Pre-lecture brain teaser

Given a directed graph (G), propose an algorithm that finds a vertex that is contained within the source SCC of the meta-graph of G.

→ max post numbering in DFS(G)

ECE-374-B: Lecture 16 - Shortest Paths [BFS, Djikstra]

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March 21, 2024

University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

Given a directed graph (G), propose an algorithm that finds a vertex that is contained within the source SCC of the meta-graph of G.

Breadth First Search

Breadth First Search (BFS)

Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a queue data structure.

(B) It processes the vertices in the graph in the order of their shortest distance from the vertex s (the start vertex).

As such...

- **DFS** good for exploring graph structure
- **BFS** good for exploring distances

Queue Data Structure

Queues
A queue is a list of elements which supports the operations:

- **enqueue**: Adds an element to the end of the list
- **dequeue**: Removes an element from the front of the list

Elements are extracted in first-in first-out (FIFO) order, i.e., elements are picked in the order in which they were inserted.

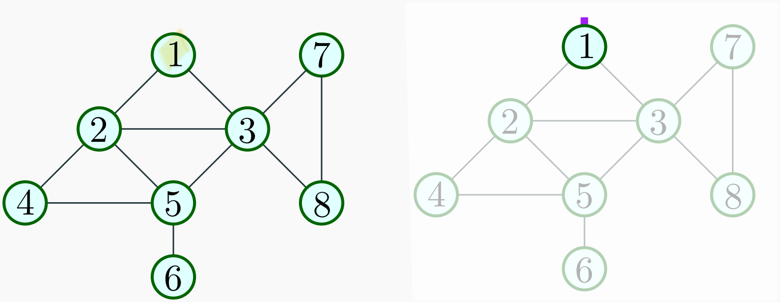
BFS Algorithm

Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

```
BFS(s)
    Mark all vertices as unvisited
    Initialize search tree T to be empty
    Mark vertex s as visited
    set Q to be the empty queue
    enqueue(Q, s)
    while Q is nonempty do
        u = dequeue(Q)
        for each vertex v ∈ Adj(u)
            if v is not visited then
                add edge (u, v) to T
                Mark v as visited and enqueue(v)
```

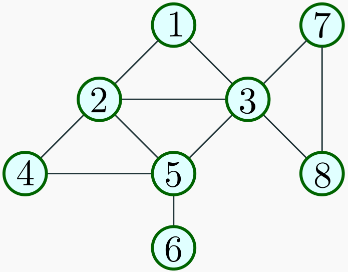
Proposition
BFS(s) *runs in* $O(n + m)$ *time.*

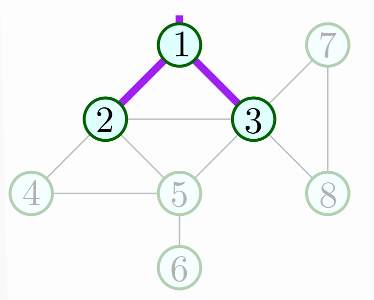
BFS: An Example in Undirected Graphs



T1. [1]

BFS: An Example in Undirected Graphs

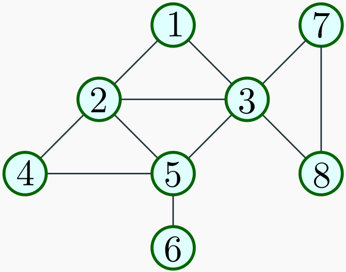


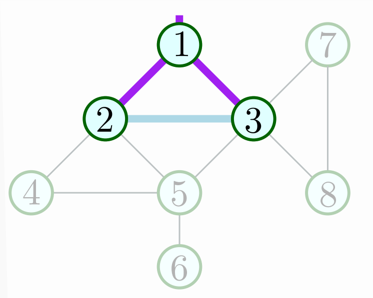


T1. [1]

T2. [2,3]

BFS: An Example in Undirected Graphs

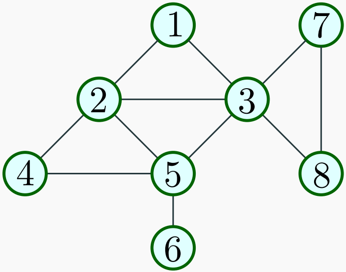


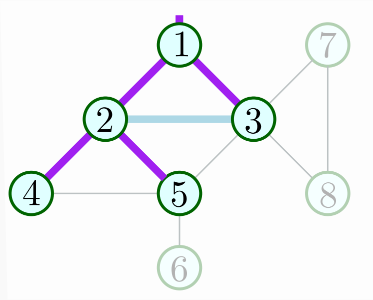


T1. [1]

T2. [2,3]

BFS: An Example in Undirected Graphs



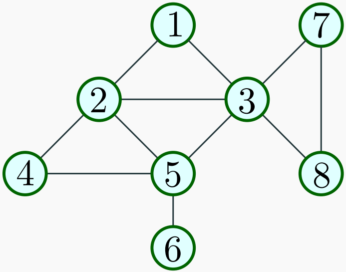


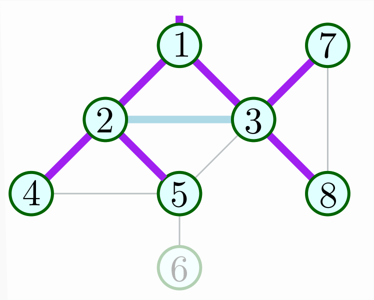
T1. [1]

T2. [2,3]

T3. [3,4,5]

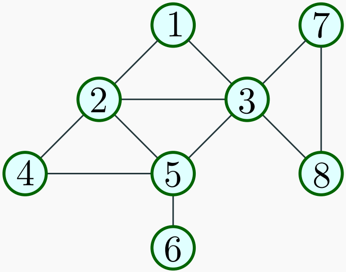
BFS: An Example in Undirected Graphs

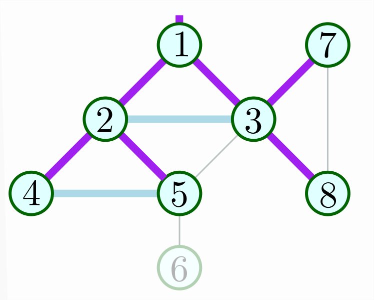




T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]

BFS: An Example in Undirected Graphs

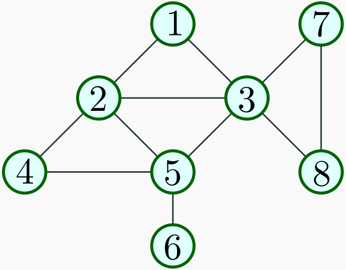


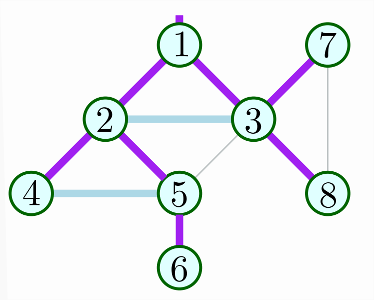


6

T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]

BFS: An Example in Undirected Graphs





T1. [1]

T2. [2,3]

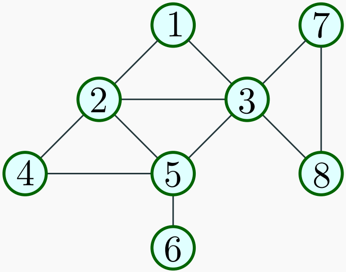
T3. [3,4,5]

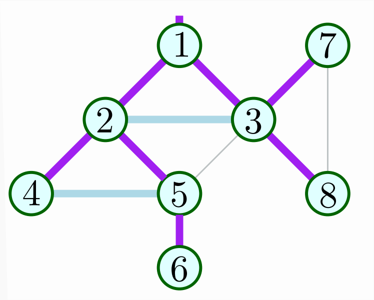
T4. [4,5,7,8]

T5. [5,7,8]

T6. [7,8,6]

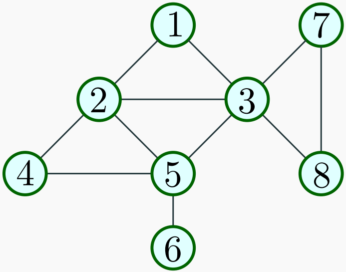
BFS: An Example in Undirected Graphs

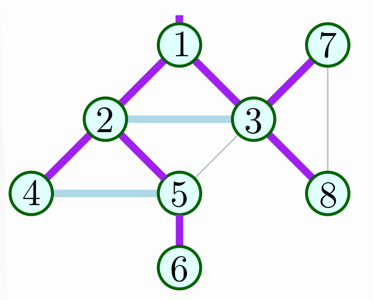




T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]
T6. [7,8,6]
T7. [8,6]

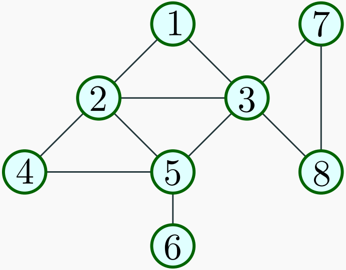
BFS: An Example in Undirected Graphs

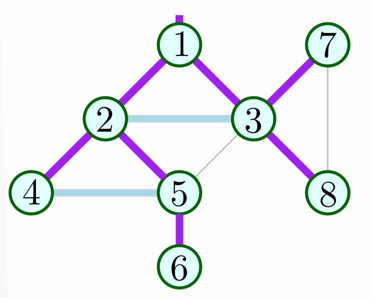




T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]
T6. [7,8,6]
T7. [8,6]
T8. [6]

BFS: An Example in Undirected Graphs

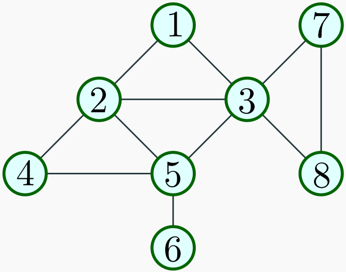


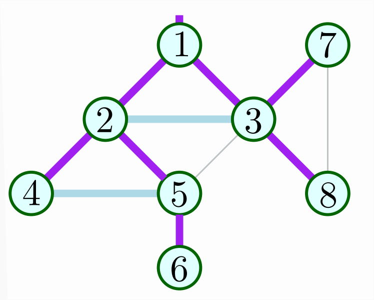


T1. [1]
T2. [2,3]
T3. [3,4,5]
T4. [4,5,7,8]
T5. [5,7,8]
T6. [7,8,6]
T7. [8,6]
T8. [6]
T9. []

BFS tree is the set of purple edges.

BFS: An Example in Undirected Graphs

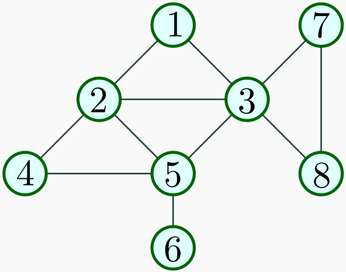


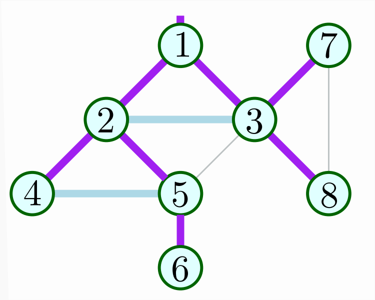


T1.	[1]	T4.	[4,5,7,8]	T7.	[8,6]
T2.	[2,3]	T5.	[5,7,8]	T8.	[6]
T3.	[3,4,5]	T6.	[7,8,6]	T9.	[]

BFS tree is the set of purple edges.

BFS: An Example in Undirected Graphs





T1.	[1]	T4.	[4,5,7,8]	T7.	[8,6]
T2.	[2,3]	T5.	[5,7,8]	T8.	[6]
T3.	[3,4,5]	T6.	[7,8,6]	T9.	[]

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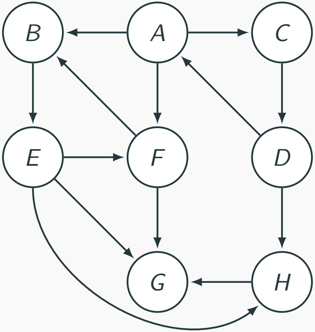
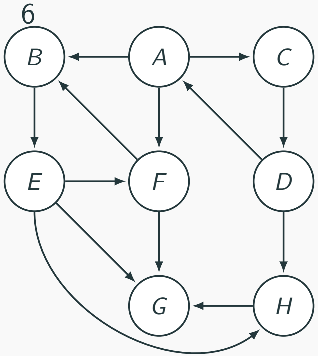
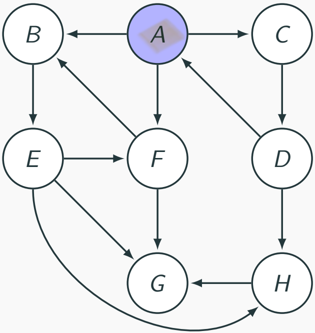
BFS: An Example in Directed Graphs



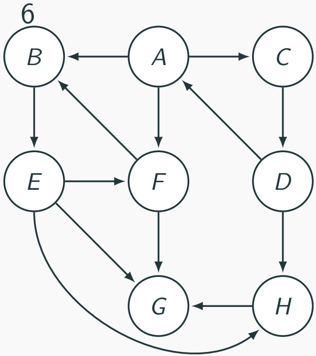
BFS: An Example in Directed Graphs

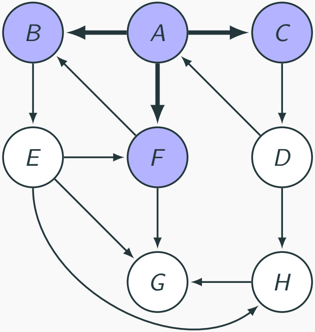




T1. [A]

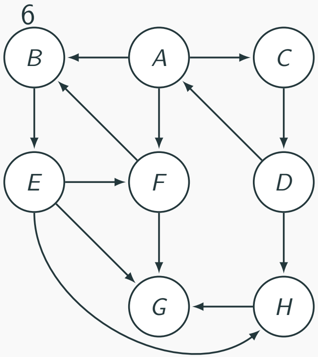
BFS: An Example in Directed Graphs

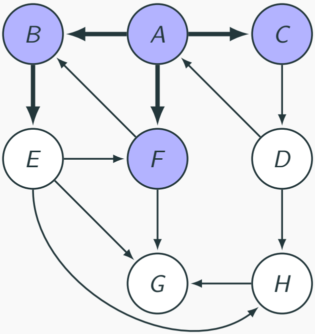




T1. [A]
T2. [B,C,F]

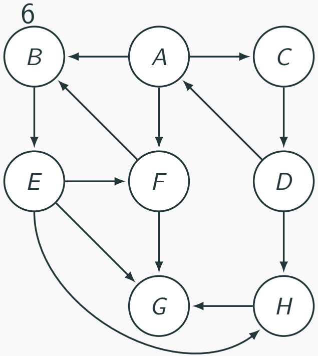
BFS: An Example in Directed Graphs

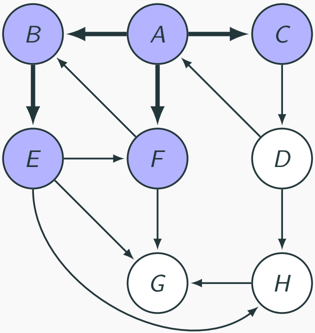




T1. [A]
T2. [B,C,F]

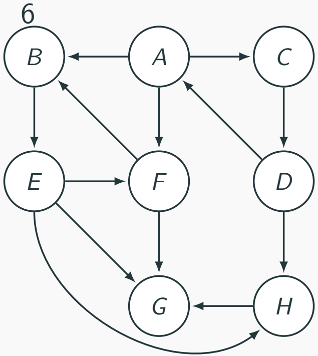
BFS: An Example in Directed Graphs

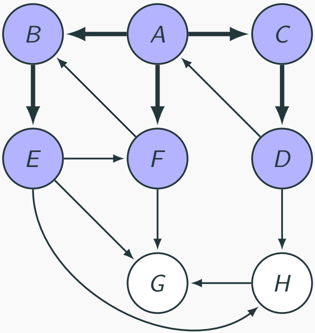




T1. [A]
T2. [B,C,F]
T3. [C,F,E]

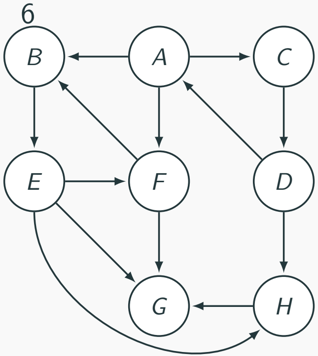
BFS: An Example in Directed Graphs

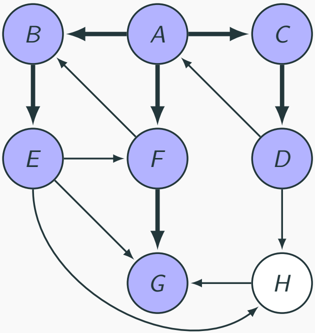




T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]

BFS: An Example in Directed Graphs





T1. [A]

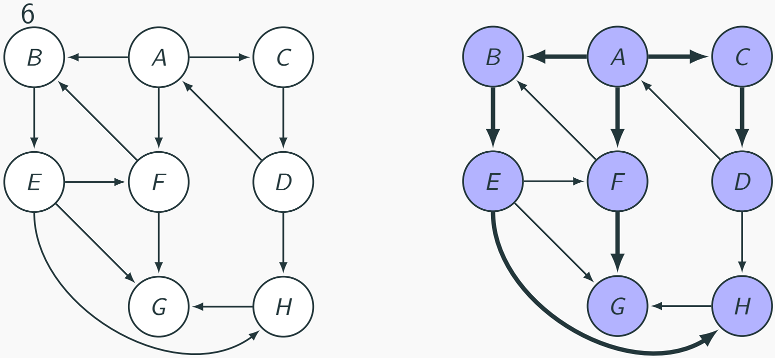
T2. [B,C,F]

T3. [C,F,E]

T4. [F,E,D]

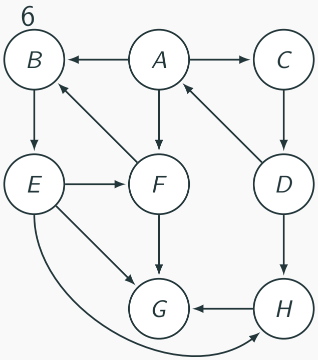
T5. [E,D,G]

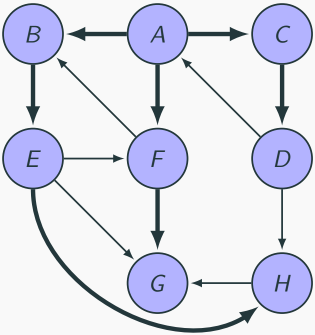
BFS: An Example in Directed Graphs



T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]
T6. [D,G,H]

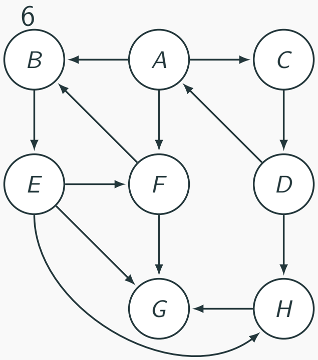
BFS: An Example in Directed Graphs

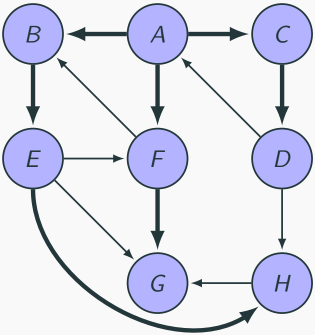




T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]
T6. [D,G,H]
T7. [G,H]

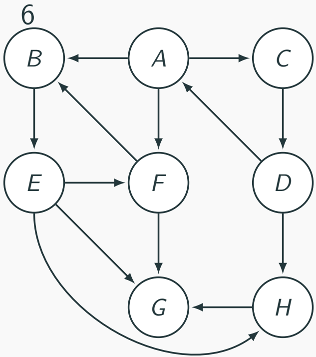
BFS: An Example in Directed Graphs

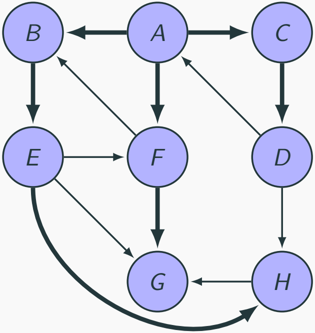




T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]
T6. [D,G,H]
T7. [G,H]
T8. [H]

BFS: An Example in Directed Graphs





T1. [A]
T2. [B,C,F]
T3. [C,F,E]
T4. [F,E,D]
T5. [E,D,G]
T6. [D,G,H]
T7. [G,H]
T8. [H]
T9. []

BFS with distances and layers

BFS with distances

BFS(s)

  Mark all vertices as unvisited; for each v set $\text{dist}(v) = \infty$

  Initialize search tree T to be empty

  Mark vertex s as visited and set $\text{dist}(s) = 0$

  set Q to be the empty queue

  enqueue(s)

  while Q is nonempty **do**

    u = **dequeue**(Q)

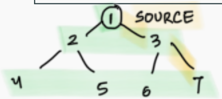
    for each vertex $v \in \text{Adj}(u)$ **do**

      if v is not visited **do**

        add edge (u, v) to T

        Mark v as visited, **enqueue**(v)

        and set $\text{dist}(v) = \text{dist}(u) + 1$



dist(v) = distance of v from s

dist(7) = 2

Properties of BFS: Undirected Graphs

Theorem
The following properties hold upon termination of **BFS**(s)

(A) *Search tree contains exactly the set of vertices in the connected component of* s.

(B) *If* $\mathrm{dist}(u) < \mathrm{dist}(v)$ *then* u *is visited before* v.

(C) *For every vertex* u, $\mathrm{dist}(u)$ *is the length of a shortest path (in terms of number of edges) from* s *to* u.

(D) *If* u, v *are in connected component of* s *and* $e = \{u, v\}$ *is an edge of* G, *then* $|\mathrm{dist}(u) - \mathrm{dist}(v)| \leq 1$.

Think about it!

Properties of BFS: Directed Graphs

Theorem
The following properties hold upon termination of **BFS**(s)*:*

(A) *The search tree contains exactly the set of vertices reachable from s*

(B) *If* $\text{dist}(u) < \text{dist}(v)$ *then u is visited before v*

(C) *For every vertex u,* $\text{dist}(u)$ *is indeed the length of shortest path from s to u*

(D) *If u is reachable from s and* $e = (u, v)$ *is an edge of G, then* $\text{dist}(v) - \text{dist}(u) \leq 1$. *Not necessarily the case that* $\text{dist}(u) - \text{dist}(v) \leq 1$.

e : u → v

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
            initialize L_{i+1} to be an empty list
            for each u in L_i do
                for each edge (u, v) ∈ Adj(u) do
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

BFS with Layers

```
BFSLayers(s):
    Mark all vertices as unvisited and initialize T to be empty
    Mark s as visited and set L_0 = {s}
    i = 0
    while L_i is not empty do
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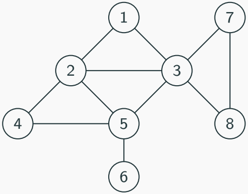
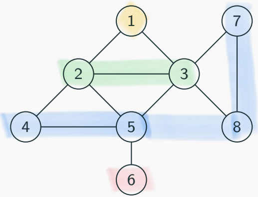
                if v is not visited
                        mark v as visited
                        add (u, v) to tree T
                        add v to L_{i+1}
            i = i + 1
```

Running time: $O(n + m)$

Example



Example



source = 1

Layer 0: 1

Layer 1: 2, 3

Layer 2: 4, 5, 7, 8

Layer 3: 6

dist(1) = 0

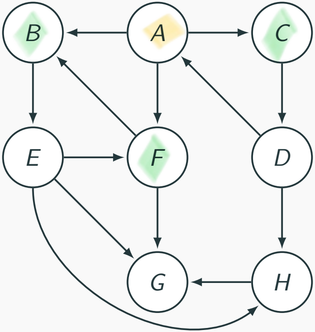
dist(2) = 1 = dist(3)

(RIY)

Proposition
The following properties hold on termination of **BFSLayers**(s).

- **BFSLayers**(s) *outputs a* **BFS** *tree*
- L_i *is the set of vertices at distance exactly* i *from* s
- *If* G *is undirected, each edge* $e = \{u, v\}$ *is one of three types:*
  - *tree edge between two consecutive layers*
  - *non-tree forward/backward edge between two consecutive layers*
  - *non-tree cross-edge with both* u, v *in same layer*
  - $\implies$ *Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.*

Example



Layer 0: A

Layer 1: B, F, C

Layer 2: E, G, D

Layer 3: H

BFS with Layers: Properties for directed graphs

Proposition
The following properties hold on termination of **BFSLayers**(s), *if* G *is directed.*

For each edge $e = (u, v)$ *is one of four types:*

- *a tree edge between consecutive layers,* $u \in L_i, v \in L_{i+1}$ *for some* $i \geq 0$
- *a non-tree forward edge between consecutive layers*
- *a non-tree backward edge*
- *a cross-edge with both* u, v *in same layer*

Shortest Paths and Dijkstra's Algorithm

Problem definition

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes s, t find shortest path from s to t.
- Given node s find shortest path from s to all other nodes.
- Find shortest paths for all pairs of nodes.

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Many applications!

Single-Source Shortest Paths: Non-Negative Edge Lengths

- Single-Source Shortest Path Problems
  - Input: A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.
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  - Restrict attention to directed graphs
  - Undirected graph problem can be reduced to directed graph problem - how?
    - Given undirected graph G, create a new directed graph G' by replacing each edge $\{u, v\}$ in G by (u, v) and (v, u) in G'.
    - set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$
    - Exercise: show reduction works. Relies on non-negativity!

Shortest path in the weighted case using BFS

Single-Source Shortest Paths via BFS

- **Special case:** All edge lengths are 1.

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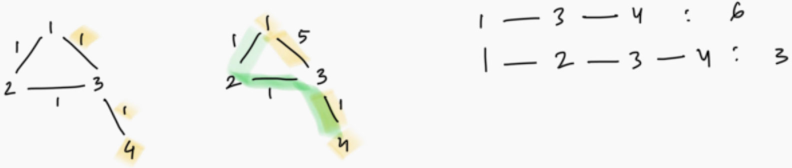
  Can we use **BFS**?



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  - Run **BFS**(s) to get shortest path distances from s to all other nodes.
  - $O(m+n)$ time algorithm.
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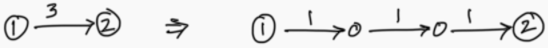
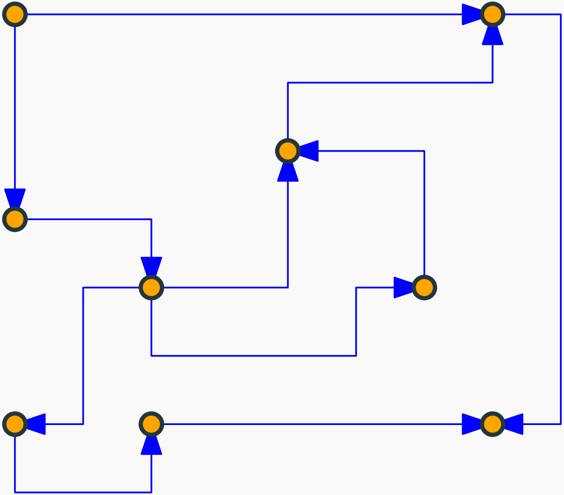
  Can we use **BFS**? Reduce to unit edge-length problem by placing $\ell(e)-1$ dummy nodes on e.



Example of edge refinement



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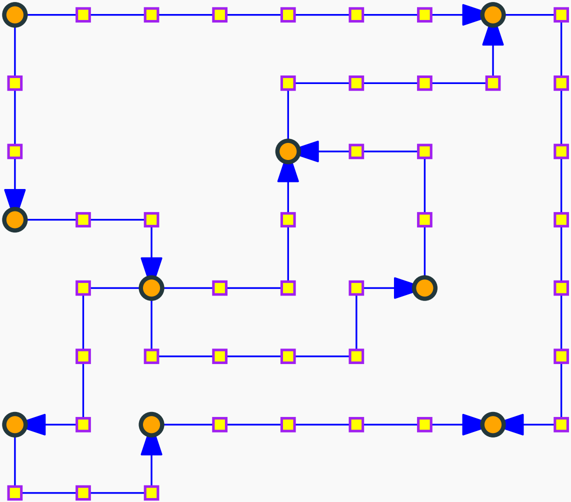

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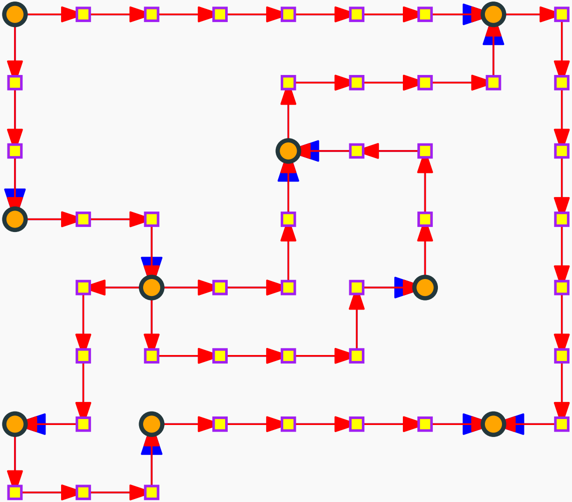


Shortest path using BFS

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. **BFS** takes $O(mL + n)$ time. Not efficient if L is large.

On the hereditary nature of shortest paths

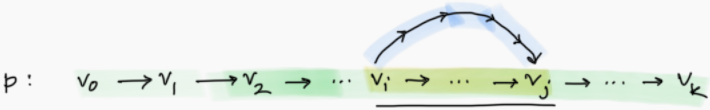
You can not shortcut a shortest path

Lemma
G: directed graph with non-negative edge lengths.

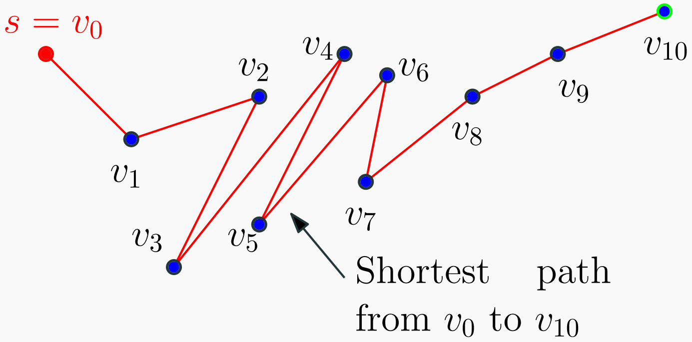
$\text{dist}(s, v)$: *shortest path length from s to v.*

If $p = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ shortest path from s to v_k then for any $0 \leq i < j \leq k$:

$v_i \to v_{i+1} \to \ldots \to v_j$ is shortest path from v_i to v_j



A proof by picture



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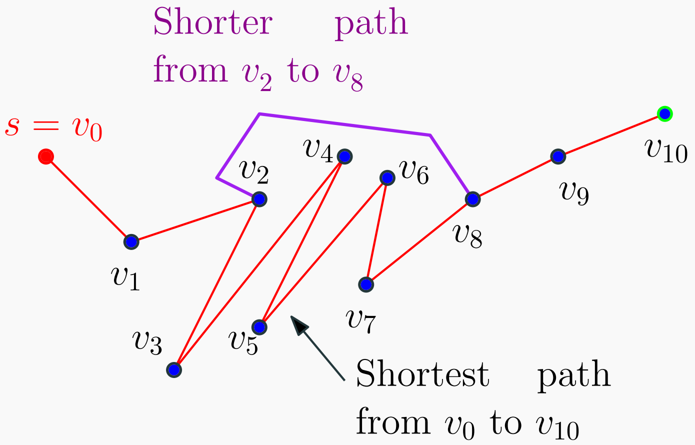
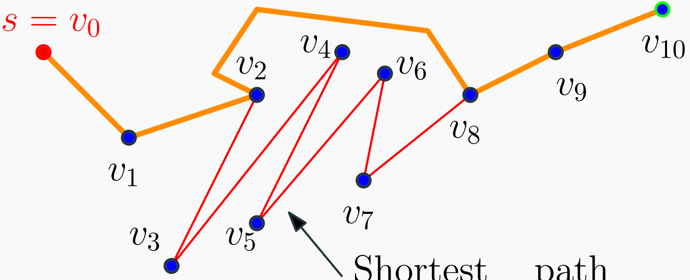


A proof by picture

"OPTIMAL SUBSTRUCTURE"

A shorter path from v_0 to v_{10}. A contradiction.



Shortest path from v_0 to v_{10}

What we really need...

Corollary
G: directed graph with non-negative edge lengths.

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If $p = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ shortest path from s to v_k then for any $0 \leq i \leq k$:

- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ *is shortest path from s to v_i*
- $\mathrm{dist}(s, v_i) \leq \mathrm{dist}(s, v_k)$. *Relies on non-neg edge lengths.*

The basic algorithm: Find the i^{th} closest vertex (to the source s)

A Basic Strategy

Explore vertices in increasing order of distance from s:
(For simplicity assume that nodes are at different distances from s and that no edge has zero length)

```
Initialize for each node v, dist(s, v) = ∞
Initialize X = {s},
for i = 2 to |V| do
    (* Invariant: X contains the i − 1 closest nodes to s *)
    Among nodes in V − X, find the node v that is the
          i^th closest to s
    Update dist(s, v)
    X = X ∪ {v}
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How can we implement the step in the for loop?

Finding the i^{th} closest node

- X contains the $i-1$ closest nodes to s
- Want to find the i^{th} closest node from $V-X$.

What do we know about the i^{th} closest node?

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Let P be a shortest path from s to v where v is the i^{th} closest node. Then, all intermediate nodes in P belong to X.

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What do we know about the i^{th} closest node?

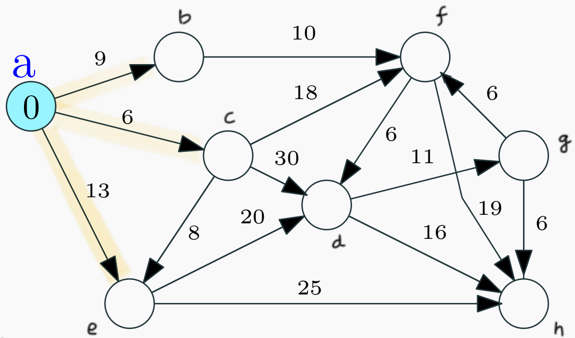
Claim

Let P be a shortest path from s to v where v is the i^{th} closest node. Then, all intermediate nodes in P belong to X.

Proof.

If P had an intermediate node u not in X then u will be closer to s than v. Implies v is not the i^{th} closest node to s - recall that X already has the $i-1$ closest nodes. □

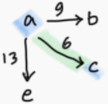
Finding the i^{th} closest node repeatedly



source $s = a$

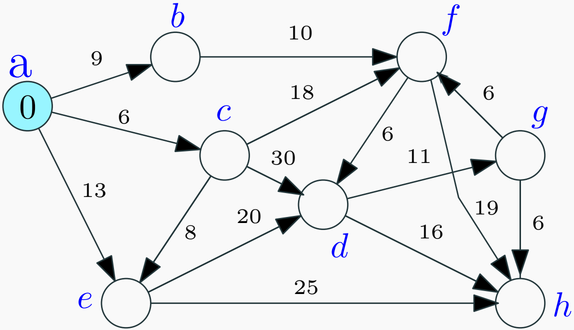
Step 1.

$X = \{a\}$



2nd closest vertex to a from the set $V - X = V - \{a\}$ is c

Finding the i^{th} closest node repeatedly



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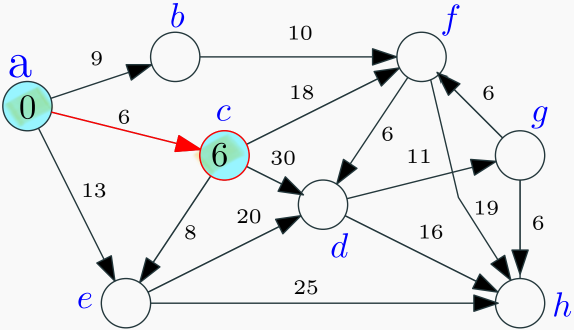
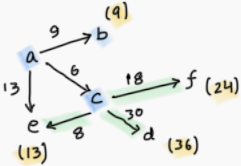


step 2.

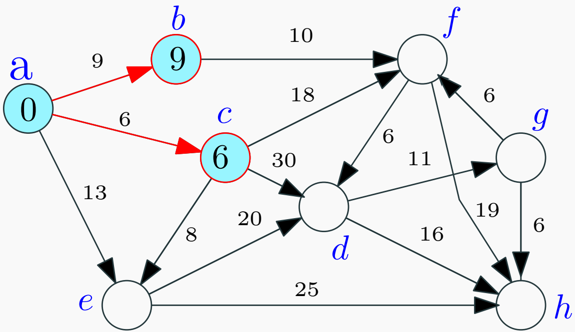
$X = \{a, c\}$

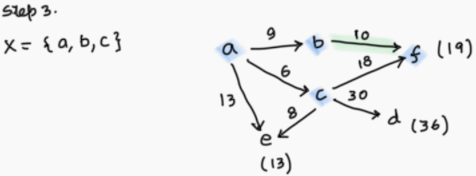


3rd closest vertex to a from $V - X$
$= V - \{a, c\}$

is b !

Finding the i^{th} closest node repeatedly





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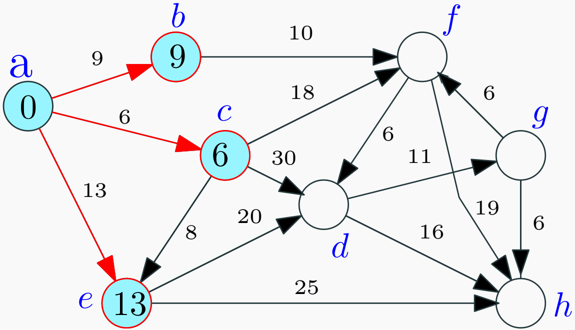
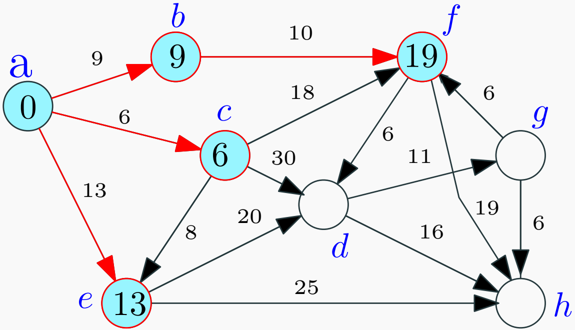


Finding the ith closest node repeatedly



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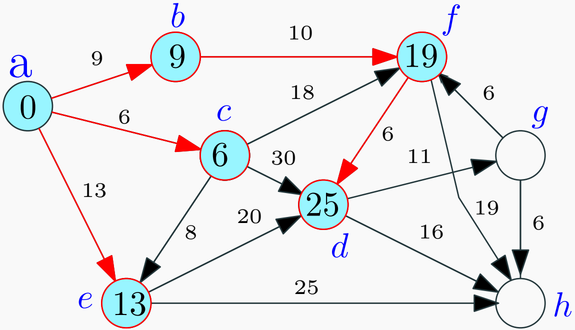

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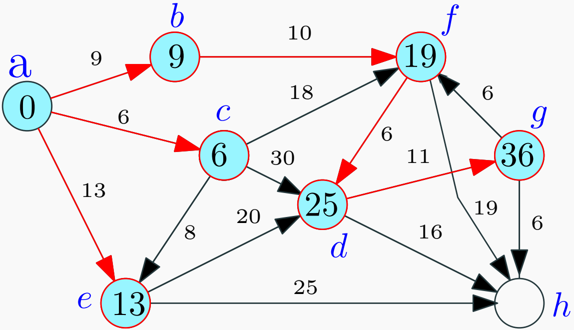

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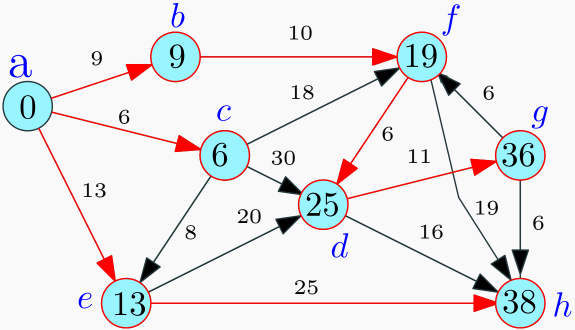

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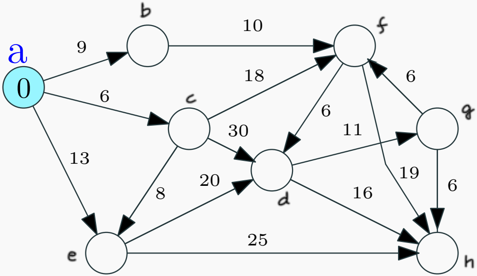

Corollary

The i^{th} closest node is adjacent to X.

Algorithm

Initialize for each node v: $\text{dist}(s, v) = \infty$ ← O(n) (i)

Initialize $X = \emptyset$, $d'(s, s) = 0$

for $i = 1$ to $|V|$ **do** ← O(n) (ii)

(* Invariant: X contains the $i - 1$ closest nodes to s *)

(* Invariant: $d'(s, u)$ is shortest path distance from u to s using only X as intermediate nodes*)

Let v be such that $d'(s, v) = \min_{u \in V - X} d'(s, u)$ ← O(n)

$\text{dist}(s, v) = d'(s, v)$

$X = X \cup \{v\}$

for each node u in $V - X$ **do**

$d'(s, u) = \min_{t \in X} \left(\text{dist}(s, t) + \ell(t, u) \right)$

We are looking at all the edges going out of X ← O(m)

O(m+n) (iii)

Runtime: $O(n) + O(n \cdot (m + n)) = O(n(n+m))$

(i), (ii), (iii)

Algorithm

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Initialize for each node v:  dist(s, v) = ∞
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    for each node u in V − X do
        d'(s, u) = min_{t∈X} (dist(s, t) + ℓ(t, u))
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Running time:

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Running time: $O(n \cdot (n + m))$ time.

- n outer iterations. In each iteration, $d'(s, u)$ for each u by scanning all edges out of nodes in X; $O(m + n)$ time/iteration.

Dijkstra's algorithm

Edsger W. Dijkstra in 1956

Turing Award in 1974

Main Idea

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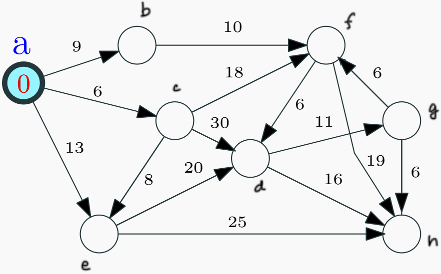
1. Maintain a set, S of vertices whose shortest path distance from s is known.
2. At each step, add to S a vertex v in V - S whose distance estimate is minimum.
3. Update distance estimates of vertices adjacent to v.

Example: Dijkstra algorithm in action



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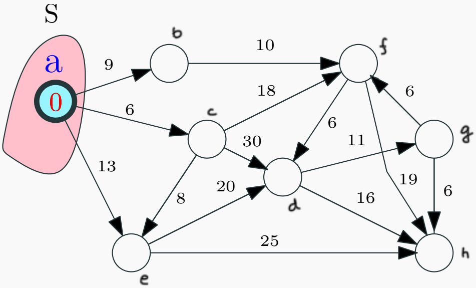

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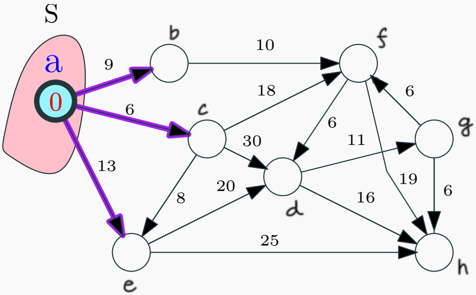

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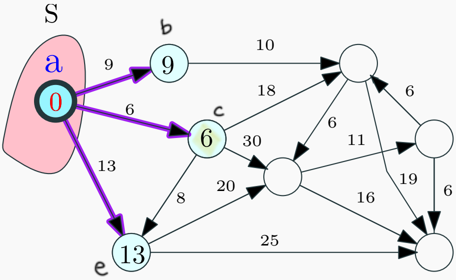

S = {a, c}

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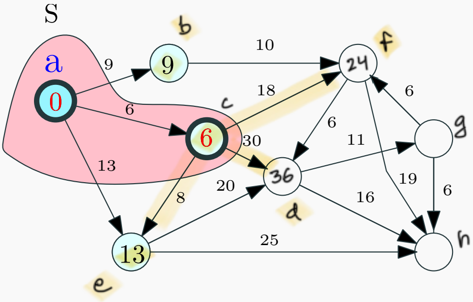

s = {a, c, b}

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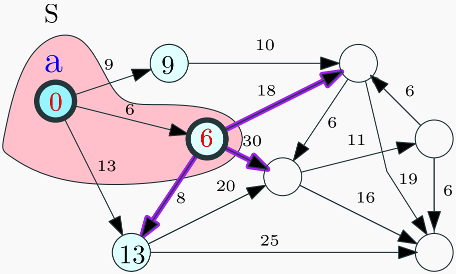

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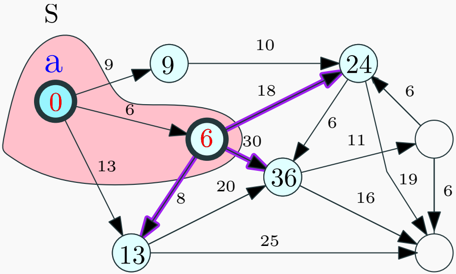

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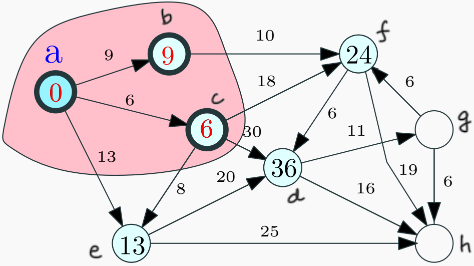

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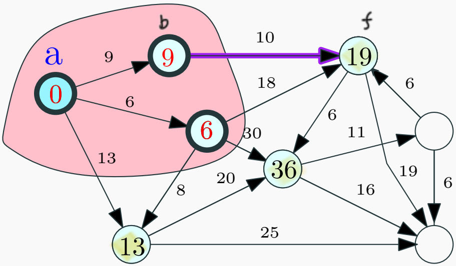

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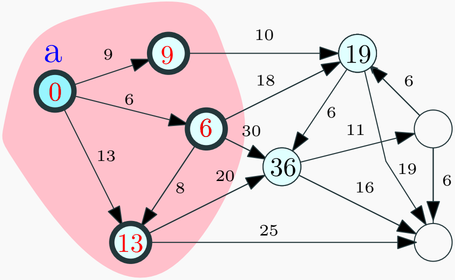

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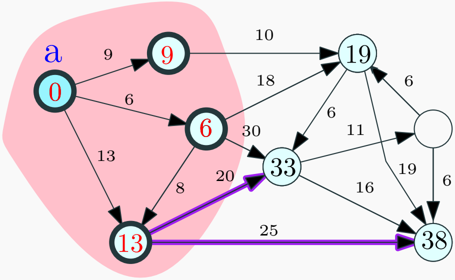

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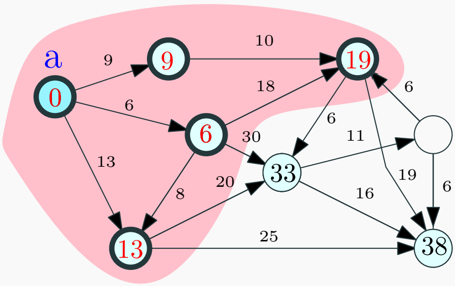

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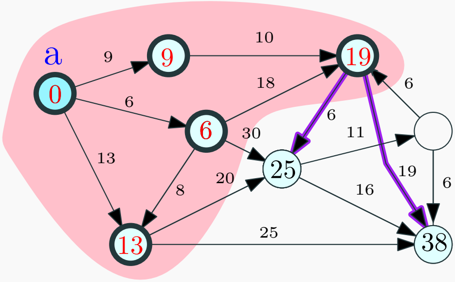

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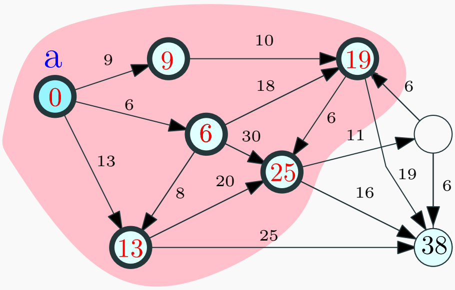

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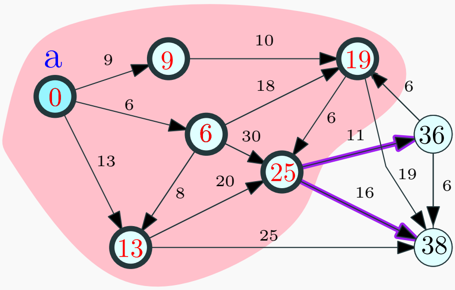

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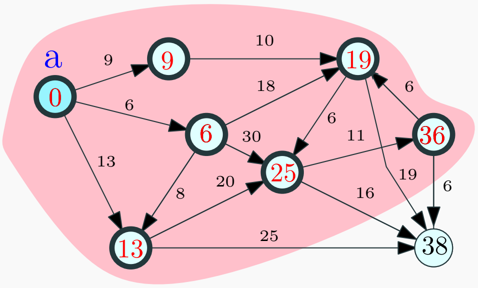

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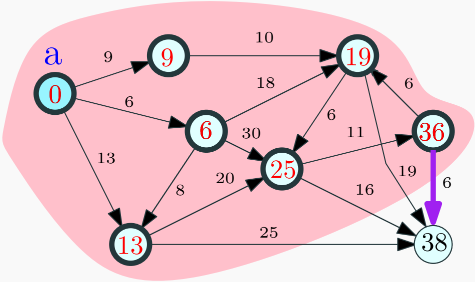

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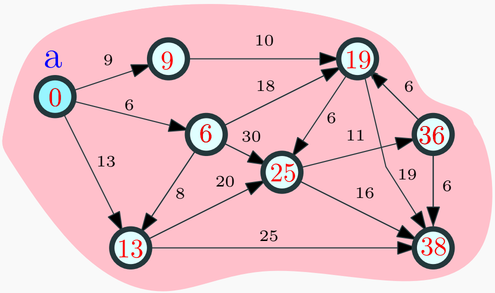

Improved Algorithm

- Main work is to compute the $d'(s,u)$ values in each iteration
- $d'(s,u)$ changes from iteration i to $i+1$ only because of the node v that is added to X in iteration i.

Improved Algorithm

- Main work is to compute the $d'(s,u)$ values in each iteration
- $d'(s,u)$ changes from iteration i to $i+1$ only because of the node v that is added to X in iteration i.

```
Initialize for each node v, dist(s,v) = d'(s,v) = ∞      ← O(n)
Initialize X = ∅, d'(s,s) = 0
for i = 1 to |V| do   ← O(n)                              O(n²) (i)
    // X contains the i − 1 closest nodes to s,
    //        and the values of d'(s,u) are current
    Let v be node realizing d'(s,v) = min_{u∈V−X} d'(s,u)   ← O(n)
    dist(s,v) = d'(s,v)
    X = X ∪ {v}
    Update d'(s,u) for each u in V − X as follows:          O(m) (ii)
        d'(s,u) = min(d'(s,u), dist(s,v) + ℓ(v,u))   ← relaxation step
```

Running time: $O(m + n^2)$ (i) (ii)

Improved Algorithm

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```

Running time: $O(m + n^2)$ time.

- n outer iterations and in each iteration following steps
- updating $d'(s, u)$ after v is added takes $O(deg(v))$ time so total work is $O(m)$ since a node enters X only once
- Finding v from $d'(s, u)$ values is $O(n)$ time

Dijkstra's Algorithm

- eliminate $d'(s,u)$ and let $\text{dist}(s,u)$ maintain it
- update *dist* values after adding v by scanning edges out of v

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Priority Queues to maintain *dist* values for faster running time

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```

Priority Queues to maintain *dist* values for faster running time

- Using heaps and standard priority queues: $O((m+n)\log n)$
- Using Fibonacci heaps: $O(m + n\log n)$.

Dijkstra using priority queues

Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations:

- **makePQ**: create an empty queue.
- **findMin**: find the minimum key in S.
- **extractMin**: Remove $v \in S$ with smallest key and return it.
- **insert**$(v, k(v))$: Add new element v with key $k(v)$ to S.
- **delete**(v): Remove element v from S.

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- **decreaseKey**$(v, k'(v))$: decrease key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$.
- **meld**: merge two separate priority queues into one.

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- **meld**: merge two separate priority queues into one.

All operations can be performed in $O(\log n)$ time.

decreaseKey is implemented via **delete** and **insert**.

Dijkstra's Algorithm using Priority Queues

```
Q ← makePQ()
insert(Q, (s,0))
for each node u ≠ s do
    insert(Q, (u,∞))
X ← ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        decreaseKey(Q, (u, min(dist(s, u), dist(s, v) + ℓ(v, u)))).
```

Priority Queue operations:

- $O(n)$ **insert** operations
- $O(n)$ **extractMin** operations
- $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps
Store elements in a heap based on the key value

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Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.

Fibonacci Heaps

- **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time
- **decreaseKey** in $O(1)$ amortized time:

average runtime defined as the runtime of the worst case input of an operation in the long run

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

- **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time
- **decreaseKey** in $O(1)$ amortized time: ℓ **decreaseKey** operations for $\ell \geq n$ take together $O(\ell)$ time
- Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra's algorithm)

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$(n+m)\log n = n\log n + m\log n$

- Dijkstra's algorithm can be implemented in $O(n\log n + m)$ time. If $m = \Omega(n\log n)$, running time is linear in input size.

→ $O(n\log n + m) = O(m+m)$: linear in input size.

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

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- Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.
- Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps, ...
- Boost library implements both Fibonacci heaps and rank-pairing heaps.

Shortest path trees and variants (RIY)

Shortest Path Tree

Dijkstra's alg. finds the shortest path distances from s to V.

Question: How do we find the paths themselves?

Shortest Path Tree

Dijkstra's alg. finds the shortest path distances from s to V.

Question: How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s,0))
prev(s) ← null
for each node u ≠ s do
    insert(Q, (u,∞) )
    prev(u) ← null

X = ∅
for i = 1 to |V| do
    (v, dist(s,v)) = extractMin(Q)
    X = X ∪ {v}
    for each u in Adj(v) do
        if (dist(s,v) + ℓ(v,u) < dist(s,u)) then
            decreaseKey(Q, (u, dist(s,v) + ℓ(v,u)))
            prev(u) = v
```

Shortest Path Tree

Lemma
The edge set $(u, \mathrm{prev}(u))$ *is the reverse of a shortest path tree rooted at* s*. For each* u*, the reverse of the path from* u *to* s *in the tree is a shortest path from* s *to* u*.*

Proof Sketch.

- The edge set $\{(u, \mathrm{prev}(u)) \mid u \in V\}$ induces a directed in-tree rooted at s (Why?)
- Use induction on $|X|$ to argue that the tree is a shortest path tree for nodes in V.

□

Shortest paths to s

Dijkstra's alg. gives shortest paths from s to all nodes in V.

How do we find shortest paths from all of V to s?

Shortest paths to s

Dijkstra's alg. gives shortest paths from s to all nodes in V.

How do we find shortest paths from all of V to s?

- In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
- In directed graphs, use Dijkstra's algorithm in G^{rev}!