Pre-lecture brain teaser

Write a (very simple) recursive algorithm that calcuates the Fibonnacci n^{th} number.

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_0 = \underline{0}, F_1 = \underline{1}$$

Indian Mathematician
Acharya Pingala in 200 BC

Named after Italian Mathematician
Leonardo of Pisa aka Fibonacci (1202)

0
1
1
2
3
5
8
13
21
34
55
⋮

ECE-374-B: Lecture 12 - Dynamic Programming I

Instructor: Abhishek Kumar Umrawal

October 05, 2022

University of Illinois at Urbana-Champaign

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Learning Objectives

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At the end of the lecture, you should be able to understand

- the concepts of the memoizationand dynamic programming,
- how to improve the time and space complexities of recursive algorithms using the above concepts,
- dynamic programming for the fibonacci numbers and longest increasing subsequence problem, and
- where and how to use dynamic programming to refine recursive algorithms.

Recursion and Memoization

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

These numbers have many interesting properties. A journal The Fibonacci Quarterly[1]!

Fibonacci Numbers

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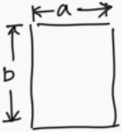
These numbers have many interesting properties. A journal The Fibonacci Quarterly[1]!

- Binet's formula: $F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \approx \frac{1.618^n - (-0.618)^n}{\sqrt{5}} \approx \frac{1.618^n}{\sqrt{5}}$
  φ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
- $\lim_{n\to\infty} F(n+1)/F(n) = \varphi$



$$\frac{b}{a} = \phi$$

→ Most visually appealing
photo frame / window / door, ...

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib($n$):
    if ($n = 0$)
        return 0
    else if ($n = 1$)
        return 1
    else
        return Fib($n - 1$) + Fib($n - 2$)
```

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

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    if ($n = 0$)
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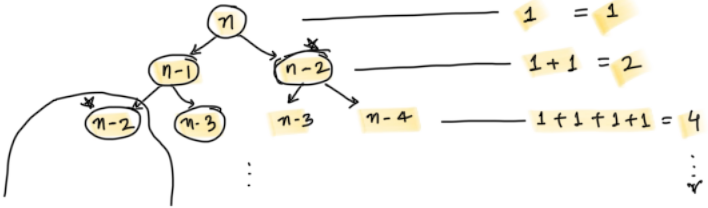
        return 1
    else
        return Fib($n-1$) + Fib($n-2$)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n-1) + T(n-2) + O(1)$$



$\Rightarrow$ # of leaves : $\underline{O(2^n)}$

$\Rightarrow$ $T(n) = 1 \cdot O(2^n)$ Additions

$T(n) \le 1 + 2 + 4 + \cdots + 2^n$

$= O(2^n)$

Exact bound:

$T(n) = \Theta(\phi^n)$

$\phi = 1.6 < 2$

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib($n$):
    if ($n = 0$)
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Recursive Algorithm for Fibonacci Numbers

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Running time? Let $T(n)$ be the number of additions in Fib(n).

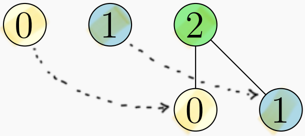
$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$: $T(n) = \Theta(\varphi^n)$.

The number of additions is exponential in n. Can we do better?

Recursion tree for the Recursive Fibonacci

Recursion tree for the Recursive Fibonacci



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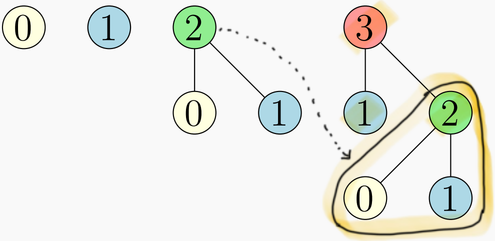

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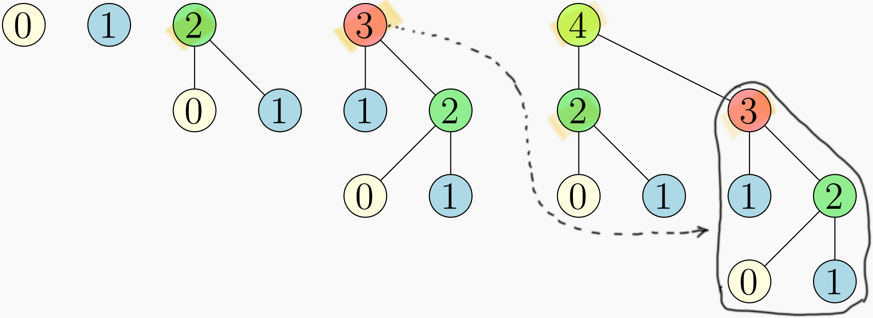

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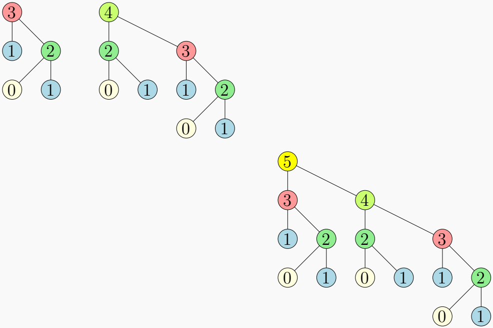

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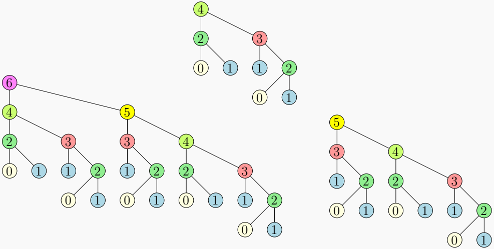
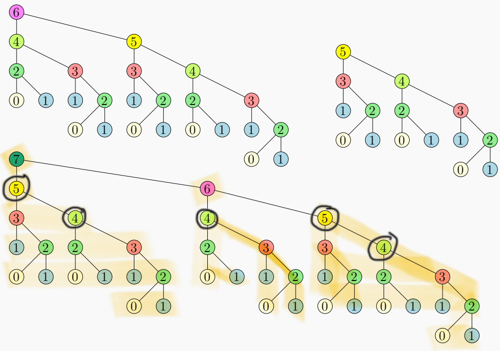


Recursion tree for the Recursive Fibonacci



Observation: OPTIMAL SUBSTRUCTURE

'A recursive solution contains a small number of distinct subproblems repeated many times.'

An iterative algorithm for Fibonacci numbers

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

F stores the values so far!

Tail recursion: Just one call!

Iterative bottom-up calculation.

$T(n) = O(n)$ additions

An iterative algorithm for Fibonacci numbers

```
FibIter($n$):
    if ($n = 0$) then
        return 0
    if ($n = 1$) then
        return 1
    $F[0] = 0$
    $F[1] = 1$
    for $i$ = 2 to $n$ do
        $F[i] = F[i-1] + F[i-2]$
    return $F[n]$
```

What is the running time of the algorithm?

$O(n)$ additions!

An iterative algorithm for Fibonacci numbers

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    if (n = 0) then
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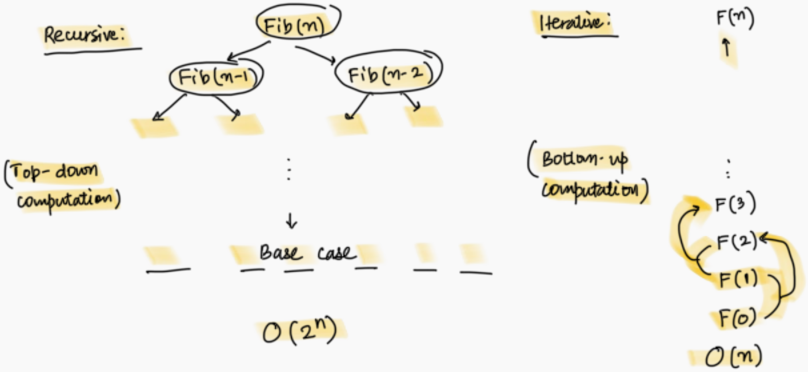
What is the running time of the algorithm? $O(n)$ additions.

What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value.



What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

What is the difference?

- Recursive algorithm is computing the same numbers again and again.
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Dynamic Programming: Finding a recursion that can be effectively/efficiently memorized.

$O(n^k)$ runtime for some constant k

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

Implicit vs. explicit memoization

Implicit or automatic memoization

Can we convert recursive algorithm into an 'efficient' algorithm without explicitly doing an iterative algorithm?

Compiler will do it!

refers to a polynomial time algorithm, i.e., runtime = $O(n^k)$ for some k constant independent of n.

SIDE NOTE: Divide-and-conquer recurrences are fundamentally different from what we would like for dynamic programming (DP). In DP, we want the 'smaller' instances to be repeated.

Avoid recomputations.

For instance: In merge sort, smaller subproblems are not repeated.

Implicit or automatic memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n − 1) + Fib(n − 2)
```

Compiler takes care of storing and retrieving of the previously computed values.

Implicit or automatic memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib($n$):
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How do we keep track of previously computed values?

Implicit or automatic memoization

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```

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure)

Implicit or automatic memoization

Initialize a (dynamic) dictionary data structure D to empty

```
Fib($n$):
        if ($n = 0$)
            return 0
        if ($n = 1$)
            return 1
        if ($n$ is already in $D$)
            return value stored with $n$ in $D$
        $val \Leftarrow$ Fib($n-1$) + Fib($n-2$)
        Store ($n, val$) in $D$
        return $val$
```

Use hash-table or a map to remember which values were already computed.

Compiler will do it! Key value pair. For instance: Python dictionary.

Made by the compiler. Ask the compiler to do the memoization.

Explicit (not automatic) memoization

Do it yourself memoization!

\# of subproblems

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.

Explicit (not automatic) memoization

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.
- Resulting code:

```
Fib($n$):
        if ($n = 0$)
            return 0
        if ($n = 1$)
            return 1
        if ($M[n] \neq -1$) // $M[n]$:  stored value of Fib($n$)
            return $M[n]$
        $M[n] \Leftarrow$ Fib($n-1$) + Fib($n-2$)
        return $M[n]$
```

You are explicitly writing what the compiler may do implicitly.

Explicit (not automatic) memoization

- Initialize table/array M of size n: $M[i] = -1$ for $i = 0, \ldots, n$.
- Resulting code:

```
Fib(n):
        if (n = 0)
            return 0
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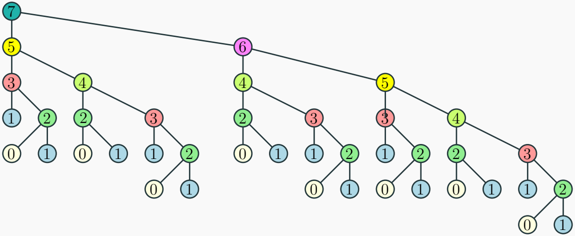
        if (M[n] ≠ −1) // M[n]: stored value of Fib(n)
            return M[n]
        M[n] ⇐ Fib(n − 1) + Fib(n − 2)
        return M[n]
```

- Need to know upfront the number of sub-problems to allocate memory.

Recursion tree for the memorized Fib...



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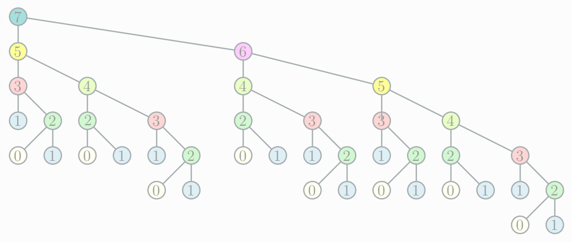

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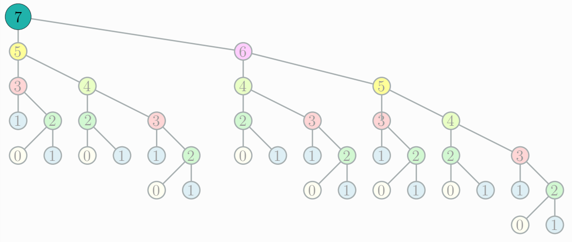

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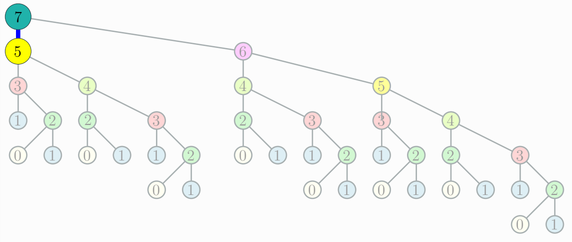

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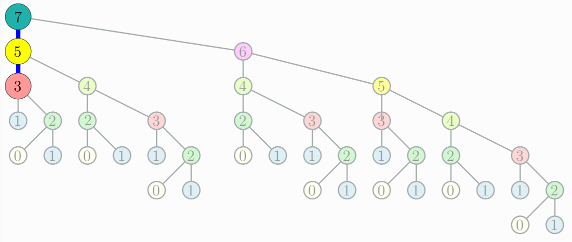

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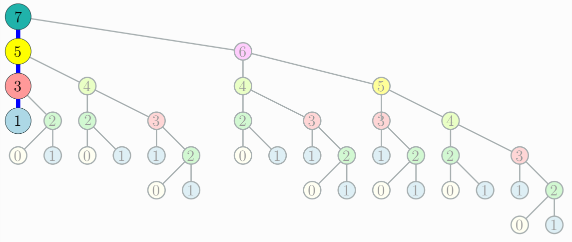

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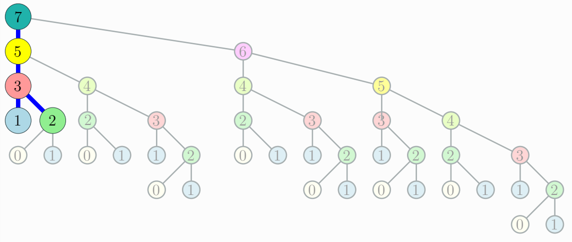

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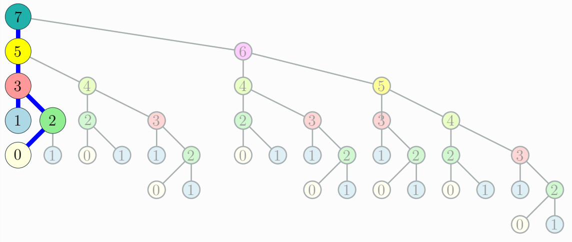

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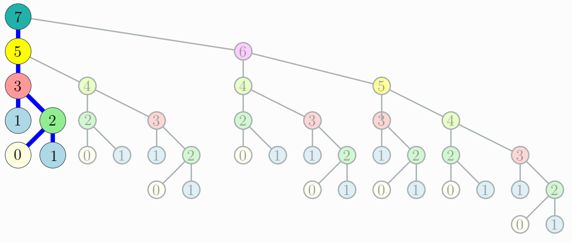

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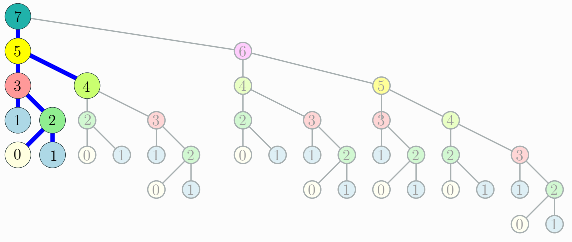

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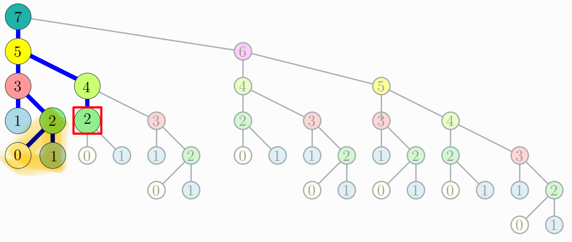

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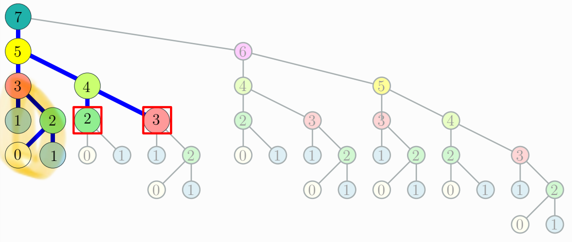

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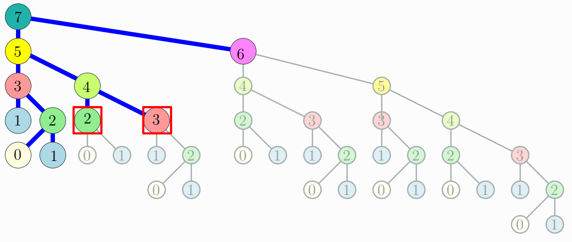

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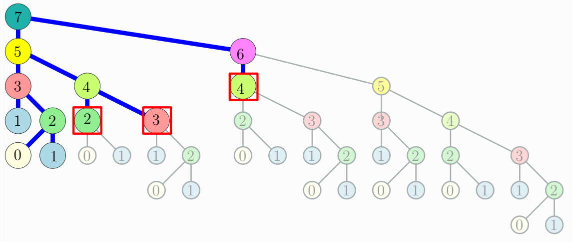

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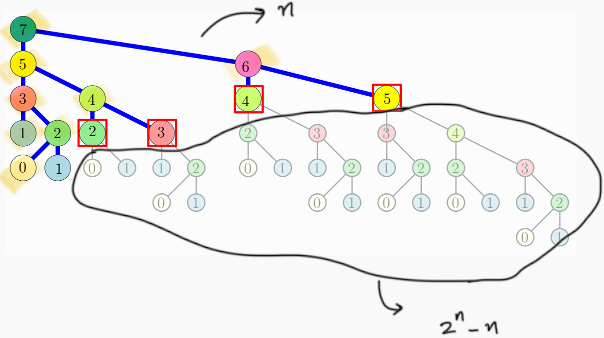

Implicit or automatic memoization

(RIY)

- Recursive version:

```
$f(x_1, x_2, \ldots, x_d)$:
        CODE
```

- Recursive version with memoization:

```
$g(x_1, x_2, \ldots, x_d)$:
        if $f$ already computed for $(x_1, x_2, \ldots, x_d)$ then
            return value already computed
        NEW_CODE
```

- NEW_CODE:
  - Replaces any "**return** α" with
  - Remember "$f(x_1, \ldots, x_d) = \alpha$"; **return** α.

Explicit vs Implicit Memoization

- Explicit memoization (on the way to iterative algorithm) preferred:
    - analyze problem ahead of time

Overlapping Subproblems: A recursive solution contains a "small" number of distinct subproblems repeated many times.

Explicit vs Implicit Memoization

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  - analyze problem ahead of time
  - Allows for efficient memory allocation and access.
- Implicit (automatic) memoization:
  - problem structure or algorithm is not well understood.
  - Need to pay overhead of data-structure.
  - Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.

Explicit/implicit memoization for Fibonacci

```
Init:  $M[i] = -1,\ i = 0, \ldots, n.$

Fib($k$):
    if ($k = 0$)
        return 0
    if ($k = 1$)
        return 1
    if ($M[k] \neq -1$)
        return $M[n]$
    $M[k] \Leftarrow$ Fib($k - 1$) + Fib($k - 2$)
    return $M[k]$
```

Explicit memoization

```
Init:  Init dictionary $D$

Fib($n$):
    if ($n = 0$)
        return 0
    if ($n = 1$)
        return 1
    if ($n$ is already in $D$)
        return value stored
            with $n$ in $D$
        $val \Leftarrow$ Fib($n - 1$) + Fib($n - 2$)
    Store ($n, val$) in $D$
    return $val$
```

Implicit memoization

Dynamic programming

Removing the recursion by filling the table in the right order

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (M[n] ≠ -1)
        return M[n]
    M[n] ⇐ Fib(n - 1) + Fib(n - 2)
    return M[n]
```

Explicit Memoization

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    F[0] = 0
    F[1] = 1
    for i = 2 to n do
        F[i] = F[i - 1] + F[i - 2]
    return F[n]
```

Iterative Algorithm

Dynamic programming: Saving space!

Saving space. Do we need an array of n numbers? Not really.

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    $F[0] = 0$
    $F[1] = 1$
    for i = 2 to n do
        $F[i] = F[i-1] + F[i-2]$
    return $F[n]$
```

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    $prev2 = 0$
    $prev1 = 1$
    for i = 2 to n do
        $temp = prev1 + prev2$
        $prev2 = prev1$
        $prev1 = temp$

    return $prev1$
```

Dynamic programming – quick review

Dynamic Programming is smart recursion

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memoization

Dynamic programming – quick review

Dynamic Programming is smart recursion

+ explicit memoization

+ filling the table in right order

+ removing recursion.

Analyzing memorized recursive function

Suppose we have a recursive program *foo*$(\overset{|x|=n}{x})$ that takes an input x.

- On input of size n the number of distinct sub-problems that *foo*(x) generates is at most $A(n)$
- *foo*(x) spends at most $B(n)$ time not counting the time for its recursive calls.

Eg. Fib(n) :

$$\left.\begin{array}{l} A(n) = O(n) \\ B(n) = 1 \end{array}\right\} \rightarrow A(n)\,B(n)$$

$$\uparrow$$

Runtime of Fib(n) [Memoized] = $O(n)$

Analyzing memorized recursive function

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Suppose we memorize the recursion.
Assumption: Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

Analyzing memorized recursive function

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Q: What is an upper bound on the running time of memorized version of *foo*(x) if $|x| = n$?

Analyzing memorized recursive function

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Q: What is an upper bound on the running time of memorized version of $foo(x)$ if $|x| = n$? $O(A(n)B(n))$.

Longest Increasing Sub-sequence Revisited

$$\underbrace{O(n2^n)}_{\text{"Brute-force"}} \xrightarrow[\text{recursion}]{?} O(2^n) \xrightarrow[\text{memoization}]{?} O(n^2)$$

Sequences

Definition
Sequence: an ordered list $a_1, a_2, \ldots, a_n$. Length of a sequence is number of elements in the list.

Definition
$a_{i_1}, \ldots, a_{i_k}$ is a sub-sequence of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition
A sequence is increasing if $a_1 < a_2 < \ldots < a_n$. It is non-decreasing if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly decreasing and non-increasing.

Sequences - Example...

Example

- Sequence: $6, 3, 5, 2, 7, 8, 1$
- Subsequence of above sequence: $5, 2, 1$
- Increasing sequence: $3, 5, 9, 17, 54$
- Decreasing sequence: $34, 21, 7, 5, 1$
- Increasing subsequence of the first sequence: $2, 7, 8$.
- *Longest* Increasing subsequence of the first sequence: $3, 5, 7, 8$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

Goal Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

Longest Increasing Subsequence Problem

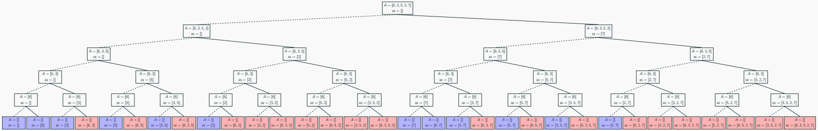
Input A sequence of numbers $a_0, a_1, \ldots, a_{n-1}$

Goal Find an increasing subsequence $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ of maximum length

Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

Naive Recursion Enumeration - State Tree



- This is just for [6,3,5,2,7]! (Tikz won't print larger trees)
- How many leafs are there for the full [6,3,5,2,7, 8, 1] sequence
- What is the running time?

Naive Recursion Enumeration - Code

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive($A[1..n]$):
    $max = 0$
    for each subsequence $B$ of $A$ do
        if $B$ is increasing and $|B| > max$ then
            $max = |B|$

    Output $max$
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

Backtracking Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[0..n-1]$):

Backtracking Approach: LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

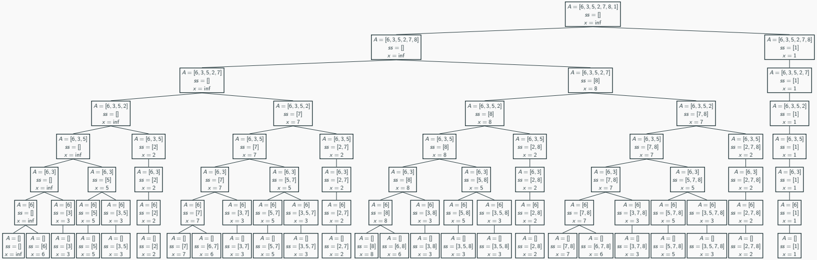
LIS($A[0..n-1]$):

- Case 1: Does not contain $A[n-1]$ in which case LIS($A[0..n-1]$) = LIS($A[0..(n-1)]$)
- Case 2: contains $A[n-1]$ in which case LIS($A[0..n-1]$) is not so clear.

Observation
For second case we want to find a subsequence in $A[1..(n-2)]$ that is restricted to numbers less than $A[n-1]$. This suggests that a more general problem is **LIS_smaller**($A[0..n-1], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Example

Sequence: $A[0..6] = 6, 3, 5, 2, 7, 8, 1$



Recursive Approach

$LIS(A[1..n])$: the length of longest increasing subsequence in A

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller($A[1..i], x$) :
    if $i = 0$ then return 0
    $m$ = LIS_smaller($A[1..i-1], x$)
    if $A[i] < x$ then
        $m = max(m, 1 + $LIS_smaller$(A[1..i-1], A[i]))$
    Output $m$
```

```
LIS($A[1..n]$) :
    return LIS_smaller($A[1..n], \infty$)
```

$O(2^n)$: Don't have to check for the increasing nature

Recursive Approach

```
LIS_smaller($(A[1..i], x)$):
    if $i = 0$ then return 0
    $m = $ LIS_smaller$(A[1..i-1], x)$
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```
LIS($(A[1..n])$):
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```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate?

Recursive Approach

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$\textbf{LIS\_smaller}(A[1..i], x)$:
    if $i = 0$ then return 0
    $m = \textbf{LIS\_smaller}(A[1..i-1], x)$
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```
$\textbf{LIS}(A[1..n])$:
    return $\textbf{LIS\_smaller}(A[1..n], \infty)$
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- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$

Recursive Approach

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LIS_smaller($A[1..i], x$):
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- What is the running time if we memorize recursion?

Recursive Approach

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```

```
**LIS**(A[1..n]):
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- What is the running time if we memorize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.

Recursive Approach

```
**LIS_smaller**$(A[1..i], x)$:
    **if** $i = 0$ **then** **return** 0
    $m =$ **LIS_smaller**$(A[1..i-1], x)$
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```

```
**LIS**$(A[1..n])$:
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- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memorize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization?

Recursive Approach

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LIS_smaller($A[1..i]$, $x$):
    if $i = 0$ then return 0
    $m$ = LIS_smaller($A[1..i-1], x$)
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    Output $m$
```

```
LIS($A[1..n]$):
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```

- How many distinct sub-problems will **LIS_smaller**$(A[1..n], \infty)$ generate? $O(n^2)$
- What is the running time if we memorize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization? $O(n^2)$

Naming sub-problems and recursive equation

After seeing that number of sub-problems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

$\underline{\mathit{LIS}(i,j)}$: length of longest increasing sequence in $\underline{A[1..i]}$ among numbers less than $\underline{A[j]}$ (defined only for $i < j$)

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$\boxed{LIS(i,j)}$: length of longest increasing sequence in $A[1..\overset{n}{i}]$ among numbers less than $A[\underset{n+1}{j}]$ (defined only for $i < j$)

$LIS(n, n+1)$

Base case: $LIS(0, j) = 0$ for $1 \leq j \leq n+1$ ← $A[0, \ldots, i]$: empty A

Recursive relation:

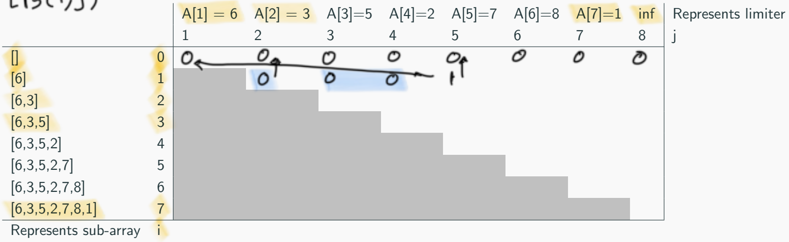
- $LIS(i,j) = LIS(\underline{i-1}, \underline{j})$ if $\underline{A[i]} \geq \underline{A[j]}$ ← If we don't include A[i] in LIS.
- $LIS(i,j) = \max\{\underline{LIS(i-1,j)}, 1 + LIS(i-1, i)\}$ if $A[i] < A[j]$

Output: $LIS(n, n+1)$.

↳ Please spend some time here!

How to order bottom up computation?

LIS(i,j)



		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1				
[6,3]	2									
[6,3,5]	3									
[6,3,5,2]	4									
[6,3,5,2,7]	5									
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Represents sub-array	i									

Recursive relation:

$$LIS(i,j) = \begin{cases} 0 & i = 0 \\ \underline{LIS(i-1,j)} & \underline{A[i] \geq A[j]} \\ \max \begin{cases} \underline{LIS(i-1,j)} \\ 1 + \underline{LIS(i-1,i)} \end{cases} & \underline{A[i] < A[j]} \end{cases}$$

Sequence:

$$A[1\dots 7] = [6,3,5,2,7,8,1]$$

$LIS(1,2)$:

$A[1] = 6$ $A[2] = 3$

$6 > 3$

$$\Rightarrow \underline{LIS(1,2)} = 0 = \underline{LIS(0,2)}$$

$\vdots$

$LIS(1,5)$ $A[1] = 6$ $A[5] = 7$

$\Rightarrow$ $6 < 7$

$$LIS(1,5) = \max\begin{cases} LIS(0,5) \\ 1 + LIS(0,1) \end{cases} = \max\begin{cases} 0 \\ 1 + 0 \end{cases} = 1$$

How to order bottom up computation?

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[]	0	0	0	0	0	0	0	0	0	
[6]	1	▨								
[6,3]	2	▨	▨							
[6,3,5]	3	▨	▨	▨						
[6,3,5,2]	4	▨	▨	▨	▨					
[6,3,5,2,7]	5	▨	▨	▨	▨	▨				
[6,3,5,2,7,8]	6	▨	▨	▨	▨	▨	▨			
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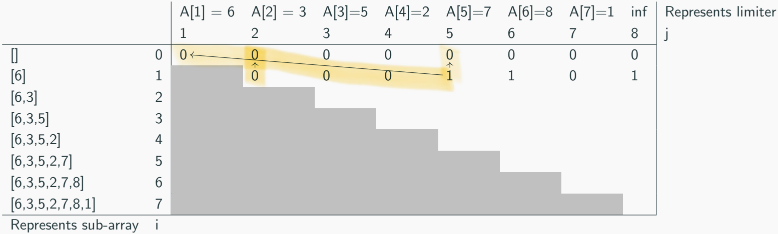

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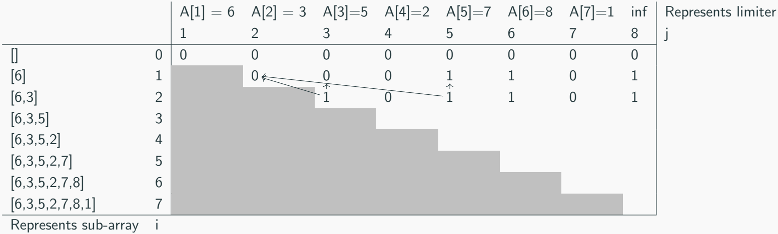

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Recursive relation:

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Represents sub-array	i									

$LIS(\overset{7}{\overset{\shortparallel}{n}}, \overbrace{n+1}^{8}) = 4$

Recursive relation:

Sequence:

$A[1 \ldots 7] = [6, 3, 5, 2, 7, 8, 1]$

$LIS = [3, 5, 7, 8]$

$$LIS(i,j) =$$

$$\begin{cases} 0 & i = 0 \\ LIS(i-1,j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1,j) \\ 1 + LIS(i-1,i) \end{cases} & A[i] < A[j] \end{cases}$$

diagonal + 1st off diagonal + ···· upper right corner

$$= (n+1) + n + (n-1) + \cdots + \overset{\text{upper right corner}}{1}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \frac{n^2 + 3n + 2}{2}$$

$$= O(n^2)$$

Iterative algorithm

The dynamic program for longest increasing subsequence

```
**LIS-Iterative**($A[1..n]$):
    $A[n+1] = \infty$
    int $LIS[0..n-1, 0..n]$
    for $j = 0 \ldots n)$ if A[i] $\leq$ A[j] then $LIS[0][j] = 1$

    for $i = 1 \ldots n-1$ do
        for $j = i \ldots n-1$ do
            if $(A[i] \geq A[j])$
                $LIS[i,j] = LIS[i-1,j]$
            else
                $LIS[i,j] = \max(LIS[i-1,j], 1 + LIS[i-1,i])$

    Return $LIS[n, n+1]$
```

Running time: $O(n^2)$

Space: $O(n^2)$

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The dynamic program for longest increasing subsequence

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    for $i = 1 \ldots n-1$ do
        for $j = i \ldots n-1$ do
            if $(A[i] \geq A[j])$
                $LIS[i,j] = LIS[i-1,j]$
            else
                $LIS[i,j] = \max(LIS[i-1,j], 1 + LIS[i-1,i])$

    Return $LIS[n, n+1]$
```

Running time: $O(n^2)$

Space: $O(n^2)$ Can be done in linear space. How? → Think about it!

Two comments

Question: Can we compute an optimum solution and not just its value?

Two comments

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Yes! See notes.

Finding the sub-sequence

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[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5						3	0	3	
[6,3,5,2,7,8]	6							0	4	
[6,3,5,2,7,8,1]	7								4	
Represents sub-array	i									

Sequence:

$$A[1\ldots7] = [6, 3, 5, 2, 7, 8, 1]$$

We know the LIS length (4) but how do we find the LIS itself?

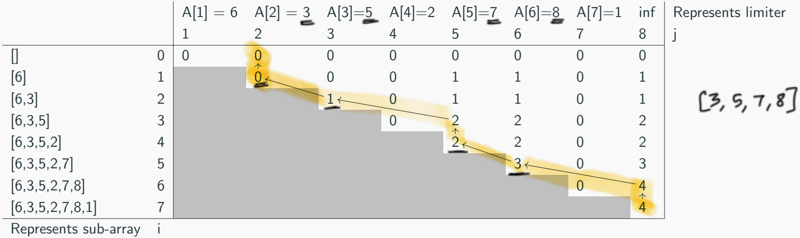
$$LIS = [3, 5, 7, 8]$$

Recursive relation:

$$LIS(i, j) =$$

$$\begin{cases} 0 & i = 0 \\ LIS(i-1, j) & A[i] \geq A[j] \\ \max \begin{cases} LIS(i-1, j) \\ 1 + LIS(i-1, i) \end{cases} & A[i] < A[j] \end{cases}$$

Finding the sub-sequence



		A[1] = 6	A[2] = 3	A[3]=5	A[4]=2	A[5]=7	A[6]=8	A[7]=1	inf	Represents limiter
		1	2	3	4	5	6	7	8	j
[]	0	0	0	0	0	0	0	0	0	
[6]	1		0	0	0	1	1	0	1	
[6,3]	2			1	0	1	1	0	1	
[6,3,5]	3				0	2	2	0	2	
[6,3,5,2]	4					2	2	0	2	
[6,3,5,2,7]	5						3	0	3	
[6,3,5,2,7,8]	6							0	4	
[6,3,5,2,7,8,1]	7								4	
Represents sub-array	i									

[3, 5, 7, 8]

Sequence:

$$A[1\dots7] = [6,3,5,2,7,8,1]$$

We know the LIS length (4) but how do we find the LIS itself?

$$LIS = [3,5,7,8]$$

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Two comments

Question: Can we compute an optimum solution and not just its value?

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Yes!

Question: Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $\boxed{O(n \log n)}$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.

$$O(n2^n) \xrightarrow{\checkmark} O(2^n) \xrightarrow{\checkmark} O(n^2) \overset{\text{beyond our class}}{\xrightarrow{\uparrow}} O(n\log n)$$

How to come up with dynamic programming algorithm: summary

(RIY)

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- Get rich!