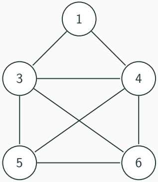
Pre-lecture brain teaser

Consider the following algorithm which takes in an undirected graph (G) and a vertex s.

```
FindClique (G, s)
    C = s
    for each vertex v ∈ V
        flag = 1
        for each vertex u ∈ C
            if (u, v) ∉ E
                flag = 0
        if flag == 1
            C = C ∪ {v}
    return C
```

The algorithm represents a greedy algorithm which finds a clique depending on a start vertex s.

- How fast is this algorithm?



ECE-374-B: Lecture 20 - P/NP and NP-completeness

Instructor: Abhishek Kumar Umrawal

April 09, 2024

University of Illinois at Urbana-Champaign

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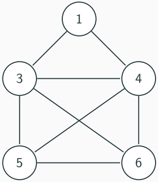

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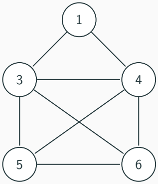

The Clique-problem is NP-complete. But this algorithm provides us with the maximal clique containing s. If we run it $|V|$ times, does that solve the clique-problem.

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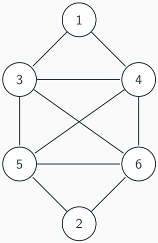

The Satisfiability Problem (SAT)

Propositional Formulas

Definition
Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

- A literal is either a boolean variable x_i or its negation $\neg x_i$.
- A clause is a disjunction of literals.
  For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
- A formula in conjunctive normal form (CNF) is propositional formula which is a conjunction of clauses.
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.

Propositional Formulas

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  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.
- A formula φ is a 3CNF:
  A CNF formula such that every clause has **exactly** 3 literals.
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a 3CNF formula, but $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

CNF is universal

Every boolean formula $f : \{0,1\}^n \to \{0,1\}$ can be written as a CNF formula.

x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \ldots, x_6)$	$\overline{x_1} \vee x_2 \overline{x_3} \vee x_4 \vee \overline{x_5} \vee x_6$
0	0	0	0	0	0	$f(0, \ldots, 0, 0)$	1
0	0	0	0	0	1	$f(0, \ldots, 0, 1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	1	0	0	1	?	1
1	0	1	0	1	0	0	0
1	0	1	0	1	1	?	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	1	1	1	1	$f(1, \ldots, 1)$	1

<u>How?</u> For every row such that f is zero, compute corresponding CNF clause. Then take the AND ($\wedge$) of all the CNF clauses computed. The resulting CNF formula is equivalent to f.

Satisfiability

Problem: SAT

Instance: A CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

Instance: A 3CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Satisfiability

SAT
Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example

- $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is satisfiable; take $x_1, x_2, \dots x_5$ to be all true
- $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$ is not satisfiable.

3SAT
Given a 3CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Importance of SAT and 3SAT

- **SAT** and **3SAT** are basic constraint satisfaction problems.
- Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
- Arise naturally in many applications involving hardware and software verification and correctness.
- As we will see, it is a fundamental problem in theory of NP-Completeness.

$z = \overline{x}$

Given two bits x, z which of the following **SAT** formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \vee x) \wedge (z \vee \overline{x})$.

(B) $(z \vee x) \wedge (\overline{z} \vee \overline{x})$.

(C) $(\overline{z} \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$.

(D) $z \oplus x$.

(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

Answer: B

$\mathbf{z} = \overline{\mathbf{x}}$: Solution

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(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

x	y	$z = \overline{x}$
0	0	0
0	1	1
1	0	1
1	1	0

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.

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Answer: C

$\mathbf{z = x \wedge y}$

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x	y	z	$z = x \wedge y$
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0	0	1	0
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1	0	1	0
1	1	0	0
1	1	1	1

Reducing SAT to 3SAT

How SAT is different from 3SAT?
In **SAT** clauses might have arbitrary length: $1, 2, 3, \ldots$ variables:

$$\Big(x \vee y \vee z \vee w \vee u\Big) \wedge \Big(\neg x \vee \neg y \vee \neg z \vee w \vee u\Big) \wedge \Big(\neg x\Big)$$

In **3SAT** every clause must have <u>exactly</u> 3 different literals.

How SAT is different from 3SAT?
In **SAT** clauses might have arbitrary length: $1, 2, 3, \ldots$ variables:

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In **3SAT** every clause must have exactly 3 different literals.

To reduce from an instance of **SAT** to an instance of **3SAT**, we must make all clauses to have exactly 3 variables...

Basic idea

- Pad short clauses so they have 3 literals.
- Break long clauses into shorter clauses.
- Repeat the above till we have a 3CNF.

Proof of this in Prof. Har-Peled's async lectures!

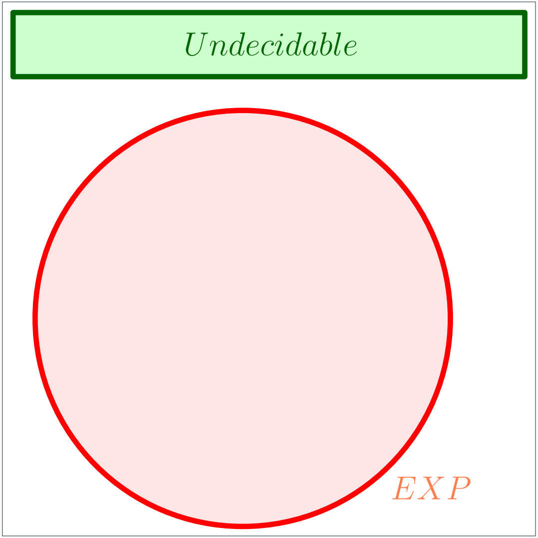
Overview of Complexity Classes

In the beginning...

In the beginning...

$\boxed{Undecidable}$

In the beginning...



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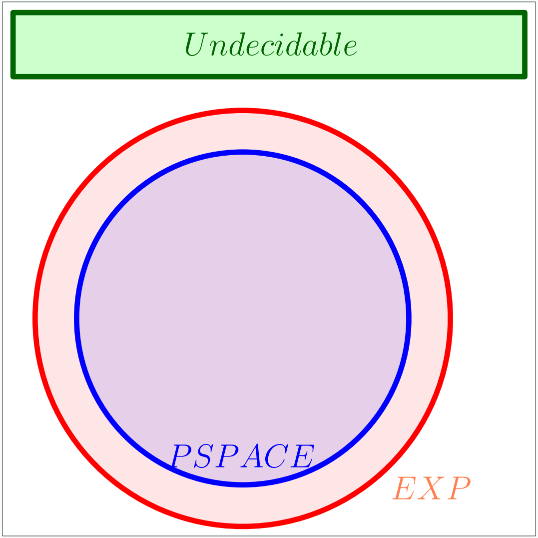

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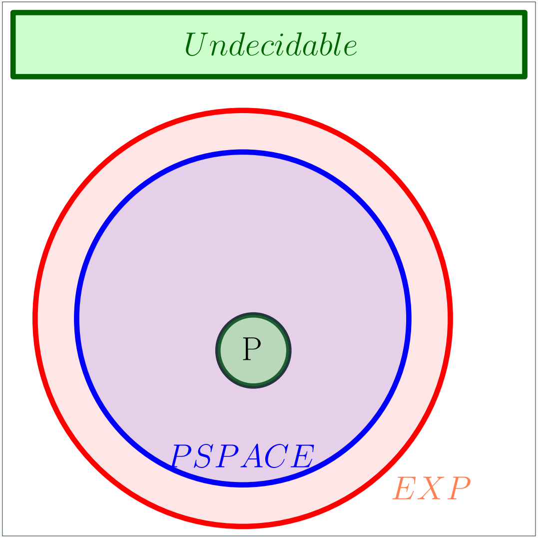

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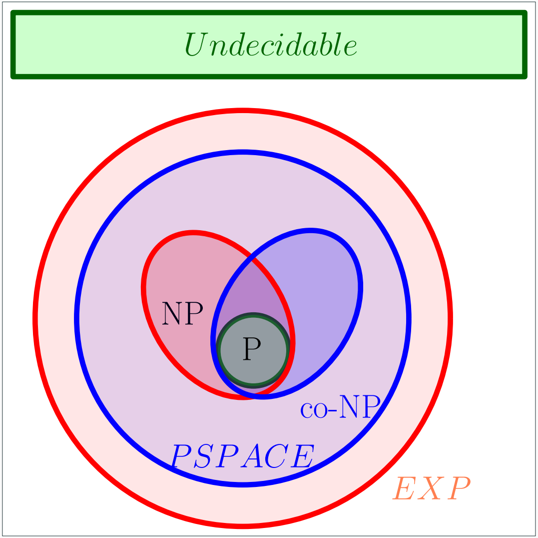

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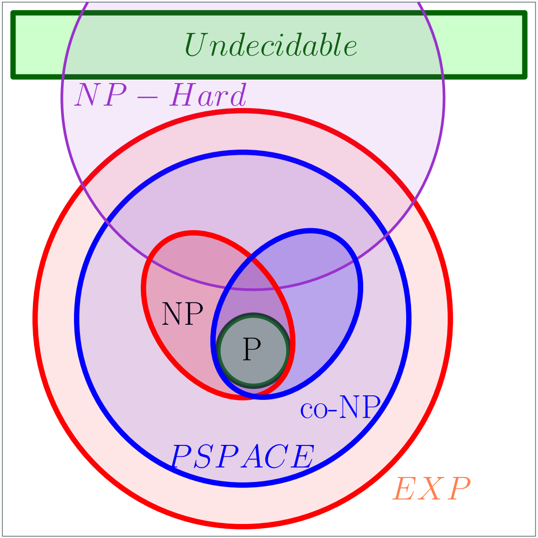

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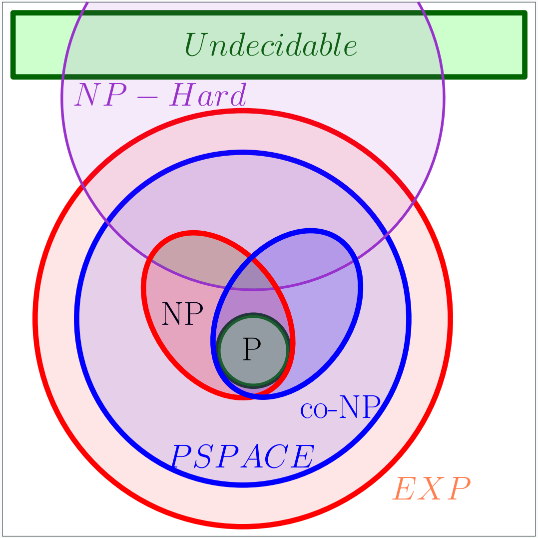

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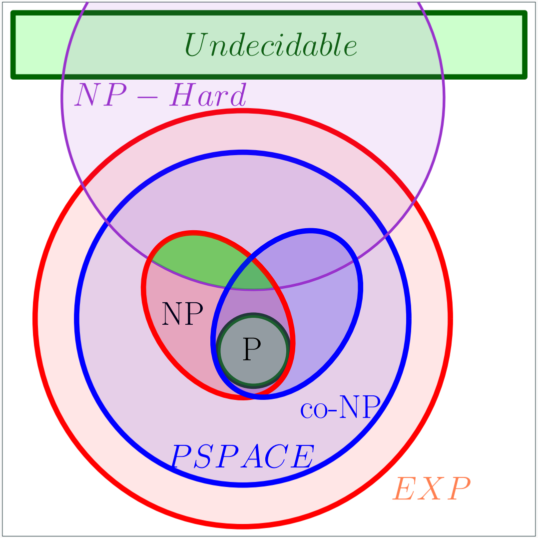

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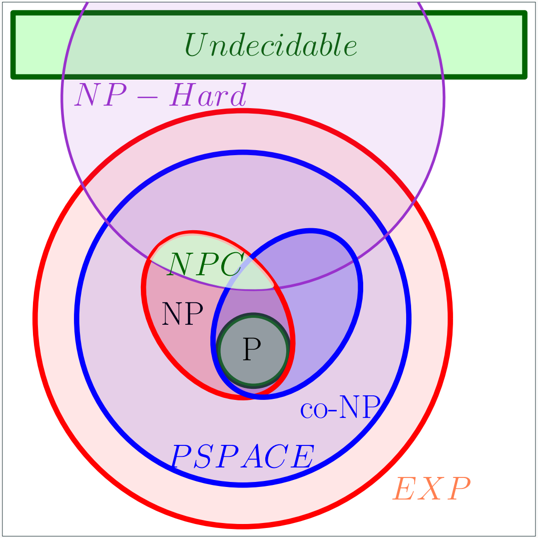

Non-deterministic polynomial time - NP

P, NP and Turing Machines

- P: set of decision problems that have polynomial time (deterministic) algorithms, i.e. efficiently solvable using a (deterministic) Turing machine (DTM).
- NP: set of decision problems that have polynomial time non-deterministic algorithms, i.e. efficiently solvable using a non-deterministic Turing machine (NTM).
- Many natural problems we would like to solve are in NP.
- Every problem in NP has an exponential time (deterministic) algorithm.
- $P \subseteq NP$.
- Some problems in NP are in P (e.g., shortest path problem).

Big Question: Does every problem in NP have an efficient algorithm? Same as asking whether $P = NP$.

Problems with no known deterministic polynomial time algorithms

Problems

- **Independent Set**
- **Vertex Cover**
- **Set Cover**
- **SAT**

There are of course undecidable problems (no algorithm at all!) but many problems that we want to solve are of similar flavor to the above.

Question: What is common to above problems?

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Question: What is common to above problems?

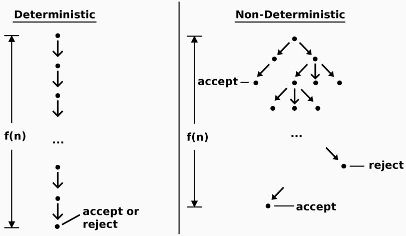
They can all be solved via a non-deterministic computer in polynomial time!

Non-determinism in computing

Non-determinism is a special property of algorithms.

An algorithm that is capable of taking multiple states concurrently. Whenever it reaches a choice, it takes both paths.

If there is a path for the string to be accepted by the machine, then the string is part of the language.



Problems with no known deterministic polynomial time algorithms

Problems

- **Independent Set** & **Vertex Cover** - Can build algorithm to check all possible collection of vertices
- **Set Cover** - Can check all possible collection of sets
- **SAT** -Can build a non-deterministic algorithm that checks every possible boolean assignment.

But we don't have access to a non-deterministic computer. So how can a deterministic computer verify that a algorithm is in NP?

Efficient Checkability

Above problems share the following feature.

Checkability

For any YES instance I_X of X there is a proof/certificate/solution that is of length poly($|I_X|$) such that given a proof one can efficiently check that I_X is indeed a YES instance.

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Examples:

- **SAT** formula φ: proof is a satisfying assignment.
- **Independent Set** in graph G and k: a subset S of vertices.
- **Homework**.

Certifiers

Definition
An algorithm $C(\cdot, \cdot)$ is a certifier for problem X if the following two conditions hold.

- For every $s \in X$ there is some string t such that $C(s, t) =$ "yes"
- If $s \notin X$, $C(s, t) =$ "no" for every t.

The string s is the problem instance. (Example: particular graph in independent set problem.) The string t is called a certificate or proof for s.

Efficient (polynomial time) Certifiers

Definition (Efficient Certifier.)
A certifier C is an efficient certifier for problem X if there is a polynomial $p(\cdot)$ such that the following conditions hold.

- For every $s \in X$ there is some string t such that $C(s,t) =$ "yes" **and** $|t| \leq p(|s|)$.
- If $s \notin X$, $C(s,t) =$ "no" for every t.
- $C(\cdot,\cdot)$ runs in polynomial time.

Example: Independent Set

- Problem: Does $G = (V, E)$ have an independent set of size $\geq k$?
  - Certificate: Set $S \subseteq V$.
  - Certifier: Check $|S| \geq k$ and no pair of vertices in S is connected by an edge.

Example: SAT

- Problem: Does formula φ have a satisfying truth assignment?
  - Certificate: Assignment a of $0/1$ values to each variable.
  - Certifier: Check each clause under a and say "yes" if all clauses are true.

Why is it called Non-deterministic Polynomial Time

A certifier is an algorithm $C(I, c)$ with the following two inputs.

- I: instance.
- c: proof/certificate that the instance is indeed a YES instance of the given problem.

One can think about C as an algorithm for the original problem if the following hold.

- Given I, the algorithm guesses (non-deterministically, and who knows how) a certificate c.
- The algorithm now verifies the certificate c for the instance I.

NP can be equivalently described using Turing machines.

Cook-Levin Theorem

“Hardest” Problems

Question
What is the hardest problem in NP? How do we define it?

Towards a definition

- Hardest problem must be in NP.
- Hardest problem must be at least as “difficult” as every other problem in NP.

NP-Complete Problems

Definition
A problem X is said to be **NP-Complete** if

- $X \in NP$, and
- (Hardness) For any $Y \in NP$, $Y \leq_P X$.

Lemma
Suppose X is NP-Complete. Then X can be solved in polynomial time if and only if $P = NP$.

Proof.

$\Rightarrow$ Suppose X can be solved in polynomial time

- Let $Y \in NP$. We know $Y \leq_P X$.
- We showed that if $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.
- Thus, every problem $Y \in NP$ is such that $Y \in P$; $NP \subseteq P$.
- Since $P \subseteq NP$, we have $P = NP$.

$\Leftarrow$ Since $P = NP$, and $X \in NP$, we have a polynomial time algorithm for X. $\square$

NP-Hard Problems

Definition
A problem Y is said to be NP-Hard if

- (Hardness) For any $X \in NP$, we have that $X \leq_P Y$.

An NP-Hard problem need not be in NP!

Example: Halting problem is NP-Hard (why?) but not NP-Complete.

Consequences of proving NP-Completeness

If X is NP-Complete

- Since we believe $P \neq NP$,
- and solving X implies $P = NP$.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for X.

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(This is proof by mob opinion — take with a grain of salt.)

NP-Complete Problems

Question
Are there any problems that are NP-Complete?

Answer
Yes! Many, many problems are NP-Complete.

Cook-Levin Theorem

Theorem (Cook-Levin)
SAT *is NP-Complete.*

Cook-Levin Theorem

Theorem (Cook-Levin)
SAT *is NP-Complete.*

Need to show the following.

- **SAT** is in NP.
- Every NP problem X reduces to **SAT**.

Steve Cook won the Turing award for his theorem.

Proving that a problem X is NP-Complete

To prove X is NP-Complete, show the following.

- Show that X is in NP.
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SAT $\leq_P X$ implies that every NP problem $Y \leq_P X$. Why?
Transitivity of reductions:

$Y \leq_P SAT$ and $SAT \leq_P X$ and hence $Y \leq_P X$.

3-SAT is NP-Complete

- **3-SAT** is in *NP*.
- **SAT** $\leq_P$ **3-SAT** as we saw.

NP-Completeness via Reductions

- **SAT** is NP-Complete due to Cook-Levin theorem.
- **SAT** $\leq_P$ **3-SAT**
- **3-SAT** $\leq_P$ **Independent Set**
- **Independent Set** $\leq_P$ **Vertex Cover**
- **Independent Set** $\leq_P$ **Clique**
- **3-SAT** $\leq_P$ **3-Color**
- **3-SAT** $\leq_P$ **Hamiltonian Cycle**

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Hundreds and thousands of different problems from many areas of science and engineering have been shown to be NP-Complete.

A surprisingly frequent phenomenon!

Reducing 3-SAT to Independent Set

Problem: Independent Set

Instance: A graph G, integer k.

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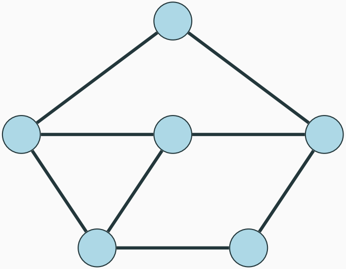

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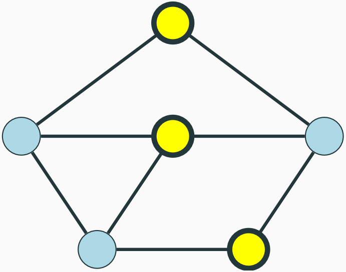

Interpreting **3SAT**

There are two ways to think about **3SAT**.

1. Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$.

We will take the second view of **3SAT** to construct the reduction.

The Reduction

1. G_φ will have one vertex for each literal in a clause.
2. Connect the literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true.
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict.
4. Take k to be the number of clauses.

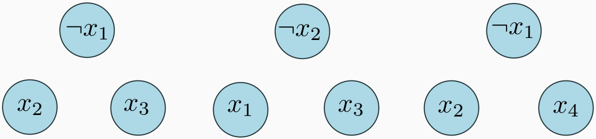

Figure 1: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$.

The Reduction

1. G_φ will have one vertex for each literal in a clause.
2. Connect the literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true.
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict.
4. Take k to be the number of clauses.

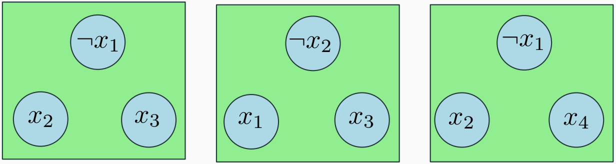

Figure 1: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$.

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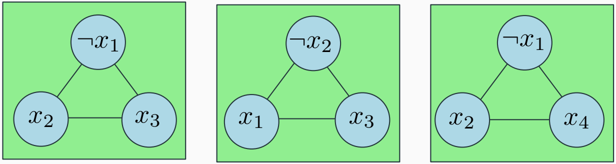

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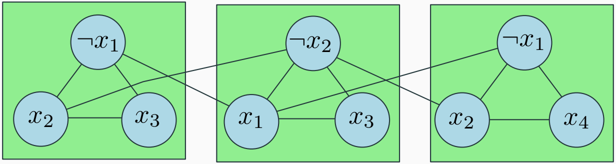

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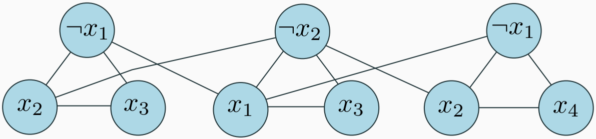

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Correctness

Lemma
φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

$\Rightarrow$ Let a be the truth assignment satisfying φ.

- Pick one of the vertices, corresponding to true literals under a, from each triangle. This is an independent set of the appropriate size. Why? $\square$

Correctness (contd)

Lemma
φ is satisfiable iff G_φ has an independent set of size k (= number of clauses in φ).

Proof.

$\Leftarrow$ Let S be an independent set of size k.

- S must contain exactly one vertex from each clause triangle.
- S cannot contain vertices labeled by conflicting literals.
- Thus, it is possible to obtain a truth assignment that makes in the literals in S true; such an assignment satisfies one literal in every clause. $\square$

Other NP-Complete problems

Graph Coloring

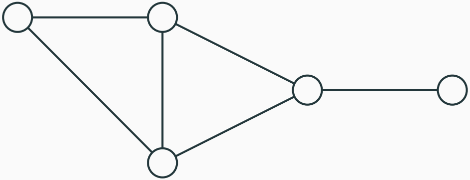
Graph Coloring

Problem: Graph Coloring

Instance: $G = (V, E)$: Undirected graph, integer k.
Question: Can the vertices of the graph be colored using k colors so that vertices connected by an edge do not get the same color?

Problem: 3 Coloring

Instance: $G = (V, E)$: Undirected graph.
Question: Can the vertices of the graph be colored using 3 colors so that vertices connected by an edge do not get the same color?



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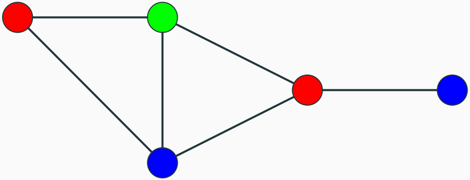

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Graph Coloring

Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G. Thus, G can be partitioned into k independent sets iff G is k-colorable.

Graph 2-Coloring can be decided in polynomial time.

G is 2-colorable iff G is bipartite! There is a linear time algorithm to check if G is bipartite using breadth first search.

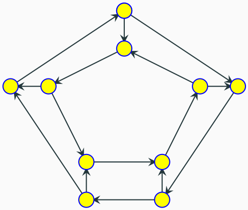
Hamiltonian Cycle

Directed Hamiltonian Cycle

Input Given a directed graph $G = (V, E)$ with n vertices

Goal Does G have a Hamiltonian cycle?

- A Hamiltonian cycle is a cycle in the graph that visits every vertex in G exactly once.



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