You have a graph $G = (V, E)$. Some of the edges are red, some are white and some are blue. You are given two distinct vertices $s$ and $t$ and want to find a walk $[s \rightarrow t]$ such that:

- a white edge must be taken after a red edge only.
- a blue edge must be taken after a white edge only.
- and a red edge may be taken after a blue edge only.
- must start on red edge.

Develop an algorithm to find a path with these edge constraints.
ECE-374-B: Lecture 17 - Bellman-Ford and Dynamic Programming on Graphs

Instructor: Abhishek Kumar Umrawal
March 28, 2024

University of Illinois at Urbana-Champaign
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Pre-lecture brain teaser
Pre-lecture brain teaser

\begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{brain_teaser_diagram}
    \caption{Diagram for the pre-lecture brain teaser.}
    \label{fig:brain_teaser}
\end{figure}
Shortest Paths with Negative Length Edges
Why Dijkstra’s algorithm fails with negative edges
Single-Source Shortest Path Problems

**Input:** A directed graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
Single-Source Shortest Path Problems

Input: A directed graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
What are the final (shortest) distances as computed by Dijkstra algorithm starting from $s$?

(a) $s = 0, x = 5, y = 1, z = 0, w = 1$.
(b) $s = 0, x = 5, y = 1, z = 2, w = 3$.
(c) IDK.
Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail.
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Dijkstra’s Algorithm and Negative Lengths

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Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail.

![Diagram](image)

**False assumption:** Dijkstra’s algorithm is based on the assumption that if $s \rightarrow v_0 \rightarrow v_1 \rightarrow v_2 \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then $dist(s, v_i) \leq dist(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.
Lemma

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$ for $1 \leq i < k$. Holds true only for non-negative edge lengths.

Cannot explore nodes in increasing order of distance! We need other strategies.
Lemma
Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
- **False**: $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$ for $1 \leq i < k$. **Holds true only for non-negative edge lengths.**
Lemma
Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
- \textbf{False:} $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$ for $1 \leq i < k$. \textit{Holds true only for non-negative edge lengths.}

Cannot explore nodes in increasing order of distance! We need other strategies.
Why can’t we just re-normalize the edge lengths!? 
Why can’t we simply add a weight to each edge so that the shortest length is 0 (or positive).
Instinctual thought

Why can’t we simply add a weight to each edge so that the shortest length is 0 (or positive).

---

Diagram: A graph with nodes labeled s, a, b, c, and t. Edges and their weights are as follows:
- From s to a: -3, 1
- From s to b: 5, 10
- From b to c: 10, 13
- From b to t: -2, 3
- From c to t: 8, 6
- From s to a: 0, 4
- From a to t: 4, 1

---
Instinctual thought

Why can’t we simply add a weight to each edge so that the shortest length is 0 (or positive).

Shortest Path: $s \rightarrow a \rightarrow c \rightarrow t$

Shortest Path: $s \rightarrow b \rightarrow t$
Why can’t we simply add a weight to each edge so that the shortest length is 0 (or positive).

Adding weights to edges penalizes paths with more edges.

Shortest Path: $s \rightarrow a \rightarrow c \rightarrow t$
Shortest Path: $s \rightarrow b \rightarrow t$
But wait! Things get worse: Negative cycles
**Definition**
A cycle $C$ is a negative length cycle if the sum of the edge lengths of $C$ is negative.
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Definition
A cycle $C$ is a negative length cycle if the sum of the edge lengths of $C$ is negative.

What is the shortest path distance between $s$ and $t$?
Reminder: Paths have to be simple ...
Given $G = (V, E)$ with edge lengths and $s, t$. Suppose

- $G$ has a negative length cycle $C$, and
- $s$ can reach $C$ and $C$ can reach $t$.

Question:

What is the shortest distance from $s$ to $t$?

Possible answers: Define shortest distance to be:

- undefined, that is $-\infty$, OR
- the length of a shortest simple path from $s$ to $t$. 
Given $G = (V, E)$ with edge lengths and $s, t$. Suppose

- $G$ has a negative length cycle $C$, and
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**Question:** What is the shortest distance from $s$ to $t$?

Possible answers: Define shortest distance to be:

- undefined, that is $-\infty$, OR
- the length of a shortest simple path from $s$ to $t$. 

Lemma
If there is an efficient algorithm to find a shortest simple $s \rightarrow t$ path in a graph with negative edge lengths, then there is an efficient algorithm to find the longest simple $s \rightarrow t$ path in a graph with positive edge lengths.

Finding the $s \rightarrow t$ longest path is difficult. NP-Hard!
Restating problem of Shortest path with negative edges
Alternatively: Finding Shortest Walks

Given a graph $G = (V, E)$:

- A **path** is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$.
- A **walk** is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. Vertices are allowed to repeat.

Define $\text{dist}(u, v)$ to be the length of a shortest walk from $u$ to $v$.

- If there is a walk from $u$ to $v$ that contains negative length cycle then $\text{dist}(u, v) = -\infty$.
- Else there is a path with at most $n - 1$ edges whose length is equal to the length of a shortest walk and $\text{dist}(u, v)$ is finite.

Helpful to think about walks.
Algorithmic Problems
Input: A directed graph $G = (V, E)$ with edge lengths (could be negative). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

Questions:

- Given nodes $s, t$, either find a negative length cycle $C$ that $s$ can reach or find a shortest path from $s$ to $t$.
- Given node $s$, either find a negative length cycle $C$ that $s$ can reach or find shortest path distances from $s$ to all reachable nodes.
- Check if $G$ has a negative length cycle or not.
Note: With negative lengths, shortest path problems and negative cycle detection in undirected graphs cannot be reduced to directed graphs by bi-directing each undirected edge. Why?

Problem can be solved efficiently in undirected graphs but algorithms are different and significantly more involved than those for directed graphs. One need to compute $T$-joins in the relevant graph. Pretty painful stuff.
Bellman Ford Algorithm
Shortest path via number of hops
Shortest Paths and Recursion

- Compute the shortest path distance from $s$ to $t$ recursively?
- What are the smaller sub-problems?
Shortest Paths and Recursion

- Compute the shortest path distance from $s$ to $t$ recursively?
- What are the smaller sub-problems?

**Lemma**

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
Shortest Paths and Recursion

- Compute the shortest path distance from $s$ to $t$ recursively?
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**Lemma**

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$

Sub-problem idea: paths of fewer hops/edges
Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source $s$.
Assume that all nodes can be reached by $s$ in $G$
Assume $G$ has no negative-length cycle (for now).

$d(v, k)$: shortest walk length from $s$ to $v$ using at most $k$ edges.
Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source $s$.
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$d(v, k)$: shortest walk length from $s$ to $v$ using at most $k$ edges.

Note: $\text{dist}(s, v) = d(v, n - 1)$. 
Hop-based Recursion: Bellman-Ford Algorithm

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Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source $s$.
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Assume $G$ has no negative-length cycle (for now).

$d(v, k)$: shortest walk length from $s$ to $v$ using at most $k$ edges.

Note: $\text{dist}(s, v) = d(v, n - 1)$. Recursion for $d(v, k)$:

$$d(v, k) = \min \begin{cases} 
\min_{u \in V} (d(u, k - 1) + \ell(u, v)) \\
    d(v, k - 1) 
\end{cases}$$

Base case: $d(s, 0) = 0$ and $d(v, 0) = \infty$ for all $v \neq s$. 
Example

The diagram shows a network with nodes labeled as follows: s, a, b, c, d, e, f. The edges have associated values, and the table is empty.
Example

\begin{center}
\begin{tikzpicture}[node distance = 2cm, thick, main node/.style = {circle, draw, font = \sffamily\Large\bfseries}]
\node[main node] (a) {$s$};
\node[main node] (b) [above of=a] {$a$};
\node[main node] (c) [right of=b] {$c$};
\node[main node] (d) [above of=c] {$d$};
\node[main node] (e) [right of=d] {$e$};
\node[main node] (f) [above of=e] {$f$};
\path
(a) edge [bend right] node [near start] {$-3$} (b)
(b) edge [bend right] node [near start] {$8$} (c)
(c) edge [bend right] node [near start] {$4$} (d)
(d) edge [bend right] node [near start] {$6$} (e)
(e) edge [bend right] node [near start] {$3$} (f)
(f) edge [bend right] node [near start] {$-3$} (a);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
round & s & a & b & c & d & e & f \\
\hline
0 & 0 & $\infty$ & $\infty$ & $\infty$ & $\infty$ & $\infty$ & $\infty$ \\
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\end{tabular}
\end{center}
Example

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Example

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round s  a  b  c  d  e  f
0  0 ∞ ∞ ∞ ∞ ∞ ∞
1 0  6 4  3 ∞ ∞ ∞
2 0  6 2  3  4 ∞ 9
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Example

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Example

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     v         v
  d ---------- f
     |         |
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     5         8

   2 ---- b ---- -1
     |         |
     3         8

   -2 ---- 0 ---- 3
     |         |
     6         4

   0 ---- s ----
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The Bellman-Ford Algorithm
Bellman-Ford Algorithm

Create in(G) list from adj(G)

for each \( u \in V \) do
    \( d(u, 0) \leftarrow \infty \)
    \( d(s, 0) \leftarrow 0 \)

for \( k = 1 \) to \( n - 1 \) do
    for each \( v \in V \) do
        \( d(v, k) \leftarrow d(v, k - 1) \)
        for each edge \( (u, v) \in in(v) \) do
            \[ d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\} \]

for each \( v \in V \) do
    \( \text{dist}(s, v) \leftarrow d(v, n - 1) \)

Running time: \( O(n(n + m)) \)
Space: \( O(m + n^2) \)
Note: Space can be reduced to \( O(m + n) \) as any row in our table depends only on the previous row.
Bellman-Ford Algorithm

Create in(G) list from adj(G)

for each $u \in V$ do
    $d(u, 0) \leftarrow \infty$
    $d(s, 0) \leftarrow 0$

for $k = 1$ to $n - 1$ do
    for each $v \in V$ do
        $d(v, k) \leftarrow d(v, k - 1)$
        for each edge $(u, v) \in \text{in}(v)$ do
            $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$

for each $v \in V$ do
    $\text{dist}(s, v) \leftarrow d(v, n - 1)$

Running time:

$O(n(n + m))$

Space:

$O(m + n^2)$

Note: Space can be reduced to $O(m + n)$ as any row in our table depends only on the previous row.
Bellman-Ford Algorithm

Create \( \text{in}(G) \) list from \( \text{adj}(G) \)

\[
\text{for each } u \in V \text{ do} \\
\quad d(u, 0) \leftarrow \infty \\
\quad d(s, 0) \leftarrow 0
\]

\[
\text{for } k = 1 \text{ to } n - 1 \text{ do} \\
\quad \text{for each } v \in V \text{ do} \\
\quad\quad d(v, k) \leftarrow d(v, k - 1) \\
\quad\quad \text{for each edge } (u, v) \in \text{in}(v) \text{ do} \\
\quad\quad\quad d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}
\]

\[
\text{for each } v \in V \text{ do} \\
\quad \text{dist}(s, v) \leftarrow d(v, n - 1)
\]

**Running time:** \( O(n(n + m)) \)
Bellman-Ford Algorithm

Create in(G) list from adj(G)

\[ \text{for each } u \in V \text{ do} \]
\[ d(u, 0) \leftarrow \infty \]
\[ d(s, 0) \leftarrow 0 \]

\[ \text{for } k = 1 \text{ to } n - 1 \text{ do} \]
\[ \text{for each } v \in V \text{ do} \]
\[ d(v, k) \leftarrow d(v, k - 1) \]
\[ \text{for each edge } (u, v) \in in(v) \text{ do} \]
\[ d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\} \]

\[ \text{for each } v \in V \text{ do} \]
\[ \text{dist}(s, v) \leftarrow d(v, n - 1) \]

Running time: \( O(n(n + m)) \) Space:
Bellman-Ford Algorithm

Create in(G) list from adj(G)

for each \( u \in V \) do
    \( d(u, 0) \leftarrow \infty \)
    \( d(s, 0) \leftarrow 0 \)

for \( k = 1 \) to \( n - 1 \) do
    for each \( v \in V \) do
        \( d(v, k) \leftarrow d(v, k - 1) \)
        for each edge \( (u, v) \in in(v) \) do
            \( d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\} \)

for each \( v \in V \) do
    \( \text{dist}(s, v) \leftarrow d(v, n - 1) \)

Running time: \( O(n(n + m)) \)  Space: \( O(m + n^2) \)
Bellman-Ford Algorithm

Create in(G) list from adj(G)

for each \( u \in V \) do

\[ d(u, 0) \leftarrow \infty \]
\[ d(s, 0) \leftarrow 0 \]

for \( k = 1 \) to \( n - 1 \) do

for each \( v \in V \) do

\[ d(v, k) \leftarrow d(v, k - 1) \]

for each edge \((u, v) \in \text{in}(v)\) do

\[ d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\} \]

for each \( v \in V \) do

\[ \text{dist}(s, v) \leftarrow d(v, n - 1) \]

Running time: \( O(n(n + m)) \) Space: \( O(m + n^2) \)

Note: Space can be reduced to \( O(m + n) \) as any row in our table depends only on the previous row.
Bellman-Ford Algorithm: Cleaner version

Create in(G) list from adj(G)

\[
\begin{align*}
\text{for each } & \ u \in V \ \text{do} \\
& d(u) \leftarrow \infty \\
& d(s) \leftarrow 0 \\
\text{for } & \ k = 1 \ \text{to} \ n - 1 \ \text{do} \\
& \text{for each } \ v \in V \ \text{do} \\
& \quad \text{for each edge } \ (u, v) \in \text{in}(v) \ \text{do} \\
& \quad \quad d(v) = \min\{d(v), d(u) + \ell(u, v)\} \\
& \text{for each } \ v \in V \ \text{do} \\
& \quad \text{dist}(s, v) \leftarrow d(v)
\end{align*}
\]

Running time: \( O(n(m + n)) \) Space: \( O(m + n) \)

Exercise: Argue that this (cleaner) version achieves the same results the one on the previous slide.
Bellman-Ford: Detecting negative cycles
Negative cycles

What happens if we run this on a graph with negative cycles?
Negative cycles

What happens if we run this on a graph with negative cycles?

![Graph diagram with nodes s, a, b and edges labeled with values 1 and -1]

<table>
<thead>
<tr>
<th>round</th>
<th>s</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>∞</td>
<td>∞</td>
</tr>
</tbody>
</table>
What happens if we run this on a graph with negative cycles?

<table>
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<tr>
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<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>∞</td>
</tr>
</tbody>
</table>

![Diagram of a graph with negative cycles](image)
Negative cycles

What happens if we run this on a graph with negative cycles?

![Graph with negative cycles]

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>∞</td>
<td>∞</td>
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<tr>
<td>1</td>
<td>0</td>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
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</tbody>
</table>
What happens if we run this on a graph with negative cycles?

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<td>1</td>
<td>∞</td>
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<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
What happens if we run this on a graph with negative cycles?

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<tr>
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<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
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What happens if we run this on a graph with negative cycles?

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<tr>
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<tr>
<td>1</td>
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<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>
Lemma
Suppose \( G \) has a negative cycle \( C \) reachable from \( s \). Then there is some node \( v \in C \) such that \( d(v, n) < d(v, n - 1) \).
Lemma
Suppose $G$ has a negative cycle $C$ reachable from $s$. Then there is some node $v \in C$ such that $d(v, n) < d(v, n - 1)$.

Proof.
Suppose not. Let $C = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_h \rightarrow v_1$ be negative length cycle reachable from $s$. $d(v_i, n - 1)$ is finite for $1 \leq i \leq h$ since $C$ is reachable from $s$. By assumption $d(v, n) \geq d(v, n - 1)$ for all $v \in C$; implies no change in $n^{th}$ iteration;

$d(v_i, n - 1) = d(v_i, n)$ for $1 \leq i \leq h$. This means

$d(v_i, n - 1) \leq d(v_{i-1}, n - 1) + \ell(v_{i-1}, v_i)$ for $2 \leq i \leq h$ and

$d(v_1, n - 1) \leq d(v_n, n - 1) + \ell(v_n, v_1)$. Adding up all these inequalities results in the inequality $0 \leq \ell(C)$ which contradicts the assumption that $\ell(C) < 0$. \qed
Proof of Lemma in more detail...

\[
\begin{align*}
    d(v_1, n) &\leq d(v_0, n - 1) + \ell(v_0, v_1) \\
    d(v_2, n) &\leq d(v_1, n - 1) + \ell(v_1, v_2) \\
    &\ldots \\
    d(v_i, n) &\leq d(v_{i-1}, n - 1) + \ell(v_{i-1}, v_i) \\
    &\ldots \\
    d(v_k, n) &\leq d(v_{k-1}, n - 1) + \ell(v_{k-1}, v_k) \\
    d(v_0, n) &\leq d(v_k, n - 1) + \ell(v_k, v_0)
\end{align*}
\]
Proof of Lemma in more detail...

\[
d(v_1, n) \leq d(v_0, n) + \ell(v_0, v_1)
\]
\[
d(v_2, n) \leq d(v_1, n) + \ell(v_1, v_2)
\]
\[
\ldots
\]
\[
d(v_i, n) \leq d(v_{i-1}, n) + \ell(v_{i-1}, v_i)
\]
\[
\ldots
\]
\[
d(v_k, n) \leq d(v_{k-1}, n) + \ell(v_{k-1}, v_k)
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\[
d(v_0, n) \leq d(v_k, n) + \ell(v_k, v_0)
\]
Proof of Lemma in more detail...

\[ d(v_1, n) \leq d(v_0, n) + \ell(v_0, v_1) \]
\[ d(v_2, n) \leq d(v_1, n) + \ell(v_1, v_2) \]
\[ \ldots \]
\[ d(v_i, n) \leq d(v_{i-1}, n) + \ell(v_{i-1}, v_i) \]
\[ \ldots \]
\[ d(v_k, n) \leq d(v_{k-1}, n) + \ell(v_{k-1}, v_k) \]
\[ d(v_0, n) \leq d(v_k, n) + \ell(v_k, v_0) \]

\[
\sum_{i=0}^{k} d(v_i, n) \leq \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0)
\]
Proof of Lemma in more detail...

\[
\sum_{i=0}^{k} d(v_i, n) \leq \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0)
\]

\[
0 \leq \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0).
\]
Proof of Lemma in more detail...

\[\sum_{i=0}^{k} d(v_i, n) \leq \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0)\]

\[0 \leq \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0) = \text{len}(C).\]
Proof of Lemma in more detail...

\[ \sum_{i=0}^{k} d(v_i, n) \leq \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0) \]

\[ 0 \leq \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0) = \text{len}(C). \]

\( C \) is a not a negative cycle. Contradiction. \( \square \)
**Lemma restated**
If $G$ does not have a negative length cycle reachable from $s \implies \forall v: d(v, n) = d(v, n - 1)$.

Also, $d(v, n - 1)$ is the length of the shortest path between $s$ and $v$.

Put together are the following:

**Lemma**
$G$ has a negative length cycle reachable from $s \iff$ there is some node $v$ such that $d(v, n) < d(v, n - 1)$. 
Bellman-Ford: Negative Cycle Detection - final version

\begin{verbatim}
for each \( u \in V \) do
    \( d(u) \leftarrow \infty \)
\end{verbatim}

\begin{verbatim}
d(s) \leftarrow 0
\end{verbatim}

\begin{verbatim}
for \( k = 1 \) to \( n - 1 \) do
    for each \( v \in V \) do
        for each edge \( (u, v) \in in(v) \) do
            \( d(v) = \min\{d(v), d(u) + \ell(u, v)\} \)
\end{verbatim}

(* One more iteration to check if distances change *)

\begin{verbatim}
for each \( v \in V \) do
    for each edge \( (u, v) \in in(v) \) do
        if \( (d(v) > d(u) + \ell(u, v)) \)
            Output ‘‘Negative Cycle’’
\end{verbatim}

\begin{verbatim}
for each \( v \in V \) do
    dist(s, v) \leftarrow d(v)
\end{verbatim}
Variants on Bellman-Ford
How do we find a shortest path tree in addition to distances?

- For each $v$ the $d(v)$ can only get smaller as algorithm proceeds.
- If $d(v)$ becomes smaller it is because we found a vertex $u$ such that $d(v) > d(u) + \ell(u, v)$ and we update $d(v) = d(u) + \ell(u, v)$. That is, we found a shorter path to $v$ through $u$.
- For each $v$ have a $prev(v)$ pointer and update it to point to $u$ if $v$ finds a shorter path via $u$.
- At end of algorithm $prev(v)$ pointers give a shortest path tree oriented towards the source $s$. 
Negative Cycle Detection
Given directed graph $G$ with arbitrary edge lengths, does it have a negative length cycle?

Bellman-Ford checks whether there is a negative cycle $C$ that is reachable from a specific vertex $s$. There may be negative cycles not reachable from $s$.

Run Bellman-Ford $|V|$ times, once from each node $u$. 
Negative Cycle Detection

Given directed graph $G$ with arbitrary edge lengths, does it have a negative length cycle?

- Bellman-Ford checks whether there is a negative cycle $C$ that is reachable from a specific vertex $s$. There may be negative cycles not reachable from $s$.
- Run Bellman-Ford $|V|$ times, once from each node $u$. 
Negative Cycle Detection

- Add a new node $s'$ and connect it to all nodes of $G$ with zero length edges. Bellman-Ford from $s'$ will fill find a negative length cycle if there is one. **Exercise:** why does this work?
- Negative cycle detection can be done with one Bellman-Ford invocation.
Shortest Paths in DAGs
Shortest Paths in a DAG

Single-Source Shortest Path Problems

**Input** A directed acyclic graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
Shortest Paths in a **DAG**

### Single-Source Shortest Path Problems

**Input** A directed *acyclic* graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.

### Simplification of algorithms for **DAGs**

- No cycles and hence no negative length cycles! Hence can find shortest paths even for negative length edges.
- Can order nodes using topological sort.
Algorithm for DAGs

- Want to find shortest paths from $s$. Ignore nodes not reachable from $s$.
- Let $v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of $G$. 

Observation:
- shortest path from $s$ to $v_i$ cannot use any node from $v_{i+1}, \ldots, v_n$.
- can find shortest paths in topological sort order.
Algorithm for DAGs

- Want to find shortest paths from $s$. Ignore nodes not reachable from $s$.
- Let $v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of $G$.

Observation:

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- can find shortest paths in topological sort order.
Shortest Paths for DAGs - Example
Shortest Paths for DAGs - Example
Algorithm for DAGs

for $i = 1$ to $n$ do
    $d(s, v_i) = \infty$
    $d(s, s) = 0$

for $i = 1$ to $n - 1$ do
    for each edge $(v_i, v_j)$ in Adj($v_i$) do
        $d(s, v_j) = \min\{d(s, v_j), d(s, v_i) + \ell(v_i, v_j)\}$

return $d(s, \cdot)$ values computed

Correctness: induction on $i$ and observation in previous slide.
Running time: $O(m + n)$ time algorithm! Works for negative edge lengths and hence can find longest paths in a DAG.
All Pairs Shortest Paths
Shortest Path Problems

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
- Find shortest paths for all pairs of nodes.
Single-Source Shortest Path Problems

**Input**
A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
SSSP: Single-Source Shortest Paths

Single-Source Shortest Path Problems

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.

**Dijkstra’s algorithm** for non-negative edge lengths. Running time: $O((m + n) \log n)$ with heaps and $O(m + n \log n)$ with advanced priority queues.

**Bellman-Ford algorithm** for arbitrary edge lengths. Running time: $O(n(m + n))$. 
All-Pairs Shortest Path Problem

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Find shortest paths for all pairs of nodes.
All-Pairs Shortest Path Problem

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Find shortest paths for all pairs of nodes.

Apply single-source algorithms $n$ times, once for each vertex.

- **Non-negative lengths:** $O(n(m + n) \log n)$ with heaps and $O(n(m + n \log n))$ using advanced priority queues.
- **Arbitrary edge lengths:** $O(n^2(m + n))$. If $m = \Omega(n^2)$ then $\Theta(n^4)$.

Can we do better?
All-Pairs Shortest Path Problem

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Find shortest paths for all pairs of nodes.

Apply single-source algorithms $n$ times, once for each vertex.

- Non-negative lengths: $O(n(m + n) \log n)$ with heaps and $O(n(m + n \log n))$ using advanced priority queues.
- Arbitrary edge lengths: $O(n^2(m + n))$. If $m = \Omega(n^2)$ then $\Theta(n^4)$.

Can we do better?
All Pairs Shortest Paths: A recursive solution
All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

\begin{align*}
dist(i, j, 0) &= \quad 100 \\
dist(i, j, 1) &= \quad 9 \\
dist(i, j, 2) &= \quad 8 \\
dist(i, j, 3) &= \quad 5 \\
\end{align*}
All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

$\text{dist}(i, j, 0) = 100$
$\text{dist}(i, j, 1) = $
$\text{dist}(i, j, 2) = $
$\text{dist}(i, j, 3) = $
All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

![Graph with labels and distances]

$\text{dist}(i, j, 0) = 100$
$\text{dist}(i, j, 1) = 9$
$\text{dist}(i, j, 2) = \text{dist}(i, j, 3) =$
All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

\[
\begin{align*}
\text{dist}(i, j, 0) &= 100 \\
\text{dist}(i, j, 1) &= 9 \\
\text{dist}(i, j, 2) &= 8 \\
\text{dist}(i, j, 3) &=
\end{align*}
\]
All-Pairs: Recursion on index of intermediate nodes

- Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
- $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

```
\begin{align*}
\text{dist}(i, j, 0) &= 100 \\
\text{dist}(i, j, 1) &= 9 \\
\text{dist}(i, j, 2) &= 8 \\
\text{dist}(i, j, 3) &= 5
\end{align*}
```
For the following graph, \( \text{dist}(i, j, 2) \) is...

(a) 9
(b) 10
(c) 11
(d) 12
(e) 15
All-Pairs: Recursion on index of intermediate nodes

\[
\begin{align*}
\text{dist}(i, k, k-1) & \quad \rightarrow \quad k \\
\text{dist}(k, j, k-1) & \\
\text{dist}(i, j, k-1) & \\
\end{align*}
\]

\[
dist(i, j, k) = \min \begin{cases} 
\text{dist}(i, j, k-1) \\
\text{dist}(i, k, k-1) + \text{dist}(k, j, k-1) 
\end{cases}
\]

Base case: \( \text{dist}(i, j, 0) = \ell(i, j) \) if \((i, j) \in E\), otherwise \(\infty\)

**Correctness:** If \(i \rightarrow j\) shortest walk goes through \(k\) then \(k\) occurs only once on the path — otherwise there is a negative length cycle.
If $i$ can reach $k$ and $k$ can reach $j$ and $\text{dist}(k, k, k - 1) < 0$ then $G$ has a negative length cycle containing $k$ and $\text{dist}(i, j, k) = -\infty$.

Recursion below is valid only if $\text{dist}(k, k, k - 1) \geq 0$. We can detect this during the algorithm or wait till the end.

$$
\text{dist}(i, j, k) = \min \left\{ \begin{array}{l}
\text{dist}(i, j, k - 1) \\
\text{dist}(i, k, k - 1) + \text{dist}(k, j, k - 1)
\end{array} \right\}
$$
Floyd-Warshall algorithm
Floyd-Warshall Algorithm - for All-Pairs Shortest Paths

\[ d(i, j, k) = \min \begin{cases} 
   d(i, j, k - 1) \\
   d(i, k, k - 1) + d(k, j, k - 1)
\end{cases} \]

\begin{verbatim}
for i = 1 to n do
  for j = 1 to n do
    d(i, j, 0) = l(i, j)
    (* l(i, j) = \infty if (i, j) \notin E, 0 if i = j *)

for k = 1 to n do
  for i = 1 to n do
    for j = 1 to n do
      d(i, j, k) = \min \begin{cases} 
        d(i, j, k - 1) \\
        d(i, k, k - 1) + d(k, j, k - 1)
      \end{cases}

for i = 1 to n do
  if (dist(i, i, n) < 0) then
    Output \exists negative cycle in G
\end{verbatim}
Floyd-Warshall Algorithm - for All-Pairs Shortest Paths

\[
d(i,j,k) = \min \begin{cases} 
  d(i,j, k-1) \\
  d(i, k, k-1) + d(k,j, k-1)
\end{cases}
\]

\[
\begin{align*}
\text{for} & \quad i = 1 \text{ to } n \text{ do} \\
& \quad \text{for} \quad j = 1 \text{ to } n \text{ do} \\
& \quad \quad d(i,j,0) = \ell(i,j) \\
& \quad \quad (\ast \ \ell(i,j) = \infty \text{ if } (i,j) \notin E, \ 0 \text{ if } i = j \ \ast)
\end{align*}
\]

\[
\begin{align*}
\text{for} & \quad k = 1 \text{ to } n \text{ do} \\
& \quad \text{for} \quad i = 1 \text{ to } n \text{ do} \\
& \quad \quad \text{for} \quad j = 1 \text{ to } n \text{ do} \\
& \quad \quad \quad d(i,j,k) = \min \begin{cases} 
  d(i,j, k-1), \\
  d(i, k, k-1) + d(k,j, k-1)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{for} & \quad i = 1 \text{ to } n \text{ do} \\
& \quad \text{if } \ (\text{dist}(i,i,n) < 0) \text{ then} \\
& \quad \quad \text{Output } \exists \text{ negative cycle in } G
\end{align*}
\]

Running Time: \(\Theta(n^3)\). Space: \(\Theta(n^3)\).

Correctness: via induction and recursive definition.
Floyd-Warshall Algorithm - for All-Pairs Shortest Paths

\[ d(i, j, k) = \min \begin{cases} 
    d(i, j, k - 1) \\
    d(i, k, k - 1) + d(k, j, k - 1) 
\end{cases} \]

for \( i = 1 \) to \( n \) do
  for \( j = 1 \) to \( n \) do
    \( d(i, j, 0) = \ell(i, j) \)
    (* \( \ell(i, j) = \infty \) if \( (i, j) \notin E \), 0 if \( i = j \) *)
  for \( k = 1 \) to \( n \) do
    for \( i = 1 \) to \( n \) do
      for \( j = 1 \) to \( n \) do
        \( d(i, j, k) = \min \begin{cases} 
            d(i, j, k - 1), \\
            d(i, k, k - 1) + d(k, j, k - 1) 
        \end{cases} \)
  for \( i = 1 \) to \( n \) do
    if \( \text{dist}(i, i, n) < 0 \) then
      Output \( \exists \) negative cycle in \( G \)

Running Time: \( \Theta(n^3) \). Space: \( \Theta(n^3) \).
Floyd-Warshall Algorithm - for All-Pairs Shortest Paths

\[ d(i, j, k) = \min \begin{cases} d(i, j, k - 1) \\ d(i, k, k - 1) + d(k, j, k - 1) \end{cases} \]

for \(i = 1\) to \(n\) do
    for \(j = 1\) to \(n\) do
        \(d(i, j, 0) = \ell(i, j)\)
        (* \(\ell(i, j) = \infty\) if \((i, j) \not\in E\), 0 if \(i = j\) *)

for \(k = 1\) to \(n\) do
    for \(i = 1\) to \(n\) do
        for \(j = 1\) to \(n\) do
            \(d(i, j, k) = \min \begin{cases} d(i, j, k - 1), \\ d(i, k, k - 1) + d(k, j, k - 1) \end{cases} \)

for \(i = 1\) to \(n\) do
    if \(\text{dist}(i, i, n) < 0\) then
        Output \(\exists\) negative cycle in \(G\)

Running Time: \(\Theta(n^3)\). Space: \(\Theta(n^3)\).
Correctness: via induction and recursive definition
**Question:** Can we find the paths in addition to the distances?
Floyd-Warshall Algorithm: Finding the Paths

**Question:** Can we find the paths in addition to the distances?

- Create a $n \times n$ array `Next` that stores the next vertex on shortest path for each pair of vertices.
- With array `Next`, for any pair of given vertices $i, j$ can compute a shortest path in $O(n)$ time.
Floyd-Warshall Algorithm - Finding the Paths

\[
\begin{align*}
&\text{for } i = 1 \text{ to } n \text{ do} \\
&\quad \text{for } j = 1 \text{ to } n \text{ do} \\
&\quad \quad d(i, j, 0) = \ell(i, j) \\
&\quad (* \ell(i, j) = \infty \text{ if } (i, j) \text{ not edge, } 0 \text{ if } i = j *) \\
&\quad \quad Next(i, j) = -1 \\
&\text{for } k = 1 \text{ to } n \text{ do} \\
&\quad \text{for } i = 1 \text{ to } n \text{ do} \\
&\quad \quad \text{for } j = 1 \text{ to } n \text{ do} \\
&\quad \quad \quad \text{if } (d(i, j, k - 1) > d(i, k, k - 1) + d(k, j, k - 1)) \text{ then} \\
&\quad \quad \quad \quad d(i, j, k) = d(i, k, k - 1) + d(k, j, k - 1) \\
&\quad \quad \quad \quad Next(i, j) = k \\
&\text{for } i = 1 \text{ to } n \text{ do} \\
&\quad \text{if } (d(i, i, n) < 0) \text{ then} \\
&\quad \quad \text{Output that there is a negative length cycle in } G
\end{align*}
\]

Exercise: Given Next array and any two vertices \(i, j\) describe an \(O(n)\) algorithm to find a \(i\)-\(j\) shortest path.
Summary of shortest path algorithms
## Summary of results on shortest paths

<table>
<thead>
<tr>
<th>Single source</th>
<th>No negative edges</th>
<th>Dijkstra</th>
<th>$O(n \log n + m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Edge lengths can be negative</td>
<td>Bellman Ford</td>
<td>$O(n(m + n))$</td>
</tr>
</tbody>
</table>

### All Pairs Shortest Paths

<table>
<thead>
<tr>
<th>No negative edges</th>
<th>$n \times$ Dijkstra</th>
<th>$O(n(n \log n + m))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No negative cycles</td>
<td>$n \times$ Bellman Ford</td>
<td>$O(n^2(m + n))$</td>
</tr>
<tr>
<td>No negative cycles</td>
<td>Johnson’s $^1$</td>
<td>$O(nm + n^2 \log n)$</td>
</tr>
<tr>
<td>No negative cycles</td>
<td>Floyd-Warshall</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>Unweighted</td>
<td>Matrix multiplication $^2$</td>
<td>$O(n^{2.38}), O(n^{2.58})$</td>
</tr>
</tbody>
</table>
Summary of results on shortest paths

(1): The algorithm for the case that there are no negative cycles, and doing all shortest paths, works by computing a potential function using **Bellman-Ford** and then doing **Dijkstra**. It is mentioned for the sake of completeness, but it outside the scope of the class.