Pre-lecture brain teaser

What is the running time of the following algorithm:

Consider computing $f(x, y)$ by recursive function + memoization.

$$f(x, y) = \sum_{i=1}^{\min(x,y)} x \times f(x + y - i, i - 1),$$

$$f(0, y) = y \quad f(x, 0) = x.$$ 

The resulting algorithm when computing $f(n, n)$ would take:

(a) $O(n^2)$
(b) $O(n^3)$
(c) $O(2^n)$
(d) $O(n^n)$
(e) The function is ill defined - it can not be computed.
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Recipe for Dynamic Programming

1. Develop a recursive backtracking style algorithm $A$ for given problem.
2. Identify structure of subproblems generated by $A$ on an instance $I$ of size $n$
   - 2.1 Estimate number of different subproblems generated as a function of $n$. Is it polynomial or exponential in $n$?
   - 2.2 If the number of problems is “small” (polynomial) then they typically have some “clean” structure.
3. Rewrite subproblems in a compact fashion.
4. Rewrite recursive algorithm in terms of notation for subproblems.
5. Convert to iterative algorithm by bottom up evaluation in an appropriate order.
6. Optimize further with data structures and/or additional ideas.
Edit Distance and Sequence Alignment
Given a string “exponen” that is not in the dictionary, how should a spell checker suggest a nearby string?
Spell Checking Problem

Given a string “exponen” that is not in the dictionary, how should a spell checker suggest a nearby string?

What does nearness mean?

**Question:** Given two strings $x_1x_2\ldots x_n$ and $y_1y_2\ldots y_m$ what is a distance between them?
Spell Checking Problem

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What does nearness mean?

**Question:** Given two strings $x_1x_2 \ldots x_n$ and $y_1y_2 \ldots y_m$ what is a distance between them?

**Edit Distance:** minimum number of “edits” to transform $x$ into $y$. 
**Edit Distance**

**Definition**

*Edit distance* between two words $X$ and $Y$ is the number of letter insertions, letter deletions and letter substitutions required to obtain $Y$ from $X$.

**Example**

The edit distance between FOOD and MONEY is at least 4:

$$\text{FOOD} \rightarrow \text{MOOD} \rightarrow \text{MONOD} \rightarrow \text{MONED} \rightarrow \text{MONEY}$$
Alignment
Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

```
FOO
MONEY
```
**Alignment**
Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

```
FOOD
MONEY
```

Formally, an alignment is a set $M$ of pairs $(i, j)$ such that each index appears at most once, and there is no "crossing": $i < i'$ and $i$ is matched to $j$ implies $i'$ is matched to $j' > j$. In the above example, this is $M = \{(1, 1), (2, 2), (3, 3), (4, 5)\}$. 
**Alignment**
Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

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FOOD
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Formally, an alignment is a set $M$ of pairs $(i, j)$ such that each index appears at most once, and there is no “crossing”: $i < i'$ and $i$ is matched to $j$ implies $i'$ is matched to $j'$ > $j$. In the above example, this is $M = \{(1, 1), (2, 2), (3, 3), (4, 5)\}$. Cost of an alignment is the number of mismatched columns plus number of unmatched indices in both strings.
Edit Distance Problem

**Problem**
Given two words, find the edit distance between them, i.e., an alignment of smallest cost.
Applications

- Spell-checkers and Dictionaries
- Unix diff
- DNA sequence alignment ... but, we need a new metric
Definition
For two strings $X$ and $Y$, the cost of alignment $M$ is

- **[Gap penalty]** For each gap in the alignment, we incur a cost $\delta$.
- **[Mismatch cost]** For each pair $p$ and $q$ that have been matched in $M$, we incur cost $\alpha_{pq}$; typically $\alpha_{pp} = 0$.

Edit distance is a special case when $\delta = \alpha_{pq} = 1$. 
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Edit distance as alignment
Example

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<tr>
<th>o</th>
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Cost = $\delta + \alpha_{ae}$

Alternative:

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Cost = $3\delta$

Or a really stupid solution (delete string, insert other string):

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Cost = $19\delta$. 
What is the minimum edit distance for the following two strings, if insertion/deletion/change of a single character cost 1 unit?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5
What is the minimum edit distance for the following two strings, if insertion/deletion(change of a single character) cost 1 unit?

373
473

(a) 1  
(b) 2  
(c) 3  
(d) 4  
(e) 5
What is the minimum edit distance for the following two strings, if insertion/deletion/change of a single character cost 1 unit?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5
**Sequence Alignment**

**Input**  Given two words $X$ and $Y$, and gap penalty $\delta$ and mismatch costs $\alpha_{pq}$

**Goal**  Find alignment of minimum cost
Edit distance: The algorithm
Let $X = \alpha x$ and $Y = \beta y$

$\alpha, \beta$: strings.

$x$ and $y$ single characters.

Think about optimal edit distance between $X$ and $Y$ as alignment, and consider last column of alignment of the two strings:

Prefixes must have optimal alignment!
Let $X = x_1x_2 \cdots x_m$ and $Y = y_1y_2 \cdots y_n$. If $(m, n)$ are not matched then either the $m^{th}$ position of $X$ remains unmatched or the $n^{th}$ position of $Y$ remains unmatched.

- **Case** $x_m$ and $y_n$ are matched.
  - Pay mismatch cost $\alpha_{x_my_n}$ plus cost of aligning strings $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_{n-1}$

- **Case** $x_m$ is unmatched.
  - Pay gap penalty plus cost of aligning $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_n$

- **Case** $y_n$ is unmatched.
  - Pay gap penalty plus cost of aligning $x_1 \cdots x_m$ and $y_1 \cdots y_{n-1}$
Subproblems and Recurrence

<table>
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<tr>
<th>$x_1 \ldots x_{i-1}$</th>
<th>$x_i$</th>
<th>or</th>
<th>$x_1 \ldots x_{i-1}$</th>
<th>$x_i$</th>
<th>or</th>
<th>$x_1 \ldots x_{i-1}x_i$</th>
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**Optimal Costs**

Let $\text{Opt}(i, j)$ be optimal cost of aligning $x_1 \cdots x_i$ and $y_1 \cdots y_j$.

Then

$$\text{Opt}(i, j) = \min \left\{ \begin{array}{l} 
\alpha_{x_iy_j} + \text{Opt}(i-1, j-1), \\
\delta + \text{Opt}(i-1, j), \\
\delta + \text{Opt}(i, j-1) 
\end{array} \right. $$
Subproblems and Recurrence

\[
\begin{array}{c|c}
\hline
x_1 \cdots x_{i-1} & x_i \\
\hline
y_1 \cdots y_{j-1} & y_j \\
\hline
\end{array}
\quad \text{or} \quad 
\begin{array}{c|c}
\hline
x_1 \cdots x_{i-1} & x \\
\hline
y_1 \cdots y_{j-1}y_j & y_j \\
\hline
\end{array}
\quad \text{or} \quad 
\begin{array}{c|c}
\hline
x_1 \cdots x_{i-1}x_i & y_1 \cdots y_{j-1} \\
\hline
& y_j \\
\hline
\end{array}
\]

Optimal Costs

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\]

Base Cases: \( \text{Opt}(i, 0) = \delta \cdot i \) and \( \text{Opt}(0, j) = \delta \cdot j \)
Assume $X$ is stored in array $A[1..m]$ and $Y$ is stored in $B[1..n]$. Array $COST$ stores cost of matching two chars. Thus $COST[a, b]$ give the cost of matching character $a$ to character $b$.

$$EDIST(A[1..m], B[1..n])$$

If $(m = 0)$ return $n\delta$
If $(n = 0)$ return $m\delta$
$m_1 = \delta + EDIST(A[1..(m - 1)], B[1..n])$
$m_2 = \delta + EDIST(A[1..m], B[1..(n - 1)])$
$m_3 = COST[A[m], B[n]] + EDIST(A[1..(m - 1)], B[1..(n - 1)])$
return $\min(m_1, m_2, m_3)$
Example: DEED and DREAD

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\[
\text{Opt}(i, j) = \min \begin{cases} 
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![Graph](image-url)
Example: DEED and DREAD

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Diagram showing the relationship between the symbols.
### Example: DEED and DREAD

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<tr>
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Dynamic programming algorithm for edit-distance
As part of the input...

The cost of aligning a character against another character

\( \Sigma: \text{Alphabet} \)

We are given a cost function (in a table):

\[
\forall b, c \in \Sigma \quad COST[b][c] = \text{cost of aligning } b \text{ with } c.
\]

\[
\forall b \in \Sigma \quad COST[b][b] = 0
\]

\( \delta: \) price of deletion of insertion of a single character
**Dynamic program for edit distance**

\[ EDIST(A[1..m], B[1..n]) \]

\[
\begin{align*}
int & \quad M[0..m][0..n] \\
& \quad \text{for } i = 1 \text{ to } m \text{ do } M[i, 0] = i\delta \\
& \quad \text{for } j = 1 \text{ to } n \text{ do } M[0, j] = j\delta \\
& \quad \text{for } i = 1 \text{ to } m \text{ do} \\
& \quad \quad \text{for } j = 1 \text{ to } n \text{ do} \\
& \quad \quad \quad \quad M[i][j] = \min \left\{ \begin{array}{l}
\text{COST}[A[i]][B[j]] + M[i-1][j-1], \\
\delta + M[i-1][j], \\
\delta + M[i][j-1]
\end{array} \right\}
\end{align*}
\]

- Running time is \( O(mn) \).
- Space used is \( O(mn) \).
Dynamic program for edit distance

\[
EDIST(A[1..m], B[1..n])
\]
\[
\text{int } M[0..m][0..n]
\]
\[
\text{for } i = 1 \text{ to } m \text{ do } M[i, 0] = i\delta
\]
\[
\text{for } j = 1 \text{ to } n \text{ do } M[0, j] = j\delta
\]

\[
\text{for } i = 1 \text{ to } m \text{ do }
\]
\[
\text{for } j = 1 \text{ to } n \text{ do }
\]
\[
M[i][j] = \min \begin{cases}
\text{COST}[A[i]][B[j]] + M[i-1][j-1], \\
\delta + M[i-1][j], \\
\delta + M[i][j-1]
\end{cases}
\]

Analysis

- Running time is \(O(mn)\).
- Space used is \(O(mn)\).
Reducing space for edit distance
Figure 1: Iterative algorithm in previous slide computes values in row order.
Optimizing Space

- Recall

\[ M(i, j) = \min \begin{cases} 
\alpha_{x_iy_j} + M(i - 1, j - 1), \\
\delta + M(i - 1, j), \\
\delta + M(i, j - 1) \end{cases} \]

- Entries in \( j^{th} \) column only depend on \( (j - 1)^{st} \) column and earlier entries in \( j^{th} \) column

- Only store the current column and the previous column reusing space; \( N(i, 0) \) stores \( M(i, j - 1) \) and \( N(i, 1) \) stores \( M(i, j) \)
Example: DEED vs. DREAD filled by column

<table>
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<tr>
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<th>R</th>
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Example: DEED vs. DREAD filled by column

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Figure 2: $M(i, j)$ only depends on previous column values. Keep only two columns and compute in column order.
Space Efficient Algorithm

\[
\text{for all } i \text{ do } N[i, 0] = i\delta \\
\text{for } j = 1 \text{ to } n \text{ do } \\
\quad N[0, 1] = j\delta \quad (* \text{ corresponds to } M(0, j) *) \\
\text{for } i = 1 \text{ to } m \text{ do } \\
\quad N[i, 1] = \min \left\{ \alpha_{x_i y_j} + N[i - 1, 0], \delta + N[i - 1, 1], \delta + N[i, 0] \right\} \\
\text{for } i = 1 \text{ to } m \text{ do } \\
\quad \text{Copy } N[i, 0] = N[i, 1]
\]

Analysis
Running time is $O(mn)$ and space used is $O(2m) = O(m)$
Analyzing Space Efficiency

- From the $m \times n$ matrix $M$ we can construct the actual alignment (exercise)
- Matrix $N$ computes cost of optimal alignment but no way to construct the actual alignment
Longest Common Subsequence Problem
**LCS Problem**

**Definition**

The **LCS** between two strings $X$ and $Y$ is the length of longest common subsequence between $X$ and $Y$. 

![Example](image)

Example: LCS between ABAZDC and BACBAD is 4 via ABAD

Derive a dynamic programming algorithm for the problem.
LCS Problem

Definition

**LCS** between two strings $X$ and $Y$ is the length of longest common subsequence between $X$ and $Y$.

$$
\begin{align*}
&ABAZDC & ABAZDC \\
&BACBAD & BACBAD
\end{align*}
$$

Example

LCS between ABAZDC and BACBAD is 4 via ABAD
LCS Problem

Definition

LCS between two strings $X$ and $Y$ is the length of longest common subsequence between $X$ and $Y$.

```
ABAZDC
BACBAD
```

```
ABAZDC
BACBAD
```

Example

LCS between ABAZDC and BACBAD is 4 via ABAD

Derive a dynamic programming algorithm for the problem.
How do we plan out the recursion?

Start off with $A[1...m]$ and $B[1...n]$ and reason the following:

• Assuming $A[m] \neq B[n]$

One or neither of the end characters are in the LCS. Therefore:

$$\max \left( \text{LCS} \left( A[1...m-1], B[1...n]\right), \text{LCS} \left( A[1...m], B[1...n-1]\right) \right)$$

• Assuming $A[m] = B[n]$

$A[m]$ and $B[n]$ are both in the LCS. Therefore:

$$\text{LCS} \left( A[1...m], B[1...n]\right) = 1 + \text{LCS} \left( A[1...m-1], B[1...n-1]\right)$$

• Base Case: $A$ is empty or $B$ is empty
How do we plan out the recursion?

Start off with $A[1...m]$ and $B[1...n]$ and reason the following:

- Assuming $A[m] \neq B[n]$
  - One or neither of the end characters are in the LCS. Therefore:
    $$\max \left( \text{LCS} \left( A[1...m-1], B[1...n] \right), \text{LCS} \left( A[1...m], B[1...n-1] \right) \right)$$

- Assuming $A[m] = B[n]$
  - $A[m]$ and $B[n]$ are both in the LCS. Therefore:
    $$\text{LCS} \left( A[1...m], B[1...n] \right) = 1 + \text{LCS} \left( A[1...m-1], B[1...n-1] \right)$$

- Base Case: $A$ is empty or $B$ is empty
How do we plan out the recursion?

Start off with $A[1...m]$ and $B[1...n]$ and reason the following:

- Assuming $A[m] \neq B[n]$
  - One or neither of the end characters are in the LCS. Therefore:
    \[
    \max (\text{LCS}(A[1...m - 1], B[1...n]), \text{LCS}(A[1...m], B[1...n - 1]))
    \]

- Base Case: $A$ is empty or $B$ is empty
How do we plan out the recursion?

Start off with $A[1...m]$ and $B[1...n]$ and reason the following:

- Assuming $A[m] \neq B[n]$
  - One or neither of the end characters are in the LCS. Therefore:
    \[
    \max \left( LCS(A[1...m - 1], B[1...n]), LCS(A[1...m], B[1...n - 1]) \right)
    \]
- Assuming $A[m] = B[n]$
How do we plan out the recursion?

Start off with $A[1...m]$ and $B[1...n]$ and reason the following:

- Assuming $A[m] \neq B[n]$
  - One or neither of the end characters are in the LCS. Therefore:
    $$\max(LCS(A[1...m-1], B[1...n]), LCS(A[1...m], B[1...n-1]))$$

- Assuming $A[m] = B[n]$
  - $A[m]$ and $B[n]$ are both in the LCS. Therefore:
    $$LCS(A[1...m], B[1...n]) = 1 + LCS(A[1...m-1], B[1...n-1])$$
How do we plan out the recursion?

Start off with $A[1...m]$ and $B[1...n]$ and reason the following:

- Assuming $A[m] \neq B[n]$
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    $$\max(LCS(A[1...m-1], B[1...n]), LCS(A[1...m], B[1...n-1]))$$

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  - $A[m]$ and $B[n]$ are both in the LCS. Therefore:
    $$LCS(A[1...m], B[1...n]) = 1 + LCS(A[1...m-1], B[1...n-1])$$

- Base Case: $A$ is empty or $B$ is empty
LCS recursive definition

\[ A[1..n], B[1..m] : \text{Input strings.} \]

\[
LCS(i, j) = \begin{cases}
0 & \text{i = 0 or j = 0} \\
\max \left( \begin{array}{c}
LCS(i - 1, j) \\
LCS(i, j - 1)
\end{array} \right) & A[i] \neq B[j] \\
1 + LCS(i - 1, j - 1) & A[i] = B[j]
\end{cases}
\]
LCS recursive definition

\[ A[1..n], B[1..m]: \text{Input strings.} \]

\[
LCS(i, j) = \begin{cases} 
0 & i = 0 \text{ or } j = 0 \\
\max \left( \begin{array}{c} 
LCS(i - 1, j), \\
LCS(i, j - 1) 
\end{array} \right) & A[i] \neq B[j] \\
1 + LCS(i - 1, j - 1) & A[i] = B[j]
\end{cases}
\]

Running time: Similar to edit distance... \( O(nm) \)

Space: \( O(m) \) space.
Longest common subsequence is just edit distance for the two sequences...

\[ A, B: \text{input sequences, } \Sigma: \text{“alphabet” all the different values in } A \text{ and } B \]

\[
\begin{align*}
\forall b, c \in \Sigma : b &\neq c \quad \text{COST}[b][c] = +\infty. \\
\forall b \in \Sigma &\quad \text{COST}[b][b] = 1
\end{align*}
\]

1 : price of deletion of insertion of a single character
Longest common subsequence is just edit distance for the two sequences...

\( A, B: \) input sequences, \( \Sigma: \) “alphabet” all the different values in \( A \) and \( B \)

\[
\forall b, c \in \Sigma : b \neq c \quad \text{\( COST[b][c] = +\infty. \)}
\]

\[
\forall b \in \Sigma \quad \text{\( COST[b][b] = 1 \)}
\]

1: price of deletion of insertion of a single character

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</tbody>
</table>
Longest common subsequence is just edit distance for the two sequences...

$A, B$: input sequences, $\Sigma$: "alphabet" all the different values in $A$ and $B$

\[
\forall b, c \in \Sigma : b \neq c \quad \text{COST}[b][c] = +\infty.
\]

\[
\forall b \in \Sigma \quad \text{COST}[b][b] = 1
\]

1: price of deletion of insertion of a single character

Length of longest common sub-sequence $= m + n - \text{ed}(A, B)$