# Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

(Hint) Write a recurrence to analyze the algorithm's running time if we choose a list of size $k$.

# ECE-374-B: Lecture 11 - Backtracking and memoization

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Why did we choose lists of size 5? Will lists of size 3 work?

(Hint) Write a recurrence to analyze the algorithm's running time if we choose a list of size $k$.

# Learning Objectives

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At the end of the lecture, you should be able to understand

- the details of the quickselect and medians of median algorithms,
- the idea of backtracking through the 8-queens puzzle,
- the longest increasing subsequence problem and recursive algorithms to solve it,
- the intuition behind memoization.

# Review linear time selection

Given an array $A = [0, ..., n-1]$ of $n$ numbers and an index $i$, where $0 \leq i \leq n-1$, find the $i^{th}$ smallest element of $A$.

For instance, assume $n = 20$ and $i = 10$.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 15 | 7 | 1 | 17 | 9 | 10 | 14 | 13 | 8 | 18 | 11 | 2 | 12 | 16 | 6 | 19 | 5 | 20 |

The smallest element of rank 10 would be 11. But how do we figure that out

Do median of medians.....

Call Median-of-Medians(A, 10)

# Review linear time selection

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First thing we need to do is find the pivot!

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For instance, assume $n = 20$ and $i = 10$.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 15 | 7 | 1 | 17 | 9 | 10 | 14 | 13 | 8 | 18 | 11 | 2 | 12 | 16 | 6 | 19 | 5 | 20 |

The smallest element of rank 10 would be 11. But how do we figure that out

Do median of medians.....

Call Median-of-Medians(A, 10)

First thing we need to do is find the pivot!

# Review linear time selection

First we reorganize:

| | | | |
|---|---|---|---|
| 4 | 17 | 8 | 16 |
| 3 | 9 | 18 | 6 |
| 15 | 10 | 11 | 19 |
| 7 | 14 | 2 | 5 |
| 1 | 13 | 12 | 20 |

# Review linear time selection

First we reorganize:

| | | | |
|---|---|---|---|
| 4 | 17 | 8 | 16 |
| 3 | 9 | 18 | 6 |
| 15 | 10 | 11 | 19 |
| 7 | 14 | 2 | 5 |
| 1 | 13 | 12 | 20 |

Then we sort each column:

| | | | |
|---|---|---|---|
| 1 | 9 | 2 | 5 |
| 3 | 10 | 8 | 6 |
| 4 | 13 | 11 | 16 |
| 7 | 14 | 12 | 19 |
| 15 | 17 | 18 | 20 |

# Review linear time selection

First we reorganize:

| | | | |
|---|---|---|---|
| 4 | 17 | 8 | 16 |
| 3 | 9 | 18 | 6 |
| 15 | 10 | 11 | 19 |
| 7 | 14 | 2 | 5 |
| 1 | 13 | 12 | 20 |

Then we sort each column:

| | | | |
|---|---|---|---|
| 1 | 9 | 2 | 5 |
| 3 | 10 | 8 | 6 |
| 4 | 13 | 11 | 16 |
| 7 | 14 | 12 | 19 |
| 15 | 17 | 18 | 20 |

Still need the pivot. Find median of medians

# Review linear time selection

| | | | |
|---|---|---|---|
| 1 | 9 | 2 | 5 |
| 3 | 10 | 8 | 6 |
| 4 | 13 | 11 | 16 |
| 7 | 14 | 12 | 19 |
| 15 | 17 | 18 | 20 |

# Review linear time selection

| | | | |
|---|---|---|---|
| 1 | 9 | 2 | 5 |
| 3 | 10 | 8 | 6 |
| 4 | 13 | 11 | 16 |
| 7 | 14 | 12 | 19 |
| 15 | 17 | 18 | 20 |

- Call Median-of-Medians([4,13,11,16], floor(len/2) = 2)
- Can sort this in linear time.
- Get back 13.
- **13** is our new pivot!

# Review linear time selection

Back to our original array! Use the pivot (=13) to break it up into two.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 15 | 7 | 1 | 17 | 9 | 10 | 14 | 13 | 8 | 18 | 11 | 2 | 12 | 16 | 6 | 19 | 5 | 20 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 7 | 1 | 9 | 10 | 8 | 11 | 2 | 12 | 6 | 5 |

| |
|---|
| 13 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 15 | 17 | 14 | 18 | 16 | 19 | 20 |

We know the following:

- $\text{len}(A_{Lower}) = 12$
- $\text{len}(A_{Upper}) = 7$
- Want $k = 10$

# Review linear time selection

Back to our original array! Use the pivot (=13) to break it up into two.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 15 | 7 | 1 | 17 | 9 | 10 | 14 | 13 | 8 | 18 | 11 | 2 | 12 | 16 | 6 | 19 | 5 | 20 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 7 | 1 | 9 | 10 | 8 | 11 | 2 | 12 | 6 | 5 |

| |
|---|
| 13 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 15 | 17 | 14 | 18 | 16 | 19 | 20 |

We know the following:

- $\text{len}(A_{Lower}) = 12$
- $\text{len}(A_{Upper}) = 7$
- Want $k = 10$

Call Median-of-Medians$(A_{Lower}, 10)$

# Review linear time selection

Then we do this again:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 7 | 1 | 9 | 10 | 8 | 11 | 2 | 12 | 6 | 5 |

# Review linear time selection

Then we do this again:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 7 | 1 | 9 | 10 | 8 | 11 | 2 | 12 | 6 | 5 |

First we reorganize:

| | | |
|---|---|---|
| 4 | 10 | |
| 3 | 8 | 6 |
| 7 | 11 | 5 |
| 1 | 2 | |
| 9 | 12 | |

# Review linear time selection

Then we do this again:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 7 | 1 | 9 | 10 | 8 | 11 | 2 | 12 | 6 | 5 |

First we reorganize:

| | | |
|---|---|---|
| 4 | 10 | |
| 3 | 8 | 6 |
| 7 | 11 | 5 |
| 1 | 2 | |
| 9 | 12 | |

Then we sort each column:

| | | |
|---|---|---|
| 1 | 2 | |
| 3 | 8 | 5 |
| 4 | 10 | 6 |
| 7 | 11 | |
| 9 | 12 | |

# Review linear time selection

| | | |
|---|---|---|
| 1 | 2 | |
| 3 | 8 | 5 |
| 4 | 10 | 6 |
| 7 | 11 | |
| 9 | 12 | |

# Review linear time selection

| | | |
|---|---|---|
| 1 | 2 | |
| 3 | 8 | 5 |
| 4 | 10 | 6 |
| 7 | 11 | |
| 9 | 12 | |

- Call Median-of-Medians([4,10,6], floor(len/2) = 1)
- Can sort this in linear time.
- Get back 6.
- **6** is our new pivot!

# Review linear time selection

Back to our original array! Use the pivot (=12) to break it up into two (well three).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 7 | 1 | 9 | 10 | 8 | 11 | 2 | 12 | 6 | 5 |

| | | | | |
|---|---|---|---|---|
| 4 | 3 | 1 | 2 | 5 |

| |
|---|
| 6 |

| | | | | | |
|---|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 | 12 |

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 6$
- Want $k = 10$ (pivot is of rank 6)

# Review linear time selection

Back to our original array! Use the pivot (=12) to break it up into two (well three).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 7 | 1 | 9 | 10 | 8 | 11 | 2 | 12 | 6 | 5 |

| | | | | |
|---|---|---|---|---|
| 4 | 3 | 1 | 2 | 5 |

| |
|---|
| 6 |

| | | | | | |
|---|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 | 12 |

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 6$
- Want $k = 10$ (pivot is of rank 6)

Call Median-of-Medians$(A_{Upper}, 10 - 6 = 4)$

Then we do this again:

| | | | | | |
|---|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 | 12 |

# Review linear time selection

Then we do this again:

| | | | | | |
|---|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 | 12 |

First we reorganize:

| | |
|---|---|
| 7 | |
| 9 | |
| 10 | 12 |
| 8 | |
| 11 | |

# Review linear time selection

Then we do this again:

| | | | | | |
|---|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 | 12 |

First we reorganize:

| | |
|---|---|
| 7 | |
| 9 | |
| 10 | 12 |
| 8 | |
| 11 | |

Then we sort each column:

| | |
|---|---|
| 7 | |
| 8 | |
| 9 | 12 |
| 10 | |
| 11 | |

# Review linear time selection

| | |
|---|---|
| 7 | |
| 8 | |
| 9 | 12 |
| 10 | |
| 11 | |

# Review linear time selection

| | |
|---|---|
| 7 | |
| 8 | |
| 9 | 12 |
| 10 | |
| 11 | |

- Call Median-of-Medians([9,12], floor(len/2) = 1)
- Can sort this in linear time.
- Get back 12.
- **12** is our new pivot!

# Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

| | | | | | |
|---|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 | 12 |

| | | | | |
|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 |

| |
|---|
| 12 |

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 0$
- Want $k = 4$ (pivot is of rank 5)

# Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

| | | | | | |
|---|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 | 12 |

| | | | | |
|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 |

| |
|---|
| 12 |

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 0$
- Want $k = 4$ (pivot is of rank 5)

Call Median-of-Medians$(A_{Lower}, 4)$

Final Step!

| | | | | |
|---|---|---|---|---|
| 7 | 9 | 10 | 8 | 11 |

Can sort in linear time!

| | | | | |
|---|---|---|---|---|
| 7 | 8 | 9 | 10 | 11 |

Return $Sorted(A[4]) = 11$

# Median of medians time analysis

```
Median-of-medians(A, i):
    sublists = [A[j:j+5] for j ∈range(0, len(A), 5)]
    medians = [sorted (sublist)[len (sublist)/2] for sublist ∈sublists]

    // Base Case
    if len (A) ≤ 5 return sorted (a)[i]

    // Find median of medians
    if len (medians) ≤ 5
        pivot = sorted (medians)[len (medians)/2]
    else
        pivot = Median-of-medians (medians, len/2)

    // Partitioning Step
    low = [j for j ∈A if j < pivot]
    high = [j for j ∈A if j > pivot]

    k = len (low)
    if i < k
        return Median-of-medians (low, i)
    elseif i > k
        return Median-of-medians (low, i-k-1)
    else
    return pivot
```

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```
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```

$$T(n) = T(\frac{1}{5}n) + T(\frac{7}{10}n) + cn$$

# Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

# Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

# Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

What about $k = 7$?

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$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

What about $k = 7$?

$$T(n) = T(\frac{1}{7}n) + T(\frac{10}{14}n) + cn$$

# On different techniques for recursive algorithms

**Reduction:** Reduce one problem to another

**Recursion**
A special case of reduction

- reduce problem to a smaller instance of itself
- self-reduction

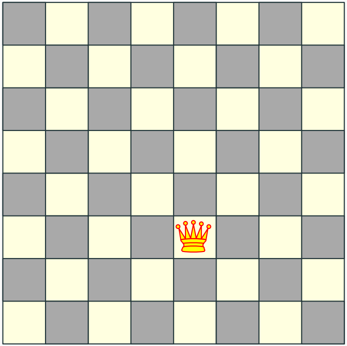
- Problem instance of size $n$ is reduced to one or more instances of size $n-1$ or less.
- For termination, problem instances of small size are solved by some other method as base cases.

# Recursion in Algorithm Design

- Tail Recursion: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.
  **Examples:** Interval scheduling, MST algorithms....
- Divide and Conquer: Problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem.
  **Examples:** Closest pair, median selection, quick sort.
- Backtracking: Refinement of brute force search. Build solution incrementally by invoking recursion to try all possibilities for the decision in each step.
- Dynamic Programming: problem reduced to multiple (typically) dependent or overlapping sub-problems. Use memoization to avoid recomputation of common solutions leading to iterative bottom-up algorithm.

# Search trees and backtracking

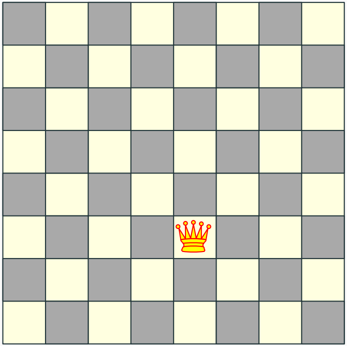
# The queens problem



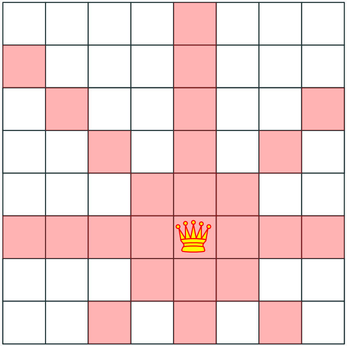
Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board?

# The queens problem



# The queens problem



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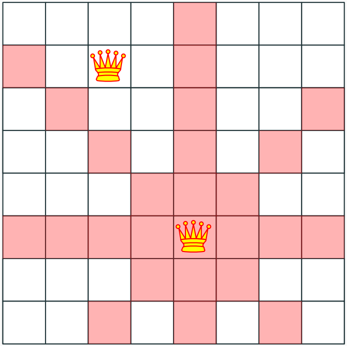

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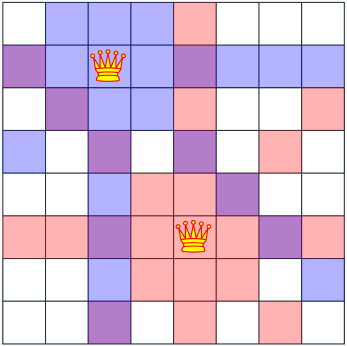

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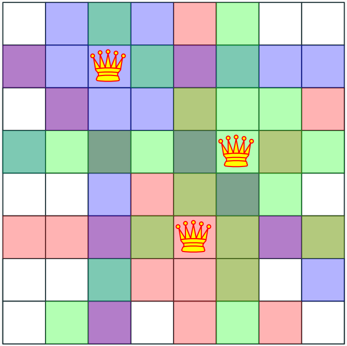
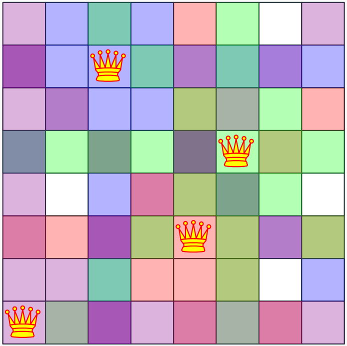
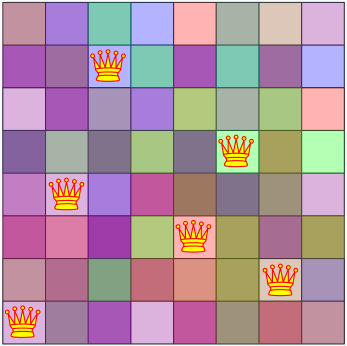


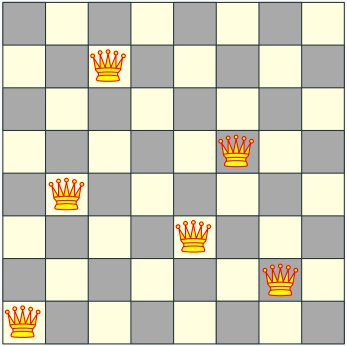
# The queens problem



# The queens problem



# The queens problem

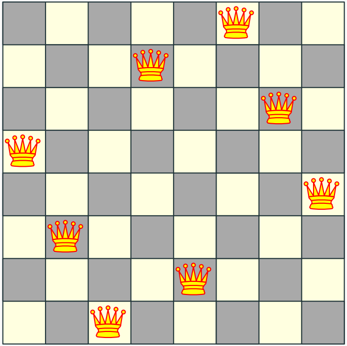


Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board? How many permutations?

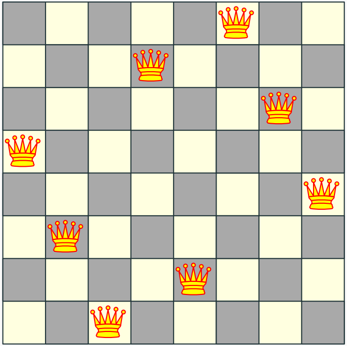
# The eight queens puzzle

Problem published in 1848, solved in 1850.

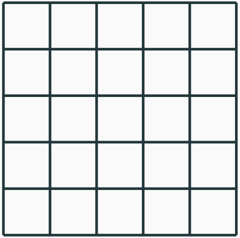


# The eight queens puzzle

Problem published in 1848, solved in 1850.

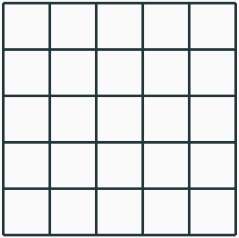


Q: How to solve problem for general *n*?



What if we attempt to find all the possible permutations and then check?

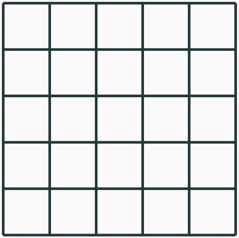
# Search tree for 5 queens

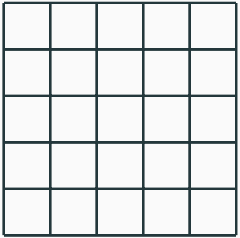


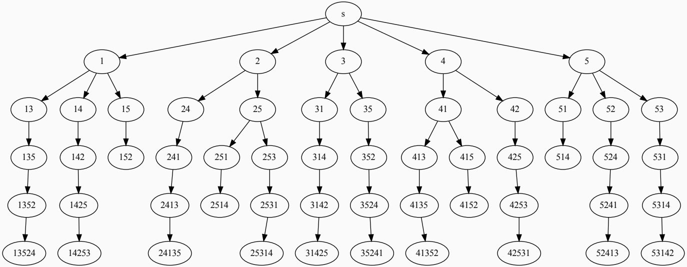
Let's be a bit smarter and recognize that:

- Queens can't be on the same row, column or diagonal
- Can have *n* queens max.

# Search tree for 5 queens







# Backtracking: Informal definition

Recursive search over an implicit tree, where we “backtrack” if certain possibilities do not work.

# n queens C++ code

```
void  generate_permutations( int  * permut, int  row, int  n )
{
   if  ( row == n ) {
      print_board( permut, n );
      return;
   }

   for  ( int  val = 1; val <= n; val++ )
      if  ( isValid( permut, row, val ) ) {
         permut[ row ] = val;
         generate_permutations( permut, row + 1, n );
      }
}

generate_permutations( permut, 0, 8 );
```

# Quick note: n queens - number of solutions

| N | Number of Solutions | Number of Unique Solutions |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 0 | 0 |
| 3 | 0 | 0 |
| 4 | 2 | 1 |
| 5 | 10 | 2 |
| 6 | 4 | 1 |
| 7 | 40 | 6 |
| 8 | 92 | 12 |
| 9 | 352 | 46 |
| 10 | 724 | 92 |
| 11 | 2,680 | 341 |
| 12 | 14,200 | 1,787 |
| 13 | 73,712 | 9,233 |
| 14 | 365,596 | 45,752 |
| 15 | 2,279,184 | 285,053 |

# Longest Increasing Sub-sequence

# Sequences

**Definition**
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

**Definition**
$a_{i_1}, \ldots, a_{i_k}$ is a subsequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

**Definition**
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

# Sequences - Example...

## Example

- Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
- Subsequence of above sequence: $5, 2, 1$
- Increasing sequence: $3, 5, 9, 17, 54$
- Decreasing sequence: $34, 21, 7, 5, 1$
- Increasing <u>subsequence</u> of the first sequence: $2, 7, 9$.

# Longest Increasing Subsequence Problem

**Input** A sequence of numbers $a_1, a_2, \ldots, a_n$

**Goal** Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

# Longest Increasing Subsequence Problem

**Input** A sequence of numbers $a_1, a_2, \ldots, a_n$

**Goal** Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

# Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array $A$

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

# Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array $A$

```
algLISNaive(A[1..n]):
    max = 0
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    Output max
```

Running time:

# Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array $A$

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

$2^n$ subsequences of a sequence of length $n$ and $O(n)$ time to check if a given sequence is increasing.

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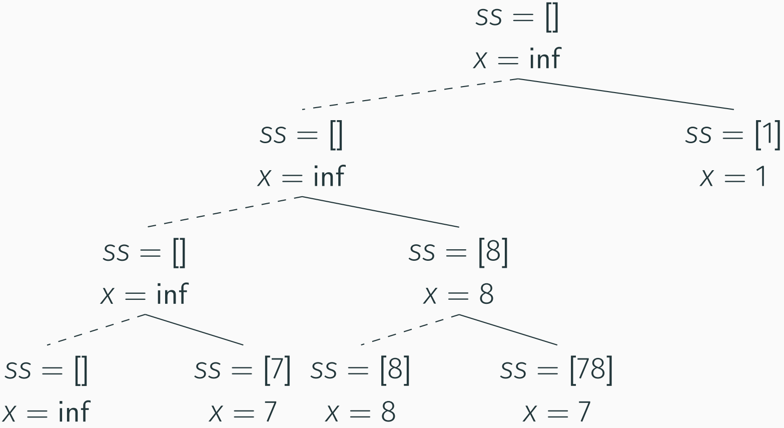
**Observation**
*For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is* LIS_smaller($A[1..n], x$) *which gives the longest increasing subsequence in $A$ where each number in the sequence is less than $x$.*

# Example

Sequence: $A[1..5] = 5, 9, 7, 8, 1$



# Recursive Approach

**LIS_smaller**$(A[1..n], x)$ : length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than $x$

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
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```

```
LIS(A[1..n]):
        return LIS_smaller(A[1..n], ∞)
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# Running time analysis

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....one can do much better using memoization!