Pre-lecture brain teaser

You are given a DFA describing the regular language L. Want to know if $|L|$ is infinite. How can we do this?

ECE-374-B: Lecture 19 - Reductions

Instructor: Abhishek Kumar Umrawal

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University of Illinois at Urbana-Champaign

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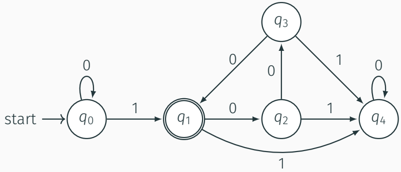

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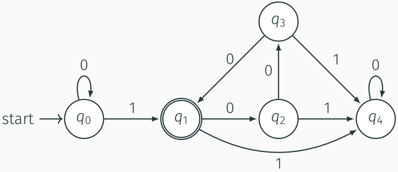

Solution:

If an accept state is within a cycle or a cycle can reach an accept state then the language is infinite.

Bigger point: [Infinite language] problem reduces to [Find cycle] problem!

Last part of the course!

Finishing touches!

- Part I: Models of computation (reg exps, DFA/NFA, CFGs, TMs)
- Part II: (Efficient) algorithm design
- Part III: Intractability via reductions
  - Undecidablity: problems that have no algorithms.
  - NP-Completeness: problems unlikely to have efficient algorithms unless $P = NP$.

Turing Machines and Church-Turing Thesis

Turing defined TMs as a machine model of computation.

Church-Turing thesis: any function that is computable can be computed by TMs.

Efficient Church-Turing thesis: any function that is computable can be computed by TMs with only a polynomial slow-down.

Computability and Complexity Theory

- What functions can and *cannot* be computed by TMs?
- What functions/problems can and cannot be solved *efficiently*?

Why?

- Foundational questions about computation.
- Pragmatic: Can we solve our problem or not?
- Are we not being clever enough to find an efficient algorithm or should we stop because there isn't one or likely to be one?

Reductions to Prove Intractability

A general methodology to prove impossibility results.

- Start with some *known* hard problem X.
- *Reduce* X to your favorite problem Y. [$X \Rightarrow Y$]

If Y can be solved then so can X. But we know X is hard, so Y has to be hard too.

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Caveat: In algorithms, we reduce a new problem to some known solved one!

Reductions to Prove Intractability

Who gives us the initial hard problem?

- Some clever person (Cantor/Gödel/Turing/Cook/Levin …) who established the hardness of a fundamental problem.
- Assume some core problem is hard because we haven't been able to solve it for a long time. This leads to *conditional* results.

Reduction Question

A general methodology to prove impossibility results.

- Start with some *known* hard problem X.
- *Reduce* X to your favorite problem Y.

If Y can be solved then so can X. But we know X is hard, so Y has to be hard too.

What if we want to prove a problem is easy?

- Start with an easy problem Y.
- *Reduce* your problem X to Y.

Decision Problems, Languages, Terminology

When proving hardness we limit attention to *decision* problems.

- A decision problem Π is a collection of instances (strings)
- For each instance I of Π, answer is either YES or NO.
- Equivalently: boolean function $f_\Pi : \Sigma^* \rightarrow \{0, 1\}$ where $f(I) = 1$ if I is a YES instance, $f(I) = 0$ if NO instance.
- Equivalently: language $L_\Pi = \{I \mid I \text{ is a YES instance}\}$.

Decision Problems, Languages, Terminology

We distinguish an object a from its encoding $\langle a \rangle$.

- n is an integer. $\langle n \rangle$ is the encoding of n in some format (could be unary, binary, decimal etc).
- G is a graph. $\langle G \rangle$ is the encoding of G in some format.
- M is a TM. $\langle M \rangle$ is the encoding of TM as a string according to some fixed convention.

Decision Problems, Languages, Terminology

Aside: Different problems can be formulated differently.
Example: Traveling salesman problem.

Common Formulation: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

Decision Formulation: Given a list of cities and the distances between each pair of cities, is there a route that visits each city exactly once and returns to the origin city **while having a shorter length than integer k.**

Examples

- Given directed graph G, is it strongly connected? $\langle G \rangle$ is a YES instance if it is, otherwise NO instance.
- Given number n, is it a prime number? $L_{PRIMES} = \{\langle n \rangle \mid n \text{ is prime}\}$.
- Given number n is it a composite number? $L_{COMPOSITE} = \{\langle n \rangle \mid n \text{ is a composite}\}$.
- Given $G = (V, E), s, t, B$ is the shortest path distance from s to t at most B? Instance is $\langle G, s, t, B \rangle$.

Reductions: Overview

Reductions for languages

For languages L_X, L_Y, a *reduction from L_X to L_Y* is:

- An algorithm.
- Input: $w \in \Sigma^*$
- Output: $w' \in \Sigma^*$
- Such that:

$$\boxed{w \in L_X} \iff \boxed{w' \in L_Y}$$

Reductions for decision problems

For decision problems X, Y, a *reduction from X to Y* is:

- An algorithm.
- Input: I_X, an instance of X.
- Output: I_Y an instance of Y.
- Such that:

$$\boxed{I_Y \text{ is YES instance of } Y} \iff \boxed{I_X \text{ is YES instance of } X}$$

Using reductions to solve problems

- $\mathcal{R}$: Reduction $X \Rightarrow Y$.
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Using reductions to solve problems

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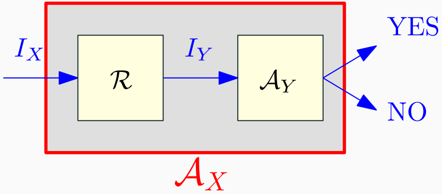
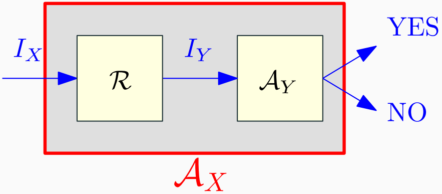


In particular, if $\mathcal{R}$ and $\mathcal{A}_Y$ are polynomial-time algorithms, $\mathcal{A}_X$ is also polynomial-time.

Reductions and running time



$R(n)$: running time of $\mathcal{R}$.

$Q(n)$: running time of $\mathcal{A}_Y$.

Question: What is running time of $\mathcal{A}_X$?

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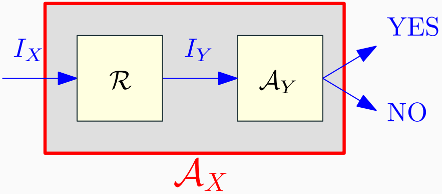

$R(n)$: running time of $\mathcal{R}$.

$Q(n)$: running time of $\mathcal{A}_Y$.

Question: What is running time of $\mathcal{A}_X$? $O(R(n) + Q(R(n)))$. Why?

- If I_X has size n, $\mathcal{R}$ creates an instance I_Y of size at most $R(n)$.
- $\mathcal{A}_Y$'s time on I_Y is by definition at most $Q(|I_Y|) \leq O(R(n) + Q(R(n)))$.

Example: If $R(n) = n^2$ and $Q(n) = n^{1.5}$ then $\mathcal{A}_X$ is $O(n^2 + n^3)$.

Comparing Problems

- Reductions allow us to formalize the notion of "Problem X is no harder to solve than Problem Y".
- If Problem X reduces to Problem Y (we write $X \leq Y$), then X cannot be harder to solve than Y.
- More generally, if $X \leq Y$, we can say that X is no harder than Y, or Y is at least as hard as X. $X \leq Y$:
    - X is no harder than Y, or
    - Y is at least as hard as X.

Examples of Reductions

Given a graph G, a set of vertices V' is:

Independent Sets and Cliques

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- An *independent set*: if no two vertices of V' are connected by an edge of G.

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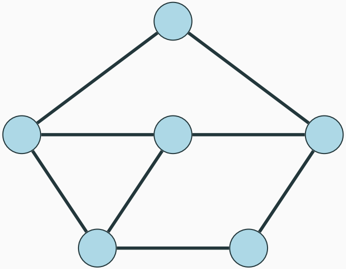

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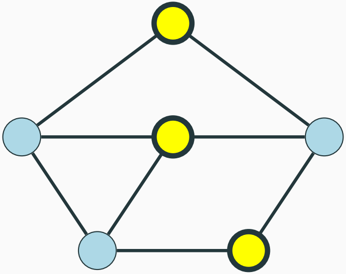

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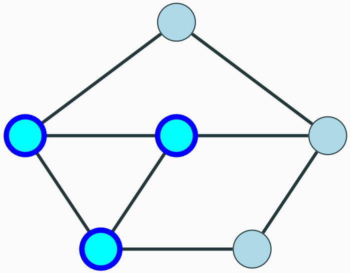


Problem: Independent Set

Instance: A graph G and an integer k.

Question: Does G has an independent set of size $\geq k$?

The Independent Set and Clique Problems

Problem: Independent Set

Instance: A graph G and an integer k.
Question: Does G has an independent set of size $\geq k$?

Problem: Clique

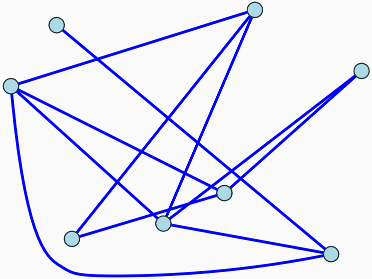
Instance: A graph G and an integer k.
Question: Does G has a clique of size $\geq k$?

Recall

For decision problems X, Y, a reduction from X to Y is:

- An algorithm ...
- that takes I_X, an instance of X as input ...
- and returns I_Y, an instance of Y as output ...
- such that the solution (YES/NO) to I_Y is the same as the solution to I_X.

An instance of **Independent Set** is a graph G and an integer k.



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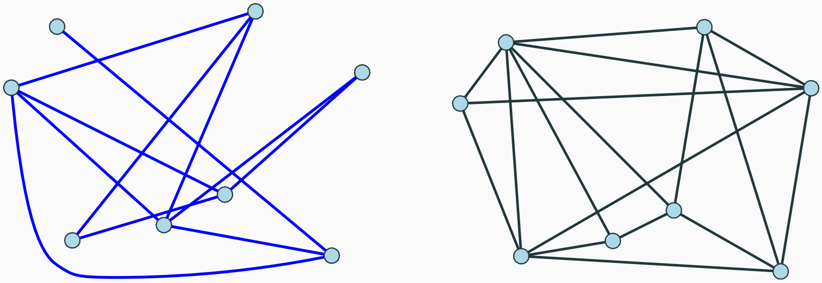
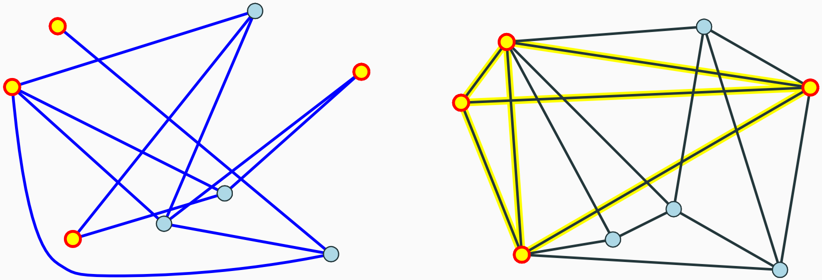


Reducing Independent Set to Clique

An instance of **Independent Set** is a graph G and an integer k.

Reduction given $\langle G, k \rangle$ outputs $\langle \overline{G}, k \rangle$ where $\overline{G}$ is the *complement* of G. $\overline{G}$ has an edge $uv \iff uv$ is not an edge of G.



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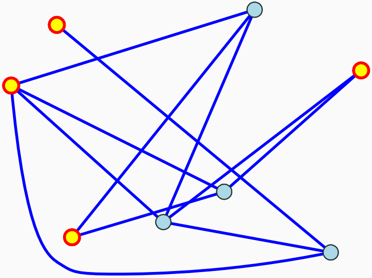

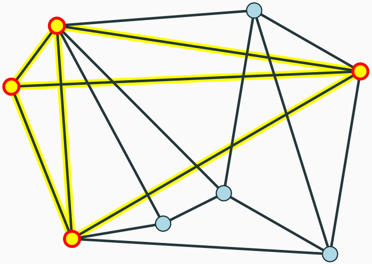

A independent set of size k in $G \iff$ A clique of size k in $\overline{G}$

Correctness of reduction

Lemma
G has an independent set of size $k \iff \overline{G}$ has a clique of size k.

Proof.
Need to prove two facts:

1. G has independent set of size at least k implies that $\overline{G}$ has a clique of size at least k.
2. $\overline{G}$ has a clique of size at least k implies that G has an independent set of size at least k.

Since $S \subseteq V$ is an independent set in $G \iff S$ is a clique in $\overline{G}$. $\square$

Independent Set and Clique

- Independent Set $\leq_P$ Clique.

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- **Independent Set** $\leq_P$ **Clique**.
  What does this mean?
- If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
- **Clique** is *at least as hard as* **Independent Set**.

Independent Set and Clique

- Independent Set $\leq_P$ Clique.
  What does this mean?
- If have an algorithm for Clique, then we have an algorithm for Independent Set.
- Clique is *at least as hard as* Independent Set.
- Also...Clique $\leq_P$ Independent Set. Why? Thus Clique and Independent Set are poylnomial-time equivalent.

Visualize Clique and independent Set Reduction

I want to show **Independent Set** is atleast as hard as **Clique**.

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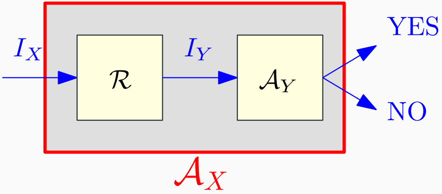
Draw reduction figure:



Visualize Clique and independent Set Reduction

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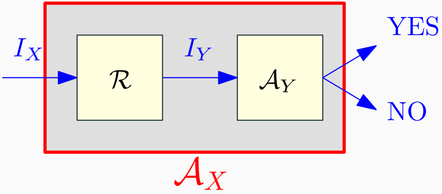

Fill in the blanks:

- $I_X = \langle \overline{G}, k \rangle$
- $\mathcal{A}_X = \text{Clique}(\overline{G}, k)$
- $I_Y = \langle G, k \rangle$
- $\mathcal{A}_Y = \text{Independent Set}(\overline{G}, k)$
- $\mathcal{R} : \overline{G} = \{V, \overline{E}\}$

Review: Independent Set and Clique

Assume you can solve the **Clique** problem in $T(n)$ time. Then you can solve the **Independent Set** problem in

(A) $O(T(n))$ time.

(B) $O(n \log n + T(n))$ time.

(C) $O(n^2 T(n^2))$ time.

(D) $O(n^4 T(n^4))$ time.

(E) $O(n^2 + T(n^2))$ time.

(F) Does not matter - all these are polynomial if $T(n)$ is polynomial, which is good enough for our purposes.

Answer: E

Independent Set and Vertex Cover

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

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- A *vertex cover* if every $e \in E$ has at least one endpoint in S.

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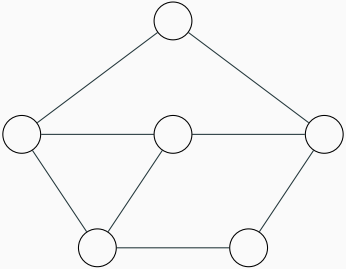

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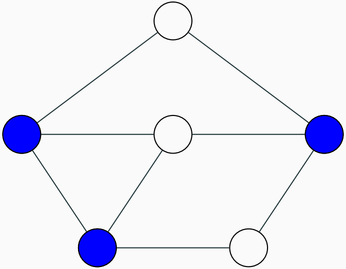

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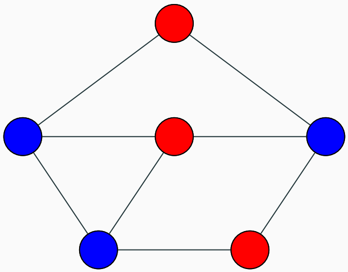

Problem (Vertex Cover)

Input: *A graph G and integer k.*

Goal: *Is there a vertex cover of size $\leq k$ in G?*

The Vertex Cover Problem

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Can we relate **Independent Set** and **Vertex Cover**?

Relationship between Vertex Cover and Independent Set

Lemma
Let $G = (V, E)$ be a graph. S is an Independent Set $\iff$ $V \setminus S$ is a vertex cover.

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Proof.

($\Rightarrow$) Let S be an independent set

- Consider any edge $uv \in E$.
- Since S is an independent set, either $u \notin S$ or $v \notin S$.
- Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
- $V \setminus S$ is a vertex cover.

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($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

- Consider $u, v \in S$
- uv is not an edge of G, as otherwise $V \setminus S$ does not cover uv.
- $\implies$ S is thus an independent set. $\square$

Independent Set $\leq_P$ Vertex Cover

- G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.

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- G has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$

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- G has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
- (G, k) is an instance of **Independent Set**, and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
- Therefore, **Independent Set** $\leq_P$ **Vertex Cover**. Also **Vertex Cover** $\leq_P$ **Independent Set**.

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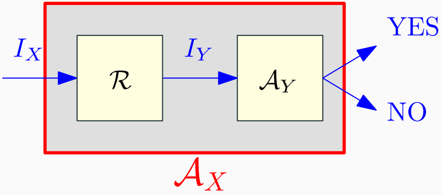


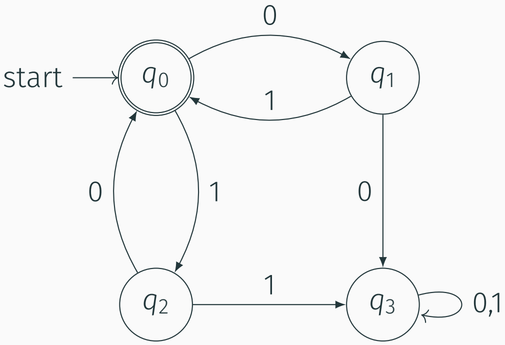
- $I_X = \langle G, k \rangle$
- $A_X = \text{Independent Set}(G, k)$
- $I_Y = \langle G, k \rangle$
- $A_Y = \text{Vertex Cover}(G, n - k)$
- $R : G' = G$

NFAs, DFAs and their Universality

DFA Accepting a String

Given DFA M and string $w \in \Sigma^*$, does M accept w?

- Instance is $\langle M, w \rangle$
- Algorithm: given $\langle M, w \rangle$, output YES if M accepts w, else NO



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Yes. Simulate M on w and output YES if M reaches a final state.

Exercise: Show a linear time algorithm. Note that linear is in the input size which includes both encoding size of M and $|w|$.

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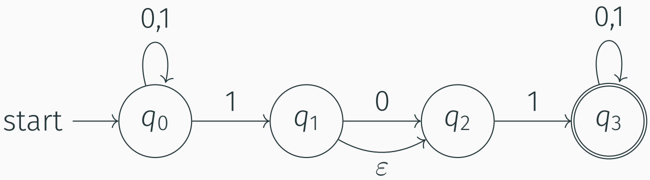

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Question: Is there an algorithm for this problem?

- Convert N to equivalent DFA M and use previous algorithm!
- Hence a reduction that takes $\langle N, w \rangle$ to $\langle M, w \rangle$
- Is this reduction efficient?

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- Hence a reduction that takes $\langle N, w \rangle$ to $\langle M, w \rangle$
- Is this reduction efficient? No, because $|M|$ is exponential in $|N|$ in the worst case.

Exercise: Describe a polynomial-time algorithm.

Hence reduction may allow you to see an easy algorithm but not necessarily best algorithm!

DFA Universality

A DFA M is universal if it accepts every string.

That is, $L(M) = \Sigma^*$, the set of all strings.

Problem (DFA universality)

Input: *A DFA M.*

Goal: *Is M universal?*

How do we solve **DFA Universality**?

We check if M has *any* reachable non-final state.

NFA Universality

An NFA N is said to be universal if it accepts every string. That is, $L(N) = \Sigma^*$, the set of all strings.

Problem (NFA universality)

Input: *A NFA M.*

Goal: *Is M universal?*

How do we solve **NFA Universality**?

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Reduce it to **DFA Universality**?

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How do we solve **NFA Universality**?

Reduce it to **DFA Universality**?

Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.

What is the problem with this reduction?

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Goal: *Is M universal?*

How do we solve **NFA Universality**?

Reduce it to **DFA Universality**?

Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.

What is the problem with this reduction? The reduction takes exponential time!

NFA Universality is known to be PSPACE-Complete.

Polynomial time reductions

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We say that an algorithm is *efficient* if it runs in polynomial-time.

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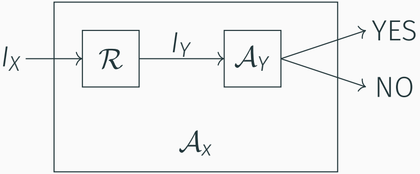

Polynomial-time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

- given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
- $\mathcal{A}$ runs in time polynomial in $|I_X|$.
- answer to I_X YES $\iff$ answer to I_Y is YES.

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- answer to I_X YES $\iff$ answer to I_Y is YES.

Lemma
If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Such a reduction is called a *Karp reduction*. Most reductions we will need are Karp reductions. Karp reductions are the same as mapping reductions when specialized to polynomial time for the reduction step.

Review question: Reductions again...

Let X and Y be two decision problems, such that X can be solved in polynomial time, and $X \leq_P Y$. Then

(A) Y can be solved in polynomial time.

(B) Y can NOT be solved in polynomial time.

(C) If Y is hard then X is also hard.

(D) None of the above.

(E) All of the above.

Answer: D

Be careful about reduction direction

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y.

That is, show that an algorithm for Y implies an algorithm for X.

The Satisfiability Problem (SAT)

Propositional Formulas

Definition
Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

- A *literal* is either a boolean variable x_i or its negation $\neg x_i$.
- A *clause* is a disjunction of literals.
  For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
- A *formula in conjunctive normal form* (CNF) is propositional formula which is a conjunction of clauses
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.

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  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.
- A formula φ is a 3CNF:
  A CNF formula such that every clause has **exactly** 3 literals.

  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a 3CNF formula, but $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

CNF is universal

Every boolean formula $f : \{0,1\}^n \to \{0,1\}$ can be written as a CNF formula.

x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \ldots, x_6)$	$\overline{x_1} \vee x_2 \overline{x_3} \vee x_4 \vee \overline{x_5} \vee x_6$
0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	1
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	1	0	0	1	?	1
1	0	1	0	1	0	0	0
1	0	1	0	1	1	?	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	1	1	1	1	$f(1, \ldots, 1)$	1

For every row that f is zero compute corresponding CNF clause.

Take the and ($\wedge$) of all the CNF clauses computed

Satisfiability

Problem: SAT

> **Instance:** A CNF formula φ.
> **Question:** Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

> **Instance:** A 3CNF formula φ.
> **Question:** Is there a truth assignment to the variable of φ such that φ evaluates to true?

Satisfiability

SAT
Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example

- $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is satisfiable; take $x_1, x_2, \ldots x_5$ to be all true
- $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$ is not satisfiable.

3SAT
Given a 3CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

(More on **2SAT** in a bit...)

Importance of SAT and 3SAT

- SAT and 3SAT are basic constraint satisfaction problems.
- Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
- Arise naturally in many applications involving hardware and software verification and correctness.
- As we will see, it is a fundamental problem in theory of NPCompleteness.

Given two bits x, z which of the following SAT formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \vee x) \wedge (z \vee \overline{x})$.

(B) $(z \vee x) \wedge (\overline{z} \vee \overline{x})$.

(C) $(\overline{z} \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$.

(D) $z \oplus x$.

(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

Answer: B

$z = \overline{x}$: Solution

Given two bits x, z which of the following SAT formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \vee x) \wedge (z \vee \overline{x})$.

(B) $(z \vee x) \wedge (\overline{z} \vee \overline{x})$.

(C) $(\overline{z} \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$.

(D) $z \oplus x$.

(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

x	y	$z = \overline{x}$
0	0	0
0	1	1
1	0	1
1	1	0

$z = x \wedge y$

Given three bits x, y, z which of the following SAT formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\bar{z} \vee x \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(B) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(C) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(D) $(z \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(E) $(z \vee x \vee y) \wedge (z \vee x \vee \bar{y}) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y}) \wedge$
$(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee x \vee \bar{y}) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (\bar{z} \vee \bar{x} \vee \bar{y})$.

Answer: C

$z = x \wedge y$

Given three bits x, y, z which of the following SAT formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\bar{z} \vee x \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(B) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(C) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(D) $(z \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(E) $(z \vee x \vee y) \wedge (z \vee x \vee \bar{y}) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y}) \wedge (\bar{z} \vee x \vee y) \wedge (\bar{z} \vee x \vee \bar{y}) \wedge$

x	y	z	$z = x \wedge y$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Exercise

What is a non-satisfiable SAT assignment?