76	35	
76	$34 + 1$	76
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1216	2	

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608	4	
1216	2	
2432	1	2432

Egyptian multiplication: 1850BC (3870 years ago?)

76	35	
76	34 + 1	76
76	34	
152	17	
152	16 + 1	152
152	16	
304	8	
608	4	
1216	2	
2432	1	2432
		2660

×2

/2

The problem: Multiplying Numbers

Problem Given two *n*-digit numbers $\overset{O(n)}{x}$ and $\overset{O(n)}{y}$, compute their product.

Grade School Multiplication
Compute "partial product" by multiplying each digit of y with x and adding the partial products.

$$\begin{array}{r}
3141 \\
\times 2718 \\
\hline
25128 \\
3141 \\
21987 \\
6282 \\
\hline
8537238
\end{array}$$

Runtime :

$n \cdot n$

$= O(n^2)$

Time Analysis of Grade School Multiplication

- Each partial product: $\Theta(n)$
- Number of partial products: $\Theta(n)$
- Addition of partial products: $\Theta(n^2)$
- Total time: $\Theta(n^2)$

Multiplication using Divide and Conquer

Divide and Conquer

Assume n is a power of 2 for simplicity and numbers are in decimal.

Split each number into two numbers with equal number of digits

- $b = b_{n-1}b_{n-2}\dots b_0$ and $c = c_{n-1}c_{n-2}\dots c_0$
- $b = b_{n-1}\dots b_{n/2}0\dots0 + b_{n/2-1}\dots b_0$
- $b(x) = b_L x + b_R$, where $x = 10^{n/2}$, $b_L = b_{n-1}\dots b_{n/2}$ and $b_R = b_{n/2-1}\dots b_0$
- Similarly $c(x) = c_L x + c_R$ where $c_L = c_{n-1}\dots c_{n/2}$ and $c_R = c_{n/2-1}\dots c_0$

$$b = \overset{\leftarrow n \rightarrow}{\underline{12}\,\underline{34}} = \overset{\curvearrowright n/2}{\boxed{12}}00 + \boxed{34}^{\frac{n}{2}}$$

$$\xrightarrow{\quad} 12\underbrace{00}_{\left(\frac{n}{2}\right)\text{zeros}} + \underline{34}$$

$$b(x) = b_L x + b_R$$

$$x = 10^{n/2}$$

$$c(x) = c_L x + b_R$$

Example

$$\begin{aligned} 1234 \times 5678 &= (12x + 34) \times (56x + 78) \qquad \text{for} \quad x = 1 \\ &= 12 \cdot 56 \cdot x^2 + (12 \cdot 78 + 34 \cdot 56)x + 34 \cdot 78. \end{aligned}$$

$$\begin{aligned} 1234 \times 5678 \quad &= \quad (100 \times 12 + 34) \times (100 \times 56 + 78) \\ &= \quad 10000 \times 12 \times 56 \\ & \qquad +100 \times (12 \times 78 + 34 \times 56) \\ & \qquad +34 \times 78 \end{aligned}$$

Divide and Conquer for multiplication

Assume n is a power of 2 for simplicity and numbers are in decimal.

- $b = b_{n-1} b_{n-2} \dots b_0$ and $c = c_{n-1} c_{n-2} \dots c_0$
- $b \equiv b(x) = b_L x + b_R$
  where $x = 10^{n/2}$, $b_L = b_{n-1} \dots b_{n/2}$ and $b_R = b_{n/2-1} \dots b_0$
- $c \equiv c(x) = c_L x + c_R$ where $c_L = c_{n-1} \dots c_{n/2}$ and $c_R = c_{n/2-1} \dots c_0$

Divide and Conquer for multiplication

Assume n is a power of 2 for simplicity and numbers are in decimal.

- $b = b_{n-1}b_{n-2}\dots b_0$ and $c = c_{n-1}c_{n-2}\dots c_0$
- $b \equiv b(x) = b_L x + b_R$
  where $x = 10^{n/2}$, $b_L = b_{n-1}\dots b_{n/2}$ and $b_R = b_{n/2-1}\dots b_0$
- $c \equiv c(x) = c_L x + c_R$ where $c_L = c_{n-1}\dots c_{n/2}$ and
  $c_R = c_{n/2-1}\dots c_0$

Therefore, for $x = 10^{n/2}$, we have

$$\begin{aligned} bc = b(x)c(x) &= (b_L x + b_R)(c_L x + c_R) \\ &= b_L c_L x^2 + (b_L c_R + b_R c_L)x + b_R c_R \\ &= 10^n b_L c_L + 10^{n/2}(b_L c_R + b_R c_L) + b_R c_R \end{aligned}$$

Time Analysis

$b = b_L \cdot x + b_R$ $x = 10^{n/2}$

$c = c_L \cdot x + c_R$

$$\overset{n}{bc} = 10^n \overbrace{b_L c_L}^{\frac{n}{2}} + 10^{n/2}(\overset{\frac{n}{2}}{b_L c_R} + \overset{\frac{n}{2}}{b_R c_L}) + \overset{\frac{n}{2}}{b_R c_R}$$

$\underline{4}$ recursive multiplications of number of size $\underline{n/2}$ each plus 4 additions and left shifts (adding enough 0's to the right)

Time Analysis

$$bc = 10^n b_L c_L + 10^{n/2}(b_L c_R + b_R c_L) + b_R c_R$$

4 recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

Time Analysis

$$bc = 10^n b_L c_L + 10^{n/2}(b_L c_R + b_R c_L) + b_R c_R$$

4 recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

$T(n) = \underline{\Theta(n^2)}$. No better than grade school multiplication!

Faster multiplication: Karatsuba's Algorithm

A Trick of Gauss

Carl Friedrich Gauss: 1777–1855 “Prince of Mathematicians”

Observation: Multiply two complex numbers: $(a + bi)$ and $(c + di)$

$$* \ (a + bi)(c + di) = ac - bd + (ad + bc)i$$

$\sqrt{-1}$

A Trick of Gauss

Carl Friedrich Gauss: 1777–1855 "Prince of Mathematicians"

Observation: Multiply two complex numbers: $(a + bi)$ and $(c + di)$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

How many multiplications do we need?

A Trick of Gauss

Carl Friedrich Gauss: 1777–1855 “Prince of Mathematicians”

Observation: Multiply two complex numbers: $(a + bi)$ and $(c + di)$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

How many multiplications do we need?

Only 3! If we do extra additions and subtractions.
Compute $ac, bd, (a + b)(c + d)$. Then

Gauss technique for polynomials

$$p(x) = ax + b \qquad \text{and} \qquad q(x) = cx + d.$$

$$p(x)q(x) = acx^2 + (ad + bc)x + bd.$$

Gauss technique for polynomials

$p(x) = ax + b$ and $q(x) = cx + d.$

$$p(x)q(x) = acx^2 + (ad + bc)x + bd.$$

$$p(x)q(x) = \underline{ac}x^2 + \big(\underline{(a+b)(c+d)} - ac - bd\big)x + \underline{bd}.$$

Improving the Running Time

$$bc = b(x)c(x) = (b_L x + b_R)(c_L x + c_R)$$

Improving the Running Time

$$\begin{aligned} bc = b(x)c(x) &= (b_L x + b_R)(c_L x + c_R) \\ &= b_L c_L x^2 + (b_L c_R + b_R c_L)x + b_R c_R \end{aligned}$$

Improving the Running Time

$$\begin{aligned} bc &= b(x)c(x) = (b_L x + b_R)(c_L x + c_R) \\ &= b_L c_L x^2 + (b_L c_R + b_R c_L)x + b_R c_R \\ &= (b_L * c_L)x^2 + \Big((b_L + b_R) * (c_L + c_R) - b_L * c_L - b_R * c_R\Big)x \\ &\qquad + b_R * c_R \end{aligned}$$

Improving the Running Time

$$\begin{aligned} bc = b(x)c(x) &= (b_L x + b_R)(c_L x + c_R) \\ &= b_L c_L x^2 + (b_L c_R + b_R c_L)x + b_R c_R \\ &= (b_L * c_L)x^2 + \Big((b_L + b_R) * (c_L + c_R) - b_L * c_L - b_R * c_R\Big)x \\ &\qquad + b_R * c_R \end{aligned}$$

Recursively compute only $b_L c_L$, $b_R c_R$, $(b_L + b_R)(c_L + c_R)$.

Improving the Running Time

$$\begin{aligned} bc &= b(x)c(x) = (b_L x + b_R)(c_L x + c_R) \\ &= b_L c_L x^2 + (b_L c_R + b_R c_L)x + b_R c_R \\ &= (b_L * c_L)x^2 + \Big((b_L + b_R) * (c_L + c_R) - b_L * c_L - b_R * c_R\Big)x \\ &\qquad + b_R * c_R \end{aligned}$$

Recursively compute only $b_L c_L, b_R c_R, (b_L + b_R)(c_L + c_R)$.

Time Analysis
Running time is given by

$$\boxed{T(n) = 3T(n/2) + O(n)} \qquad T(1) = O(1)$$

which means $T(n) = O(n^{\log_2 3}) = O(n^{1.585}) \le O(n^2)$

State of the Art

Schönhage-Strassen 1971: $O(\textcircled{n}\underline{\log n \log \log n})$ time using Fast-Fourier-Transform (FFT)

Martin Fürer 2007: $O(\textcircled{n}\underline{\log n 2^{O(\log^* n)}})$ time

Conjecture: There is an $O(\underline{n}\,\underline{\log n})$ time algorithm.

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