Pre-lecture brain teaser

We saw a linear time selection algorithm in the previous lecture.

Quick Select + MoM

Why did we choose lists of size 5? Will lists of size 3 work?

(Hint) Write a recurrence to analyze the algorithm's running time if we choose a list of size *k*.

ECE-374-B: Lecture 11 - Backtracking and memoization

Instructor: Abhishek Kumar Umrawal

February 27, 2024

University of Illinois at Urbana-Champaign

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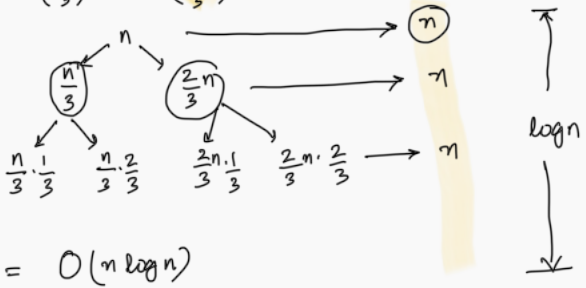
We saw a linear time selection algorithm in the previous lecture.

Why did we choose lists of size 5? Will lists of size 3 work?

(Hint) Write a recurrence to analyze the algorithm's running time if we choose a list of size *k*.

$k=3:$ $\quad T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2}{3}n\right) + O(n)$



$\Rightarrow \quad T(n) = O(n \log n)$

$k=7$: $T(n) = T\left(\frac{n}{7}\right) + T\left(\frac{5}{7}n\right) + O(n)$

Learning Objectives

Learning Objectives

At the end of the lecture, you should be able to understand

- the details of the quickselect and medians of median algorithms,
- the idea of backtracking through the 8-queens puzzle,
- the longest increasing subsequence problem and recursive algorithms to solve it,
- the intuition behind memoization.

Review linear time selection

Given an array $A = [0, ..., n-1]$ of n numbers and an index i, where $0 \leq i \leq n-1$, find the i^{th} smallest element of A.

For instance, assume $n = 20$ and $i = 10$.

4	3	15	7	1	17	9	10	14	13	8	18	11	2	12	16	6	19	5	20

The smallest element of rank 10 would be 11. But how do we figure that out

Do median of medians.....

(MoM)

Call Median-of-Medians(A, 10)

↑

Quick select + MoM

Review linear time selection

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First thing we need to do is find the pivot!

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Do median of medians.....

Call Median-of-Medians(A, 10)

First thing we need to do is find the pivot!

Review linear time selection

First we reorganize:

4	17	8	16
3	9	18	6
15	10	11	19
7	14	2	5
1	13	12	20

Review linear time selection

First we reorganize:

$4 = n/5$

5

4	17	8	16
3	9	18	6
15	10	11	19
7	14	2	5
1	13	12	20

Then we sort each column:

1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

Review linear time selection

First we reorganize:

$\frac{n}{5} = \frac{20}{5}$

4

5

4	17	8	16
3	9	18	6
15	10	11	19
7	14	2	5
1	13	12	20

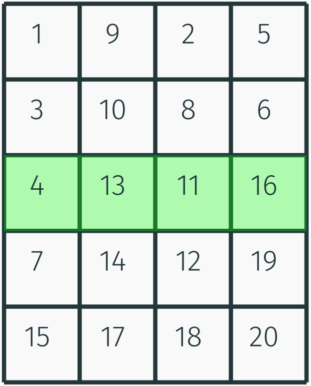
Then we sort each column:

1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

4 #s $= \frac{n}{5}$

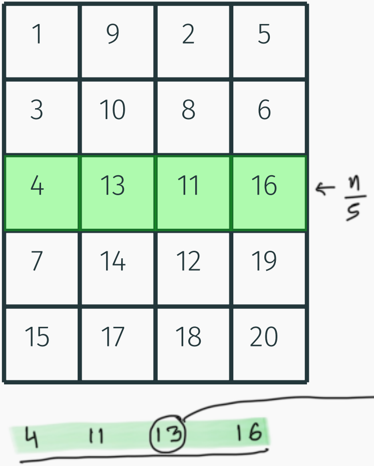
Still need the pivot. Find median of medians

Review linear time selection



1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

Review linear time selection



1	9	2	5
3	10	8	6
4	13	11	16
7	14	12	19
15	17	18	20

$\leftarrow \frac{n}{5}$

4 11 13 16

- Call Median-of-Medians([4,13,11,16], floor(len/2) = 2)
- Can sort this in linear time.
- Get back 13.
- 13 is our new pivot!

Review linear time selection

Back to our original array! Use the pivot (=13) to break it up into two.

									✓										
4	3	15	7	1	17	9	10	14	13	8	18	11	2	12	16	6	19	5	20

4	3	7	1	9	10	8	11	2	12	6	5

A_{lower}

13

15	17	14	18	16	19	20

A_{upper}

We know the following:

- $\text{len}(A_{Lower}) = 12$
- $\text{len}(A_{Upper}) = 7$
- Want $k = 10$

Review linear time selection

Back to our original array! Use the pivot (=13) to break it up into two.

4	3	15	7	1	17	9	10	14	13	8	18	11	2	12	16	6	19	5	20

4	3	7	1	9	10	8	11	2	12	6	5

13

15	17	14	18	16	19	20

We know the following:

- $\text{len}(A_{Lower}) = 12$
- $\text{len}(A_{Upper}) = 7$
- Want $k = 10$

Call Median-of-Medians(A_{Lower}, 10)

Review linear time selection

Then we do this again:

4	3	7	1	9	10	8	11	2	12	6	5

Review linear time selection

Then we do this again:

A_{Lower}:

4	3	7	1	9	10	8	11	2	12	6	5

First we reorganize:

4	10	
3	8	6
7	11	5
1	2	
9	12	

Review linear time selection

Then we do this again:

4	3	7	1	9	10	8	11	2	12	6	5

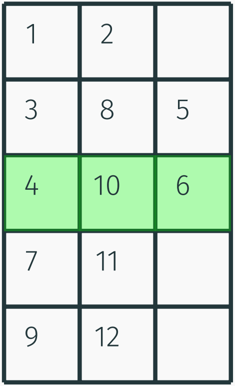
First we reorganize:

4	10	
3	8	6
7	11	5
1	2	
9	12	

Then we sort each column:

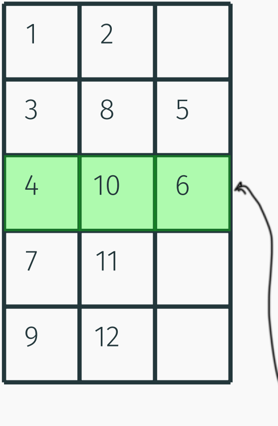
1	2	
3	8	5
4	10	6
7	11	
9	12	

Review linear time selection



1	2	
3	8	5
4	10	6
7	11	
9	12	

Review linear time selection



1	2	
3	8	5
4	10	6
7	11	
9	12	

length : f(n)

- Call Median-of-Medians([4,10,6], floor(len/2) = 1)

$\frac{n}{2}$ e.g. $\approx \frac{12}{5}$

- Can sort this in linear time.
- Get back 6.
- **6** is our new pivot!

Review linear time selection

Back to our original array! Use the pivot (=12) to break it up into two (well three).

4	3	7	1	9	10	8	11	2	12	6	5

4	3	1	2	5

6

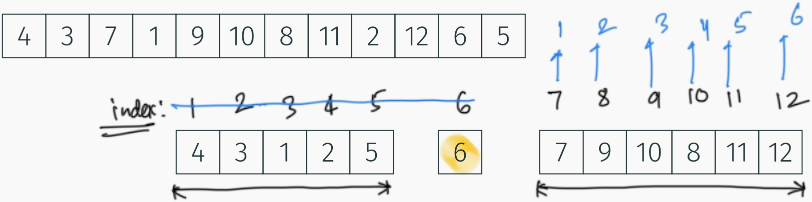
7	9	10	8	11	12

We know the following:

- $\text{len}(A_{Lower}) = 5$
- $\text{len}(A_{Upper}) = 6$
- Want $k = 10$ (pivot is of rank 6)

Review linear time selection

Back to our original array! Use the pivot (=12) to break it up into two (well three).



We know the following:

- len(A_{Lower}) $= 5$
- len(A_{Upper}) $= 6$
- Want $k = 10$ (pivot is of rank 6)

Call Median-of-Medians(A_{Upper}, $10 - 6 = 4$)

Review linear time selection

Then we do this again:

7	9	10	8	11	12

Review linear time selection

Then we do this again:

7	9	10	8	11	12

First we reorganize:

7	
9	
10	12
8	
11	

Review linear time selection

Then we do this again:

7	9	10	8	11	12

First we reorganize:

7	
9	
10	12
8	
11	

Then we sort each column:

7	
8	
9	12
10	
11	

Review linear time selection

7	
8	
9	12
10	
11	

Review linear time selection

7	
8	
9	12
10	
11	

- Call Median-of-Medians([9,12], floor(len/2) = 1)
- Can sort this in linear time.
- Get back 12.
- **12** is our new pivot!

Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

7	9	10	8	11	12

7	9	10	8	11

12

We know the following:

- len(A_{Lower}) $= 5$
- len(A_{Upper}) $= 0$
- Want $k = 4$ (pivot is of rank 5)

Review linear time selection

Back to our original array! Use the pivot (=6) to break it up into two (well three).

7	9	10	8	11	12

7	9	10	8	11

12

We know the following:

- len(A_{Lower}) $= 5$
- len(A_{Upper}) $= 0$
- Want $k = 4$ (pivot is of rank 5)

Call Median-of-Medians(A_{Lower}, 4)

Review linear time selection

Final Step!

7	9	10	8	11

Can sort in linear time!

7	8	9	10	11

Return $\mathit{Sorted}(A[4]) = 11$

Median of medians time analysis

```
Median-of-medians(A, i):
    sublists = [A[j:j+5] for j ∈range(0, len(A), 5)]
    medians = [sorted (sublist)[len (sublist)/2] for sublist ∈sublists]

    // Base Case
    if len (A) ≤ 5 return sorted (a)[i]

    // Find median of medians
    if len (medians) ≤ 5
        pivot = sorted (medians)[len (medians)/2]
    else
        pivot = Median-of-medians (medians, len/2)

    // Partitioning Step
    low = [j for j ∈A if j < pivot]
    high = [j for j ∈A if j > pivot]

    k = len (low)
    if i < k
        return Median-of-medians (low, i)
    elseif i > k
        return Median-of-medians (low, i-k-1)
    else
    return pivot
```

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✓ $$T(n) = T(\frac{1}{5}n) + T(\frac{7}{10}n) + cn$$

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$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

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$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

What about $k = 7$?

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$$T(n) = T(\frac{1}{3}n) + T(\frac{4}{6}n) + cn$$

What about $k = 7$?

$$T(n) = T(\frac{1}{7}n) + T(\frac{10}{14}n) + cn$$

On different techniques for recursive algorithms

Recursion

Reduction: Reduce one problem to another

Recursion
A special case of reduction

- reduce problem to a smaller instance of itself
- self-reduction

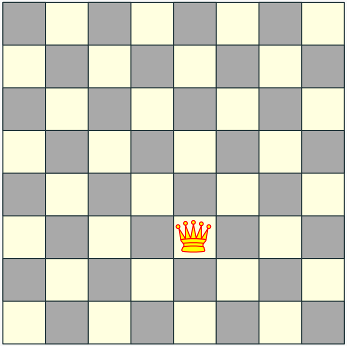
- Problem instance of size n is reduced to one or more instances of size $n - 1$ or less.
- For termination, problem instances of small size are solved by some other method as base cases.

Recursion in Algorithm Design

- Tail Recursion: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms.
  Examples: Interval scheduling, MST algorithms....
- Divide and Conquer: Problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem.
  Examples: Closest pair, median selection, quick sort.
- Backtracking: Refinement of brute force search. Build solution incrementally by invoking recursion to try all possibilities for the decision in each step.
- Dynamic Programming: problem reduced to multiple (typically) dependent or overlapping sub-problems. Use memoization to avoid recomputation of common solutions leading to iterative bottom-up algorithm.

Search trees and backtracking

The queens problem

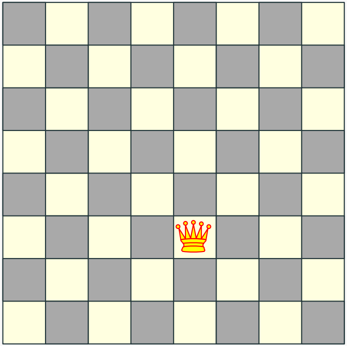


8×8 board
8 queens

Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board? YES!

The queens problem



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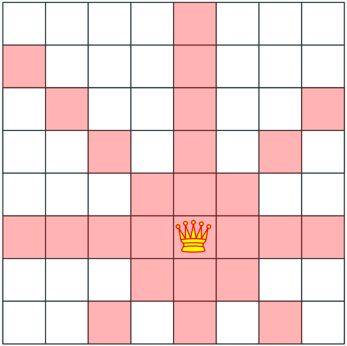

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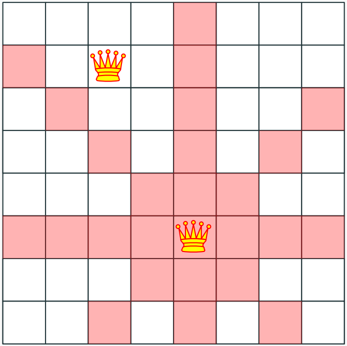

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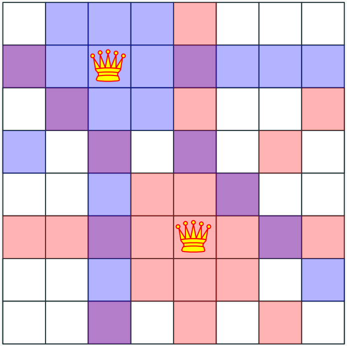
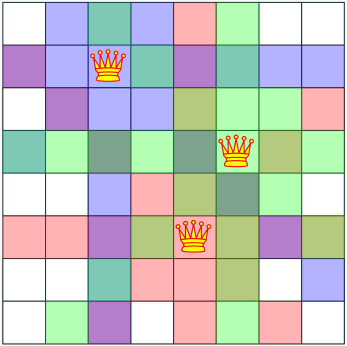


The queens problem



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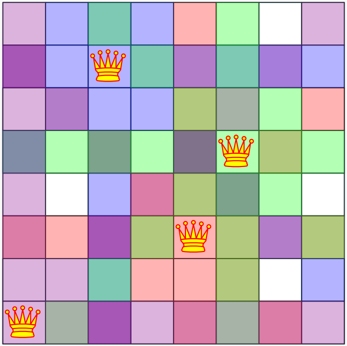

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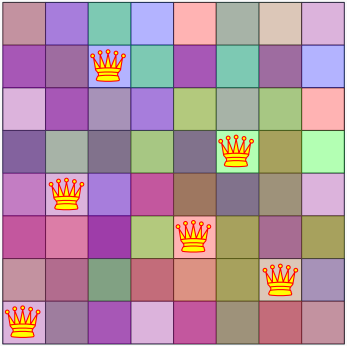
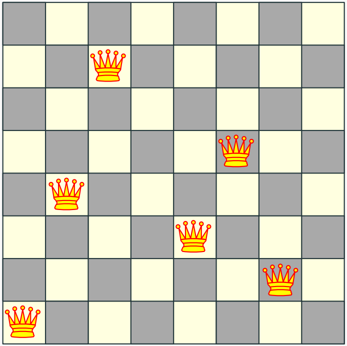


The queens problem



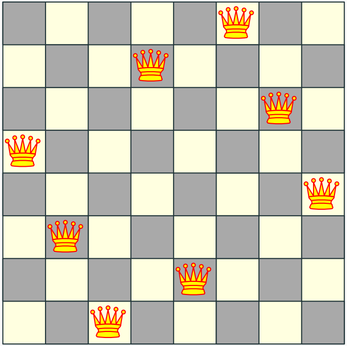
Q: How many queens can one place on the board?

Q: Can one place 8 queens on the board? YES How many permutations?

The eight queens puzzle

Max Bezzel Franz Nauck

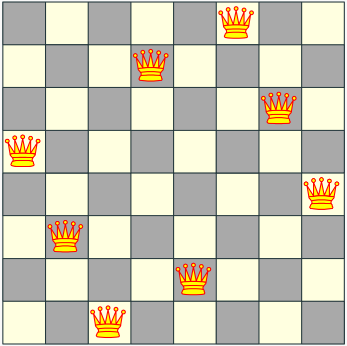
Problem published in 1848, solved in 1850.



8x8 board
8 queens
: 92 distinct solution

The eight queens puzzle

Problem published in 1848, solved in 1850.

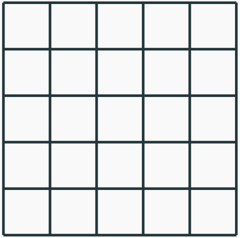


$n \times n$ board

n queens

Q: How to solve problem for general n?

Introducing concept of state tree



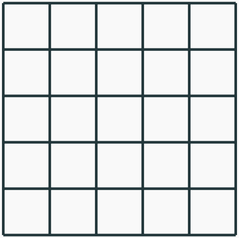
What if we attempt to find all the possible permutations and then check?

$n = 8$

$$\binom{8 \times 8 = 64}{8} = 4,426,165,368$$

$\Rightarrow$ 140 years @ 1 second/perm.

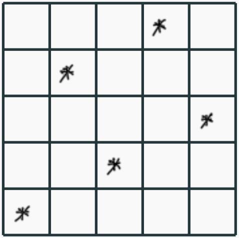
Search tree for 5 queens



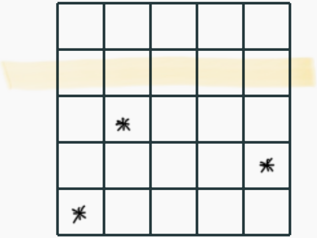
Let’s be a bit smarter and recognize that:

- Queens can’t be on the same row, column or diagonal
- Can have *n* queens max.

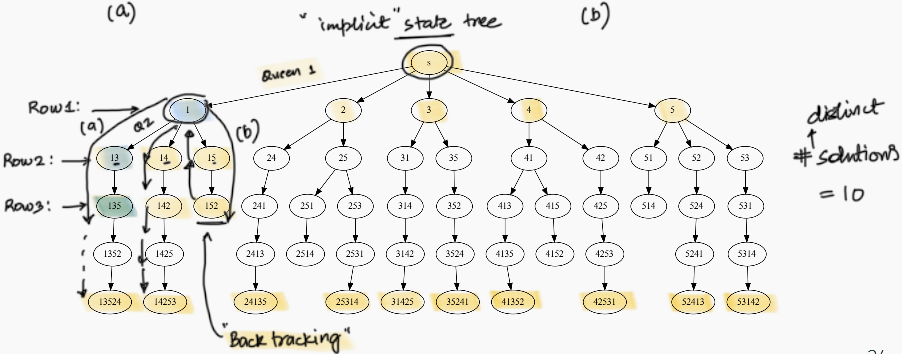
Search tree for 5 queens



(a)



(b)



" implicit " state tree

"Back tracking"

distinct
↑
solutions
= 10

Backtracking: Informal definition

Recursive search over an implicit tree, where we "backtrack" if certain possibilities do not work.

n queens C++ code

```
void  generate_permutations( int  * permut, int  row, int  n )
{
    if  ( row == n ) {
        print_board( permut, n );
        return;
    }

    for  ( int  val = 1; val <= n; val++ )
        if  ( isValid( permut, row, val ) ) {
            permut[ row ] = val;
            generate_permutations( permut, row + 1, n );
        }
}

generate_permutations( permut, 0, 8 );
```

Quick note: n queens - number of solutions

distinct

fundamental

n ~~N~~	Number of Solutions	Number of Unique Solutions
1	1	1
2	0	0
3	0	0
4	2	1
5	10	2
6	4	1
7	40	6
8	92	12
9	352	46
10	724	92
11	2,680	341
12	14,200	1,787
13	73,712	9,233
14	365,596	45,752
15	2,279,184	285,053

Longest Increasing Sub-sequence

Sequences

Definition
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

Definition
$a_{i_1}, \ldots, a_{i_k}$ is a subsequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Sequences - Example...

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1, 9
- Subsequence of above sequence: 5, 2, 1 1 2 5 (x)
- Increasing sequence: 3, 5, 9, 17, 54 : strictly increasing
- Decreasing sequence: 34, 21, 7, 5, 1
- Increasing subsequence of the first sequence: 2, 7, 9.

3 5 5 9 17 54 : non- decreasing

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

1 2 9 12 18 $n = 5$

$\underbrace{\text{LIS: } \underline{1 \quad 2 \quad 9 \quad 12 \quad 18}}$ $\underbrace{\#(\text{LIS}) = 5}$

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

Naive Enumeration

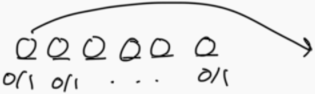
Assume $a_1, a_2, \dots, a_n$ is contained in an array A

```
            algLISNaive($A[1..n]$):
                $max = 0$
$2^n$ --->      for each subsequence $B$ of $A$ do
     $n$ --->       if $B$ is increasing and $|B| > max$ then
                        $max = |B|$

                Output $max$
```

length of the given seq. $|A| = n$

Run time: $O(n 2^n)$



$2 \; 2 \; \dots \; 2 = 2^n$

Naive Enumeration

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    $max = 0$
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        if $B$ is increasing and $|B| > max$ then
            $max = |B|$

    Output $max$
```

Running time:

Naive Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

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algLISNaive($A[1..n]$):
    $max = 0$
    for each subsequence $B$ of $A$ do
        if $B$ is increasing and $|B| > max$ then
            $max = |B|$

    Output $max$
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Recursive Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[1..n]$):

Recursive Approach: LIS: Longest increasing subsequence

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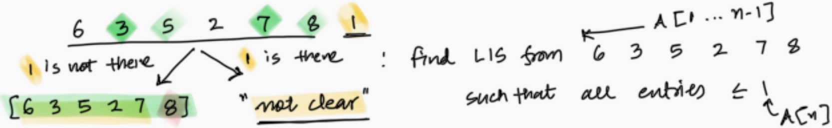
- Case 1: Does not contain $A[n]$ in which case LIS($A[1..n]$) = LIS($A[1..(n-1)]$)
- Case 2: contains $A[n]$ in which case LIS($A[1..n]$) is



Recursive Approach: LIS: Longest increasing subsequence

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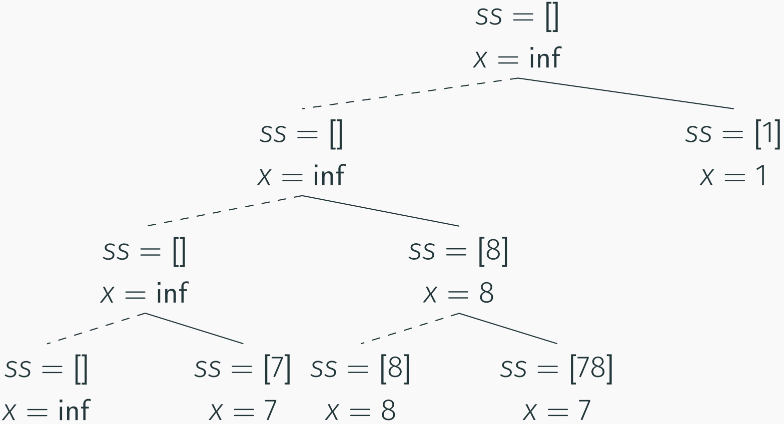
Observation

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $\underline{A[n]}$. This suggests that a more general problem is **LIS_smaller**($A[1..n], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Example

Sequence: $A[1..5] = 5, 9, 7, 8, 1$



$ss = []$
$x = \text{inf}$

$ss = []$
$x = \text{inf}$

$ss = [1]$
$x = 1$

$ss = []$
$x = \text{inf}$

$ss = [8]$
$x = 8$

$ss = []$
$x = \text{inf}$

$ss = [7]$
$x = 7$

$ss = [8]$
$x = 8$

$ss = [78]$
$x = 7$

Recursive Approach

LIS_smaller($A[1..n], x$) : length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller($A[1..n]$, $x$):
                    if ($n = 0$) then return 0
                    $m$ = LIS_smaller($A[1..(n-1)]$, $x$)   ← when we don't include the last element
we include A[n] →   if ($A[n] < x$) then
                        $m = max(m, 1 +$ LIS_smaller($A[1..(n-1)], A[n]$))
                    Output $m$
```

```
LIS($A[1..n]$):
        return LIS_smaller($A[1..n]$, $\infty$)
```

($\max(A) + 1$)

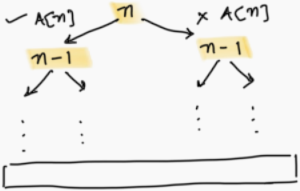
$O(n)$

Running time analysis

Running time of LIS([1..n])

```
LIS_smaller($A[1..n], x$):
    if ($n = 0$) then return 0
    $m$ = LIS_smaller($A[1..(n-1)], x$)
    if ($A[n] < x$) then
        $m = max(m, 1 +$ LIS_smaller($A[1..(n-1)], A[n]$))
    Output $m$
```

```
LIS($A[1..n]$):
        return LIS_smaller($A[1..n], \infty$)
```



2^n subproblems

Runtime : $O(2^n)$

Running time of LIS([1..n])

Lemma

LIS_smaller *runs in* $O(2^n)$ *time.*

Naive ⟶ Recursion ⟶ Can we improve ?

$O(n2^n) \longrightarrow O(2^n) \longrightarrow$

Running time of LIS([1..n])

Lemma
LIS_smaller *runs in* $O(2^n)$ *time.*

Improvement: From $O(n2^n)$ to $O(2^n)$.

Running time of LIS([1..n])

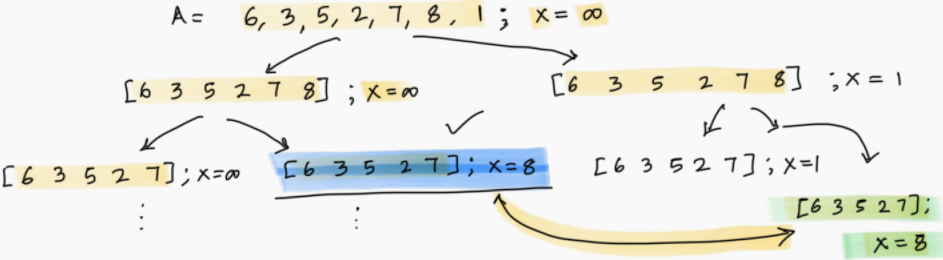
Lemma
LIS_smaller *runs in* $O(2^n)$ *time.*

$\uparrow (2^n - n^2)$
"Waste of time"

Improvement: From $O(n2^n)$ to $O(2^n)$.

$|A| = n$

....one can do much better using memoization!



$(n+1)$ prefixes of A

# of choices of x : $(n+1)$

$\rightarrow$ # of distinct subproblems:

$(n+1)(n+1)$

$= O(n^2)$