Write a (very simple) **recursive** algorithm that calculates the Fibonacci $n^{th}$ number.

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = 0, F_1 = 1$$

- **Indian Mathematician**
  - Ācārya Pingala in 200 BC

- **Named after Italian Mathematician**
  - Leonardo of Pisa aka Fibonacci (1202)
Write a (very simple) recursive algorithm that calculates the Fibonacci $n^{th}$ number.

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = 0, F_1 = 1$$
Learning Objectives
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At the end of the lecture, you should be able to understand

- the concepts of the memoization and dynamic programming,
- how to improve the time and space complexities of recursive algorithms using the above concepts,
- dynamic programming for the fibonacci numbers and longest increasing subsequence problem, and
- where and how to use dynamic programming to refine recursive algorithms.
Recursion and Memoization
Fibonacci Numbers

Fibonacci numbers defined by recurrence:

\[ F(n) = F(n - 1) + F(n - 2) \text{ and } F(0) = 0, F(1) = 1. \]

These numbers have many interesting properties. A journal The Fibonacci Quarterly!
Fibonacci Numbers

Fibonacci numbers defined by recurrence:

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These numbers have many interesting properties. A journal *The Fibonacci Quarterly*¹!

- **Binet’s formula**: \( F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \approx \frac{1.618^n - (-0.618)^n}{\sqrt{5}} \approx \frac{1.618^n}{\sqrt{5}} \)

  \( \varphi \) is the **golden ratio** \( (1 + \sqrt{5})/2 \approx 1.618. \)

- \( \lim_{n \to \infty} F(n + 1)/F(n) = \varphi \)

---

² *Most visually appealing* frame/window/door,...
Question: Given $n$, compute $F(n)$.

$$\text{Fib}(n):$$

```plaintext
if $(n = 0)$
  return 0
else if $(n = 1)$
  return 1
else
  return $\text{Fib}(n - 1) + \text{Fib}(n - 2)$
```

Running time? Let $T(n)$ be the number of additions in Fib($n$).

$$T(n) = T(n - 1) + T(n - 2) + 1$$ and $T(0) = T(1) = 0$
Question: Given $n$, compute $F(n)$.

```python
Fib(n):
    if ($n = 0$)
        return 0
    else if ($n = 1$)
        return 1
    else
        return Fib($n - 1$) + Fib($n - 2$)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + O(1)$$
\[ T(n) \leq 1 + 2 + 4 + \cdots + 2^n = O(2^n) \]

\[
\Rightarrow \quad \text{# of leaves} : \quad O(2^n)
\]

\[
T(n) = 1 \cdot O(2^n) \quad \text{Additions}
\]

\[
\text{Exact bound: } \quad T(n) = \Theta(\phi^n)
\]

\[
\phi = 1.6 < 2
\]
Recursive Algorithm for Fibonacci Numbers

Question: Given $n$, compute $F(n)$.

\[
\text{Fib}(n) :
\]
\[
\begin{cases}
    \text{if } (n = 0) & \text{return 0} \\
    \text{else if } (n = 1) & \text{return 1} \\
    \text{else} & \text{return Fib}(n - 1) + \text{Fib}(n - 2)
\end{cases}
\]

Running time? Let $T(n)$ be the number of additions in Fib(n).

\[
T(n) = T(n - 1) + T(n - 2) + 1 \text{ and } T(0) = T(1) = 0
\]
Recursive Algorithm for Fibonacci Numbers

Question: Given $n$, compute $F(n)$.

\[
\text{Fib}(n):
\]
\[
\begin{array}{l}
\text{if } (n = 0) \\
\quad \text{return } 0 \\
\text{else if } (n = 1) \\
\quad \text{return } 1 \\
\text{else} \\
\quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\end{array}
\]

Running time? Let $T(n)$ be the number of additions in Fib(n).

\[
T(n) = T(n - 1) + T(n - 2) + 1 \text{ and } T(0) = T(1) = 0
\]

Roughly same as $F(n)$: $T(n) = \Theta(\varphi^n)$.

The number of additions is exponential in $n$. Can we do better?
Recursion tree for the Recursive Fibonacci

0 1
Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci

Observation: OPTIMAL SUBSTRUCTURE
'A recursive solution contains a small number of distinct subproblems repeated many times.'
An iterative algorithm for Fibonacci numbers

\[ \text{FibIter}(n): \]
\[
\text{if } (n = 0) \text{ then return 0}
\]
\[
\text{if } (n = 1) \text{ then return 1}
\]
\[
F[0] = 0
\]
\[
F[1] = 1
\]
\[
\text{for } i = 2 \text{ to } n \text{ do}
\]
\[
F[i] = F[i - 1] + F[i - 2]
\]
\[
\text{return } F[n]
\]

**Iterative bottom-up calculation.**

\[ T(n) = O(n) \text{ additions} \]
An iterative algorithm for Fibonacci numbers

\[
\text{FibIter}(n):
\]
\[
\begin{align*}
\text{if } (n = 0) & \text{ then} \\
& \text{return } 0 \\
\text{if } (n = 1) & \text{ then} \\
& \text{return } 1 \\
F[0] &= 0 \\
F[1] &= 1 \\
& \text{for } i = 2 \text{ to } n \text{ do} \\
& \quad F[i] = F[i - 1] + F[i - 2] \\
& \text{return } F[n]
\end{align*}
\]

What is the running time of the algorithm?

\(O(n)\) additions!
An iterative algorithm for Fibonacci numbers

\[ \text{FibIter}(n): \]
\[
\begin{align*}
\text{if } (n = 0) & \text{ then return } 0 \\
\text{if } (n = 1) & \text{ then return } 1 \\
F[0] & = 0 \\
F[1] & = 1 \\
\text{for } i = 2 \text{ to } n & \text{ do } \\
& F[i] = F[i - 1] + F[i - 2] \\
\text{return } F[n]
\end{align*}
\]

What is the running time of the algorithm? \( O(n) \) additions.
What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value.
What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. Memoization.
What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. **Memoization**.

Dynamic Programming: Finding a recursion that can be effectively/efficiently memorized.

\[O(n^k) \text{ runtime for some constant } k\]

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
Implicit vs. explicit memoization
Implicit or automatic memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

Computer will do it!

refers to a polynomial time algorithm, i.e., runtime = \( O(n^k) \) for some \( k \) constant independent of \( n \).

SIDE NOTE: Divide-and-conquer recurrences are fundamentally different from what we would like for dynamic programming (DP). In DP, we want the ‘smaller’ instances to be repeated. Avoid recomputations.

For instance: In merge sort, smaller subproblems are not repeated.
Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) : \\
\begin{align*}
&\text{if } (n = 0) \\
&\quad \text{return } 0 \\
&\text{if } (n = 1) \\
&\quad \text{return } 1 \\
&\text{if } (\text{Fib}(n) \text{ was previously computed}) \\
&\quad \text{return stored value of Fib}(n) \\
&\text{else} \\
&\quad \text{return Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]
Implicit or automatic memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\( \text{Fib}(n) : \)

\[
\begin{align*}
\text{if} \quad (n = 0) & \quad \text{return} \ 0 \\
\text{if} \quad (n = 1) & \quad \text{return} \ 1 \\
\text{if} \quad (\text{Fib}(n) \text{ was previously computed}) & \quad \text{return} \ \text{stored value of Fib}(n) \\
\text{else} & \quad \text{return} \ \text{Fib}(n - 1) + \ \text{Fib}(n - 2)
\end{align*}
\]

How do we keep track of previously computed values?
Implicit or automatic memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```python
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n - 1) + Fib(n - 2)
```

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Implicit or automatic memoization

Initialize a (dynamic) dictionary data structure $D$ to empty

$$
\text{Fib}(n):
\begin{align*}
&\text{if } (n = 0) \\
&\quad \text{return } 0 \\
&\text{if } (n = 1) \\
&\quad \text{return } 1 \\
&\text{if } (n \text{ is already in } D) \\
&\quad \text{return } \text{value stored with } n \text{ in } D \\
&\quad \text{val} \leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2) \\
&\text{Store } (n, \text{val}) \text{ in } D \\
&\text{return } \text{val}
\end{align*}
$$

Use hash-table or a map to remember which values were already computed.

Compiler will do it! Key value pair. For instance: Python dictionary.

Made by the compiler. Ask the compiler to do the memoization.
Explicit (not automatic) memoization

- Initialize table/array $M$ of size $n$: $M[i] = -1$ for $i = 0, \ldots, n$. 

```
Fib(n):
if (n == 0)
    return 0
if (n == 1)
    return 1
if (M[n] != -1)
    return M[n]
else
    return Fib(n - 1) + Fib(n - 2)
```

- Need to know upfront the number of sub-problems to allocate memory.
Explicit (not automatic) memoization

- Initialize table/array $M$ of size $n$: $M[i] = -1$ for $i = 0, \ldots, n$.
- Resulting code:

  $\text{Fib}(n)$:

  ```python
  if (n == 0)
      return 0
  if (n == 1)
      return 1
  if (M[n] \neq -1) // M[n]: stored value of Fib(n)
      return M[n]
  M[n] \leftarrow \text{Fib}(n-1) + \text{Fib}(n-2)
  return M[n]
  ```

  You are explicitly writing what the compiler may do implicitly.
Explicit (not automatic) memoization

- Initialize table/array $M$ of size $n$: $M[i] = -1$ for $i = 0, \ldots, n$.
- Resulting code:

  ```python
  def Fib(n):
      if (n == 0):
          return 0
      if (n == 1):
          return 1
      if (M[n] != -1): // M[n]: stored value of Fib(n)
          return M[n]
  M[n] = Fib(n - 1) + Fib(n - 2)
  return M[n]
  ```

- Need to know upfront the number of sub-problems to allocate memory.
Recursion tree for the memorized Fib...
Recursion tree for the memorized Fib...
Recursion tree for the memorized Fib...
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Recursion tree for the memorized Fib...
Implicit or automatic memoization

- Recursive version:

  \[ f(x_1, x_2, \ldots, x_d) : \]

  CODE

- Recursive version with memoization:

  \[ g(x_1, x_2, \ldots, x_d) : \]

  if \( f \) already computed for \((x_1, x_2, \ldots, x_d)\) then
  
  return value already computed

  NEW_CODE

- NEW_CODE:
  - Replaces any “return \( \alpha \)” with
  - Remember “\( f(x_1, \ldots, x_d) = \alpha \)” ; return \( \alpha \).
Explicit vs Implicit Memoization

- **Explicit memoization** (on the way to iterative algorithm) preferred:
  - analyze problem ahead of time

Overlapping Subproblems: A recursive solution contains a "small" number of distinct subproblems repeated many times.
Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.

- Implicit (automatic) memoization:
  - problem structure or algorithm is not well understood.
  - Need to pay overhead of data-structure.
  - Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.
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  ● problem structure or algorithm is not well understood.
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  ● Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.
Explicit/implicit memoization for Fibonacci

**Explicit memoization**

Init: $M[i] = -1$, $i = 0, \ldots, n$.

**Fib**($k$):

- if ($k = 0$) return 0
- if ($k = 1$) return 1
- if ($M[k] \neq -1$) return $M[n]
- $M[k] \leftarrow \text{Fib}(k - 1) + \text{Fib}(k - 2)$
- return $M[k]$

**Implicit memoization**

Init: Init dictionary $D$

**Fib**($n$):

- if ($n = 0$) return 0
- if ($n = 1$) return 1
- if ($n$ is already in $D$) return value stored with $n$ in $D$
- val $\leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2)$
- Store $(n, \text{val})$ in $D$
- return val
Dynamic programming
Removing the recursion by filling the table in the right order

\[ \text{Fib}(n) : \]
\[
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{if } (n = 1) & \quad \text{return } 1 \\
\text{if } (M[n] \neq -1) & \quad \text{return } M[n] \\
M[n] \leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2) & \\
\text{return } M[n] \\
\end{align*}
\]

\[ \text{FibIter}(n) : \]
\[
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{if } (n = 1) & \quad \text{return } 1 \\
F[0] = 0 \\
F[1] = 1 \\
\text{for } i = 2 \text{ to } n & \quad \text{do} \\
& \quad F[i] = F[i - 1] + F[i - 2] \\
\text{return } F[n] \\
\end{align*}
\]

Explicit Memoization

Iterative Algorithm
Dynamic programming: Saving space!

Saving space. Do we need an array of $n$ numbers? Not really.

\begin{verbatim}
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i - 1] + F[i - 2]
    return F[n]
\end{verbatim}
Dynamic Programming is smart recursion
Dynamic Programming is **smart recursion**

+ **explicit memoization**
Dynamic programming – quick review

Dynamic Programming is **smart recursion**

+ explicit memoization
+ filling the table in right order
+ removing recursion.
Suppose we have a recursive program $\text{foo}(x)$ that takes an input $x$.

- On input of size $n$ the number of distinct sub-problems that $\text{foo}(x)$ generates is at most $A(n)$.
- $\text{foo}(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

**Eg.** $\text{Fib}(n)$:

$$
\begin{align*}
A(n) &= O(n) \\
B(n) &= 1
\end{align*}
$$

$$
\text{Runtime of Fib(n) [Memoized]} = O(n)
$$
Analyzing memorized recursive function

Suppose we have a recursive program $foo(x)$ that takes an input $x$.

- On input of size $n$ the number of distinct sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

Suppose we memorize the recursion.

**Assumption:** Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.
Suppose we have a recursive program \( f_{oo}(x) \) that takes an input \( x \).

- On input of size \( n \) the number of distinct sub-problems that \( f_{oo}(x) \) generates is at most \( A(n) \).
- \( f_{oo}(x) \) spends at most \( B(n) \) time not counting the time for its recursive calls.

Suppose we memorize the recursion.

**Assumption:** Storing and retrieving solutions to pre-computed problems takes \( O(1) \) time.

**Q:** What is an upper bound on the running time of memorized version of \( f_{oo}(x) \) if \( |x| = n \)?
Analyzing memorized recursive function

Suppose we have a recursive program $foo(x)$ that takes an input $x$.

- On input of size $n$ the number of distinct sub-problems that $foo(x)$ generates is at most $A(n)$
- $foo(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

Suppose we memorize the recursion.

Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Q: What is an upper bound on the running time of memorized version of $foo(x)$ if $|x| = n$? $O(A(n)B(n))$. 
Longest Increasing Sub-sequence Revisited
**Sequences**

**Definition**
**Sequence**: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is the number of elements in the list.

**Definition**
$a_{i_1}, \ldots, a_{i_k}$ is a **sub-sequence** of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

**Definition**
A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.
Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Subsequence of above sequence: 5, 2, 1
- Increasing sequence: 3, 5, 9, 17, 54
- Decreasing sequence: 34, 21, 7, 5, 1
- Increasing subsequence of the first sequence: 2, 7, 8.
- Longest Increasing subsequence of the first sequence: 3, 5, 7, 8.
Longest Increasing Subsequence Problem

**Input**  A sequence of numbers \( a_0, a_1, \ldots, a_{n-1} \)

**Goal**  Find an increasing subsequence \( a_{i_0}, a_{i_1}, \ldots, a_{i_k} \) of maximum length
Longest Increasing Subsequence Problem

**Input**  A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

**Goal**  Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

**Example**

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8
This is just for [6,3,5,2,7]! (Tikz won’t print larger trees)

- How many leaves are there for the full [6,3,5,2,7, 8, 1] sequence
- What is the running time?
Assume \( a_1, a_2, \ldots, a_n \) is contained in an array \( A \)

\[
\text{algLISNaive}(A[1..n]):\n\]
\[
\begin{align*}
max &= 0 \\
\text{for each subsequence } B \text{ of } A &\text{ do} \\
&\quad \text{if } B \text{ is increasing and } |B| > max \text{ then} \\
&\qquad max = |B|
\end{align*}
\]

Output \( max \)

Running time: \( O(n2^n) \).

\( 2^n \) subsequences of a sequence of length \( n \) and \( O(n) \) time to check if a given sequence is increasing.
Can we find a recursive algorithm for LIS?

\[ \text{LIS}(A[0..n-1]) : \]
Can we find a recursive algorithm for LIS?

\[ \text{LIS}(A[0..n-1]): \]

- **Case 1:** Does not contain \( A[n-1] \) in which case \( \text{LIS}(A[0..n-1]) = \text{LIS}(A[0..(n-1)]) \)

- **Case 2:** contains \( A[n-1] \) in which case \( \text{LIS}(A[0..n-1]) \) is not so clear.

**Observation**
*For second case we want to find a subsequence in \( A[1..(n-2)] \) that is restricted to numbers less than \( A[n-1] \). This suggests that a more general problem is \( \text{LIS}_\text{smaller}(A[0..n-1], x) \) which gives the longest increasing subsequence in \( A \) where each number in the sequence is less than \( x \).*
Sequence: $A[0..6] = 6, 3, 5, 2, 7, 8, 1$
**Recursive Approach**

\(LIS(A[1..n])\): the length of longest increasing subsequence in \(A\)

\(LIS_{\text{smaller}}(A[1..n], x)\): length of longest increasing subsequence in \(A[1..n]\) with all numbers in subsequence less than \(x\)

\[
\text{LIS}_{\text{smaller}}(A[1..i], x) : \\
\text{if } i = 0 \text{ then return } 0 \\
m = \text{LIS}_{\text{smaller}}(A[1..i - 1], x) \\
\text{if } A[i] < x \text{ then} \\
\quad m = \max(m, 1 + \text{LIS}_{\text{smaller}}(A[1..i - 1], A[i])) \\
\text{Output } m
\]

\[
\text{LIS}(A[1..n]) : \\
\text{return } \text{LIS}_{\text{smaller}}(A[1..n], \infty)
\]

\(O(2^n) \): Don't need to check for the increasing nature
Recursive Approach

\[ \text{LIS}\text{-smaller}(A[1..i], x) : \]
\[ \text{if } i = 0 \text{ then return } 0 \]
\[ m = \text{LIS}\text{-smaller}(A[1..i - 1], x) \]
\[ \text{if } A[i] < x \text{ then} \]
\[ m = \max(m, 1 + \text{LIS}\text{-smaller}(A[1..i - 1], A[i])) \]
\[ \text{Output } m \]

\[ \text{LIS}(A[1..n]) : \]
\[ \text{return } \text{LIS}\text{-smaller}(A[1..n], \infty) \]

- How many distinct sub-problems will \text{LIS}\text{-smaller}(A[1..n], \infty) generate?
Recursive Approach

\[
\text{LIS\_smaller}(A[1..i], x) :
\begin{align*}
& \text{if } i = 0 \text{ then return } 0 \\
& m = \text{LIS\_smaller}(A[1..i-1], x) \\
& \text{if } A[i] < x \text{ then} \\
& \quad m = \max(m, 1 + \text{LIS\_smaller}(A[1..i-1], A[i])) \\
& \text{Output } m
\end{align*}
\]

\[
\text{LIS}(A[1..n]) : \\
\quad \text{return } \text{LIS\_smaller}(A[1..n], \infty)
\]

- How many distinct sub-problems will \text{LIS\_smaller}(A[1..n], \infty) generate? \(O(n^2)\)
Recursive Approach

\[
\text{LIS\_smaller}(A[1..i], x): \\
\quad \text{if } i = 0 \text{ then return } 0 \\
\quad m = \text{LIS\_smaller}(A[1..i - 1], x) \\
\quad \text{if } A[i] < x \text{ then} \\
\quad \quad m = \max(m, 1 + \text{LIS\_smaller}(A[1..i - 1], A[i])) \\
\text{Output } m
\]

\[
\text{LIS}(A[1..n]): \\
\quad \text{return } \text{LIS\_smaller}(A[1..n], \infty)
\]

- How many distinct sub-problems will \text{LIS\_smaller}(A[1..n], \infty) generate? \(O(n^2)\)
- What is the running time if we memorize recursion?
Recursive Approach

\[
\text{LIS\textunderscore smaller}(A[1..i], x): \\
\quad \text{if } i = 0 \text{ then return } 0 \\
\quad m = \text{LIS\textunderscore smaller}(A[1..i - 1], x) \\
\quad \text{if } A[i] < x \text{ then} \\
\quad \quad m = \max(m, 1 + \text{LIS\textunderscore smaller}(A[1..i - 1], A[i])) \\
\quad \text{Output } m
\]

\[
\text{LIS}(A[1..n]): \\
\quad \text{return } \text{LIS\textunderscore smaller}(A[1..n], \infty)
\]

- How many distinct sub-problems will \text{LIS\textunderscore smaller}(A[1..n], \infty) generate? \(O(n^2)\)
- What is the running time if we memorize recursion? \(O(n^2)\) since each call takes \(O(1)\) time to assemble the answers from to recursive calls and no other computation.
Recursive Approach

```python
def LIS_smaller(A[1..i], x):
    if i == 0:
        return 0
    m = LIS_smaller(A[1..i - 1], x)
    if A[i] < x:
        m = max(m, 1 + LIS_smaller(A[1..i - 1], A[i]))
    return m

def LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

- How many distinct sub-problems will \(\text{LIS\_smaller}(A[1..n], \infty)\) generate? \(O(n^2)\)
- What is the running time if we memorize recursion? \(O(n^2)\) since each call takes \(O(1)\) time to assemble the answers from recursive calls and no other computation.
- How much space for memoization?
Recursive Approach

\[
\text{LIS\_smaller}(A[1..i], x) :
\]
\[
\text{if } i = 0 \text{ then return } 0
\]
\[
m = \text{LIS\_smaller}(A[1..i - 1], x)
\]
\[
\text{if } A[i] < x \text{ then}
\]
\[
m = \max(m, 1 + \text{LIS\_smaller}(A[1..i - 1], A[i]))
\]
\text{Output } m

\[
\text{LIS}(A[1..n]) :
\]
\[
\text{return LIS\_smaller}(A[1..n], \infty)
\]

- How many distinct sub-problems will \text{LIS\_smaller}(A[1..n], \infty) generate? \(O(n^2)\)
- What is the running time if we memorize recursion? \(O(n^2)\) since each call takes \(O(1)\) time to assemble the answers from recursive calls and no other computation.
- How much space for memoization? \(O(n^2)\)
After seeing that number of sub-problems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add $\infty$ at end of array (in position $n + 1$)

$LIS(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)
After seeing that number of sub-problems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add $\infty$ at end of array (in position $n + 1$)

$LIS(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

Base case: $LIS(0, j) = 0$ for $1 \leq j \leq n + 1$

Recursive relation:

- $LIS(i, j) = LIS(i - 1, j)$ if $A[i] \geq A[j]$ (If we don’t include $A[i]$ in LIS).
- $LIS(i, j) = \max\{LIS(i - 1, j), 1 + LIS(i - 1, i)\}$ if $A[i] < A[j]$

Output: $LIS(n, n + 1)$. Please spend some time here!
How to order bottom up computation?

Sequence:

\[ A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1] \]

Recursive relation:

\[
\text{LIS}(i, j) = \begin{cases} 
0 & \text{if } i = 0 \\
\text{LIS}(i - 1, j) & \text{if } A[i] \geq A[j] \\
\max \left\{ \text{LIS}(i - 1, j), 1 + \text{LIS}(i - 1, i) \right\} & \text{if } A[i] < A[j]
\end{cases}
\]
LIS(1, 2):


\[ 6 > 3 \]

\[ \Rightarrow LIS(1, 2) = 0 = LIS(0, 2) \]

\[ \vdots \]


\[ 6 < 7 \]

\[ LIS(1, 5) = \max \left\{ \begin{array}{c} LIS(0, 5) = \max \left\{ \begin{array}{c} 0 \\ 1 + LIS(0, 1) \end{array} \right. \end{array} \right. \]
How to order bottom up computation?

Sequence:

\[ A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1] \]

Recursive relation:

\[ LIS(i, j) = \begin{cases} 
0 & i = 0 \\
LIS(i - 1, j) & A[i] \geq A[j] \\
\max \left\{ LIS(i - 1, j), 1 + LIS(i - 1, i) \right\} & A[i] < A[j]
\end{cases} \]
How to order bottom up computation?

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\end{cases}
\]
### How to order bottom up computation?

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Represents limiter

Represents sub-array

#### Recursive relation:

\[
LIS(i, j) = \begin{cases} 
0 & i = 0 \\
LIS(i - 1, j) & A[i] \geq A[j] \\
\max \left\{ LIS(i - 1, j), 1 + LIS(i - 1, i) \right\} & A[i] < A[j]
\end{cases}
\]

**Sequence:**

\[ A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1] \]
How to order bottom up computation?

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Represents sub-array $i$

Represents limiter $j$

Recursive relation:

$$LIS(i, j) = \begin{cases} 
0 & i = 0 \\
LIS(i - 1, j) & A[i] \geq A[j] \\
\max \left\{ LIS(i - 1, j), 1 + LIS(i - 1, i) \right\} & A[i] < A[j]
\end{cases}$$

Sequence:

$A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1]$
How to order bottom up computation?

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Represents limiter

Represents sub-array

Recursive relation:

\[
LIS(i, j) =
\begin{cases}
0 & i = 0 \\
LIS(i - 1, j) & A[i] \geq A[j] \\
\max \left\{ LIS(i - 1, j), 1 + LIS(i - 1, i) \right\} & A[i] < A[j]
\end{cases}
\]

Sequence:

\[A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1]\]
How to order bottom up computation?

Sequence:

$A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1]$

Recursive relation:

$LIS(i, j) =$

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How to order bottom up computation?

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Represents limiter

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0
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Represented sub-array

### Recursive relation:

\[
LIS(i, j) =
\begin{cases}
0 & i = 0 \\
LIS(i - 1, j) & A[i] \geq A[j] \\
\max \left\{ LIS(i - 1, j), 1 + LIS(i - 1, i) \right\} & A[i] < A[j]
\end{cases}
\]

### Sequence:

\[
A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1]
\]
How to order bottom up computation?

### Sequence:

\[
\text{A[1 \ldots 7]} = [6, 3, 5, 2, 7, 8, 1]
\]

### LIS \((i, j)\) Recursive relation:

\[
\text{LIS}(i, j) = \begin{cases} 
0 & i = 0 \\
\text{LIS}(i-1, j) & A[i] \geq A[j] \\
\max \left\{ \text{LIS}(i-1, j), 1 + \text{LIS}(i-1, i) \right\} & A[i] < A[j]
\end{cases}
\]

### Represents sub-array \(i\)

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Receives limiter \(j\)
\[
diagonal + 1st \ off \ diagonal + \ldots = (n+1) + n + (n-1) + \ldots + 1
\]
\[
= \frac{(n+1)(n+2)}{2}
\]
\[
= \frac{n^2 + 3n + 2}{2}
\]
\[
= O(n^2)
\]
Iterative algorithm

The dynamic program for longest increasing subsequence

\textbf{LIS-Iterative}(A[1..n]):

\begin{align*}
A[n + 1] &= \infty \\
\text{int} &\ LIS[0..n - 1, 0..n] \\
\text{for} &\ j = 0 \ldots n \text{) if } A[i] \leq A[j] \text{ then } LIS[0][j] = 1
\end{align*}

\begin{align*}
\text{for } &\ i = 1 \ldots n - 1 \text{ do} \\
&\text{for } j = i \ldots n - 1 \text{ do} \\
&\quad \text{if } (A[i] \geq A[j]) \\
&\quad \quad LIS[i, j] = LIS[i - 1, j] \\
&\quad \text{else} \\
&\quad \quad LIS[i, j] = \max(LIS[i - 1, j], 1 + LIS[i - 1, i])
\end{align*}

Return \ LIS[n, n + 1]

\textbf{Running time: } O(n^2)

\textbf{Space: } O(n^2)
Iterative algorithm

The dynamic program for longest increasing subsequence

**LIS-Iterative**($A[1..n]$):

1. $A[n + 1] = \infty$
2. int $LIS[0..n - 1, 0..n]$
3. for $j = 0\ldots n$) if $A[i] \leq A[j]$ then $LIS[0][j] = 1$
4. for $i = 1\ldots n - 1$ do
5.   for $j = i\ldots n - 1$ do
7.       $LIS[i, j] = LIS[i - 1, j]$
8.     else
9.       $LIS[i, j] = \max(LIS[i - 1, j], 1 + LIS[i - 1, i])$
10. Return $LIS[n, n + 1]$

**Running time:** $O(n^2)$

**Space:** $O(n^2)$ Can be done in linear space. How?
Question: Can we compute an optimum solution and not just its value?
Two comments

**Question:** Can we compute an optimum solution and not just its value?
Yes! See notes.
Finding the sub-sequence

Sequence:

\[ A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1] \]

We know the LIS length (4) but how do we find the LIS itself?

\[ LIS = [3, 5, 7, 8] \]

Recursive relation:

\[
LIS(i, j) = \begin{cases} 
0 & i = 0 \\
LIS(i - 1, j) & A[i] \geq A[j] \\
\max \left\{ LIS(i - 1, j), 1 + LIS(i - 1, i) \right\} & A[i] < A[j]
\end{cases}
\]
**Finding the sub-sequence**

Sequence:

\[ A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1] \]

We know the LIS length (4) but how do we find the LIS itself?

\[ LIS = [3, 5, 7, 8] \]

**Recursive relation:**

\[
LIS(i, j) = \begin{cases} 
0 & i = 0 \\
LIS(i-1, j) & A[i] \geq A[j] \\
\max \left\{ LIS(i-1, j), 1 + LIS(i-1, i) \right\} & A[i] < A[j]
\end{cases}
\]
Two comments

**Question**: Can we compute an optimum solution and not just its value?

Yes!

**Question**: Is there a faster algorithm for LIS?
**Two comments**

**Question:** Can we compute an optimum solution and not just its value?
Yes!

**Question:** Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $O(n \log n)$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.
How to come up with dynamic programming algorithm: summary
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