Pre-lecture brain teaser

We know that SAT is NP-complete which means that it is in NP-Hard. HALT is also in NP-Hard. Is SAT reducible to HALT? How?

- Construct a Turing machine that considers all possible assignments. Using for loops.
- if satisfying assignment is solved then halt.

Clearly oracle for HALT can find if the following Turing machine halts and therefore if the CNF is satisfiable.
Is this ok? The turing machine runs in exponential time?

ECE-374-B: Lecture 23 - Decidability II

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University of Illinois at Urbana-Champaign

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Reductions

Reduction

Meta definition: Problem $\mathbf{X}$ reduces to problem $\mathbf{Y}$, if given a solution to $\mathbf{Y}$, then it implies a solution for $\mathbf{X}$. Namely, we can solve $\mathbf{Y}$ then we can solve $\mathbf{X}$. We will done this by $\mathbf{X} \implies \mathbf{Y}$.

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Definition

Oracle ORAC for language L is a function that receives as a word w, returns TRUE $\iff w \in L$.

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Definition
Oracle ORAC for language L is a function that receives as a word w, returns TRUE $\iff$ $w \in L$.

Lemma
A language X reduces to a language Y, if one can construct a TM decider for X using a given oracle ORAC_Y *for Y.*

We will denote this fact by $X \implies Y$.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.

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- Result in decider for **X** (i.e., A_{TM}).
- Contradiction **X** is not decidable.
- Thus, L must be not decidable.

Reduction implies decidability

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If Y is decidable then X is decidable.

Proof.
Let $\mathtt{T}$ be a decider for Y (i.e., a program or a TM). Since X reduces to Y, it follows that there is a procedure $\mathtt{T}_{X|Y}$ (i.e., decider) for X that uses an oracle for Y as a subroutine. We replace the calls to this oracle in $\mathtt{T}_{X|Y}$ by calls to $\mathtt{T}$. The resulting program $\mathtt{T}_X$ is a decider and its language is X. Thus X is decidable (or more formally TM decidable). $\square$

The countrapositive...

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If X is undecidable then Y is undecidable.

Halting

The halting problem

Language of all pairs $\langle M, w\rangle$ such that M halts on w:

$$A_{\mathrm{Halt}} = \left\{ \langle M, w\rangle \;\middle|\; M \text{ is a } \mathit{TM} \text{ and } M \text{ stops on } w \right\}.$$

Similar to language already known to be undecidable:

$$\mathrm{A}_{\mathit{TM}} = \left\{ \langle M, w\rangle \;\middle|\; M \text{ is a } \mathit{TM} \text{ and } M \text{ accepts } w \right\}.$$

One way to proving that Halting is undecidable...

Lemma
The language A_{TM} *reduces to* A_{Halt}. *Namely, given an oracle for* A_{Halt} *one can build a decider (that uses this oracle) for* A_{TM}.

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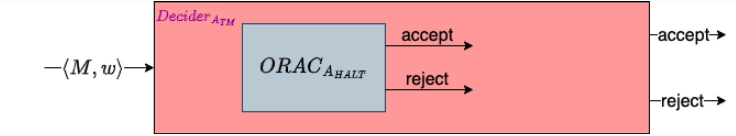


One way to proving that Halting is undecidable...

Proof.
Let ORAC_{Halt} be the given oracle for A_{Halt}. We build the following decider for A_{TM}.

AnotherDecider-$\mathrm{A}_{TM}\Big(\langle M, w\rangle\Big)$

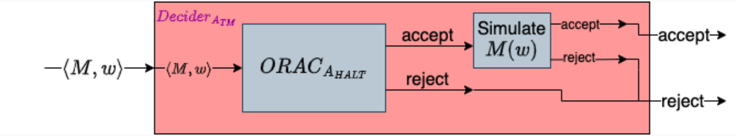
```
    res ← ORAC_Halt(⟨M, w⟩)
    // if M does not halt on w then reject.
    if res = reject then
        halt and reject.
    // M halts on w since res =accept.
    // Simulating M on w terminates in finite time.
    res_2 ←Simulate M on w.
    return res_2 .
```

This procedure always return and as such its a decider for A_{TM}. $\square$

Theorem
The language A_{Halt} is not decidable.

Proof.
Assume, for the sake of contradiction, that A_{Halt} is decidable. As such, there is a TM, denoted by TM_{Halt}, that is a decider for A_{Halt}. We can use TM_{Halt} as an implementation of an oracle for A_{Halt}, which would imply that one can build a decider for A_{TM}. However, A_{TM} is undecidable. A contradiction. It must be that A_{Halt} is undecidable. $\square$

The same proof by figure...



... if A_{Halt} is decidable, then A_{TM} is decidable, which is impossible.

Emptiness

The language of empty languages

- $E_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) = \emptyset \right\}$.
- TM_{ETM}: Assume we are given this decider for E_{TM}.
- Need to use TM_{ETM} to build a decider for A_{TM}.
- Decider for A_{TM} is given M and w and must decide whether M accepts w.
- Restructure question to be about Turing machine having an empty language.
- Somehow make the second input (w) disappear.

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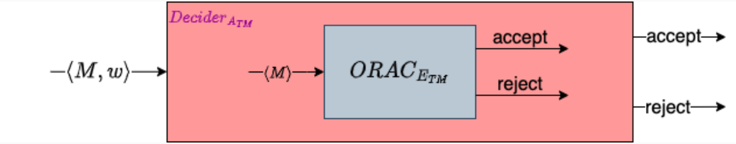

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- Restructure question to be about Turing machine having an empty language.
- Somehow make the second input (w) disappear.
- Idea: hard-code w into M, creating a TM M_w which runs M on the fixed string w.
- TM $M_w(x)$:
    1. Input $= x$ (which will be ignored)
    2. Simulate M on w.
    3. If the simulation accepts, accept. Else, reject.

Embedding strings...

- Given program $\langle M \rangle$ and input w...
- ...can output a program $\langle M_w \rangle$.
- The program M_w simulates M on w. And accepts/rejects accordingly.
- **EmbedString**($\langle M, w \rangle$) input two strings $\langle M \rangle$ and w, and output a string encoding (TM) $\langle M_w \rangle$.

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- What is $L(M_w)$?
- Since M_w ignores input x.. language M_w is either Σ^* or $\emptyset$. It is Σ^* if M accepts w, and it is $\emptyset$ if M does not accept w.

Emptiness is undecidable

Theorem
The language E_{TM} is undecidable.

- Assume (for contradiction), that E_{TM} is decidable.
- TM_{ETM} be its decider.
- Build decider **AnotherDecider-A_{TM}** for A_{TM}:

AnotherDecider-A_{TM} $(\langle M, w\rangle)$
    $\langle M_w\rangle \leftarrow$ **EmbedString** $(\langle M, w\rangle)$
    $r \leftarrow TM_{ETM}(\langle M_w\rangle)$.
    if $r =$ accept **then**
        return reject
    // $TM_{ETM}(\langle M_w\rangle)$ `rejected its input`
    return accept

Emptiness is undecidable...

Consider the possible behavior of **AnotherDecider-**A_{TM} on the input $\langle M, w\rangle$.

- If TM_{ETM} accepts $\langle M_w\rangle$, then $L(M_w)$ is empty. This implies that M does not accept w. As such, **AnotherDecider-**A_{TM} rejects its input $\langle M, w\rangle$.
- If TM_{ETM} accepts $\langle M_w\rangle$, then $L(M_w)$ is not empty. This implies that M accepts w. So **AnotherDecider-**A_{TM} accepts $\langle M, w\rangle$.

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$\implies$ **AnotherDecider-**A_{TM} is decider for A_{TM}.

But A_{TM} is undecidable...

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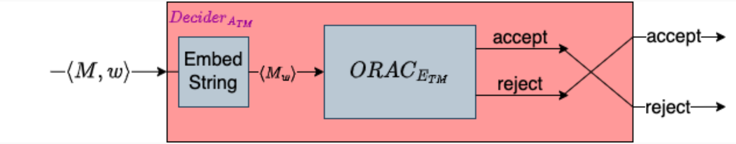
But A_{TM} is undecidable...

...must be assumption that E_{TM} is decidable is false.

Emptiness is undecidable via diagram



AnotherDecider-A_{TM} never actually runs the code for M_w. It hands the code to a function TM_{ETM} which analyzes what the code would do if run it. So it does not matter that M_w might go into an infinite loop.

Equality

Equality is undecidable

$$EQ_{TM} = \left\{ \langle M, N \rangle \;\middle|\; M \text{ and } N \text{ are TM's and } L(M) = L(N) \right\}.$$

Lemma
The language EQ_{TM} is undecidable.

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Let's try something different. We know E_{TM} is undecidable. Let's use that:

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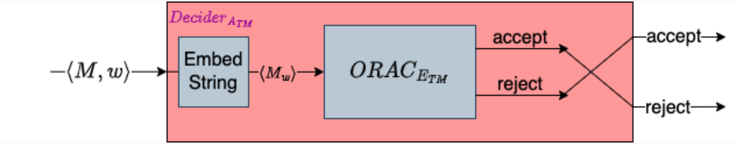
$$E_{TM} \implies EQ_{TM}$$

Equality diagram



Proof

Proof.
Suppose that we had a decider **DeciderEqual** for EQ_{TM}. Then we can build a decider for E_{TM} as follows:

TM R:

1. Input $= \langle M \rangle$
2. Include the (constant) code for a TM T that rejects all its input. We denote the string encoding T by $\langle T \rangle$.
3. Run **DeciderEqual** on $\langle M, T \rangle$.
4. If **DeciderEqual** accepts, then accept.
5. If **DeciderEqual** rejects, then reject.

$\square$

DFAs

DFAs are empty?

$$E_{DFA} = \left\{ \langle A \rangle \;\middle|\; A \text{ is a DFA and } L(A) = \emptyset \right\}.$$

What does the above language describe?

All the DFA encodings that edescribe empty languages.

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Is the language above decidable? Yes ofcourse. It's a simple DFA.

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Lemma
The language E_{DFA} is decidable:

Scratch

Proof

Proof.
Unlike in the previous cases, we can directly build a decider (**DeciderEmptyDFA**) for E_{DFA}

TM R:

1. Input $= \langle A \rangle$
2. Mark start state of A as visited.
3. Repeat until no new states get marked:
   - Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, then accept.
5. Otherwise, then reject.

$\square$

Equal DFAs

DFAs are equal?

$$EQ_{DFA} = \left\{ \langle A, b \rangle \;\middle|\; A \text{ and } B \text{ are DFAs and } L(A) = L(B) \right\}.$$

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All the DFA string pairs that represent equivalent languages

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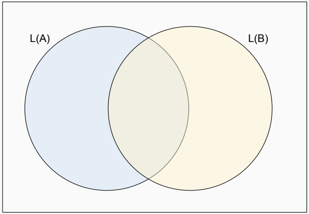
Lemma
The language E_{DFA} is decidable.

Can we show this using reductions? $EQ_{DFA} \implies E_{DFA}$

Equal DFA trick I

Need a way to determine if there any strings in one language and not the other....



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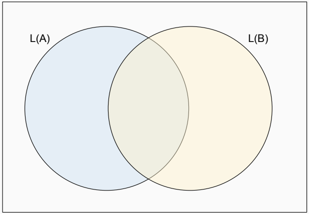

This is known as the symmetric difference. Can create a new DFA (C) which represents the symmetric difference of L_A and L_B.

$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right) \tag{1}$$

Equal DFA trick II

Notice with $L(C)$:

- If $L(A) = L(B)$ then $L(C) = \emptyset$
- If $L(A) \neq L(B)$ then $L(C)$ is not empty

Good time to use E_{DFA} proof from before.....How do we show EQ_{DFA} is decidable using a reduction?

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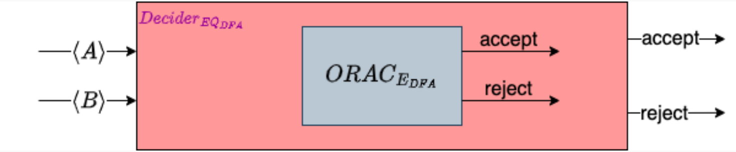

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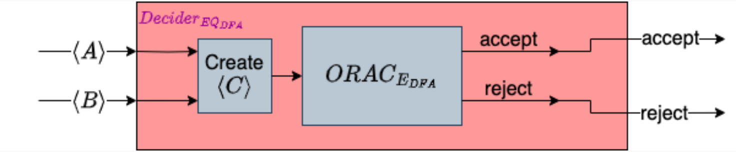


Equal DFA decider

TM F:

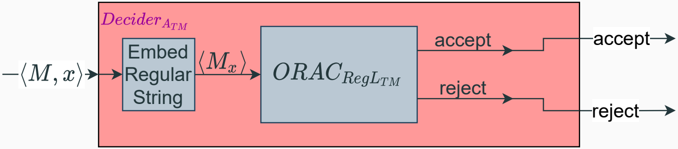
1. Input $= \langle A, B \rangle$ where A and B are DFAs
2. Construct DFA C as described before
3. Run **DeciderEmptyDFA** from previous slide on C
4. If accepts, then accept.
5. If rejects, then reject.

Regularity

Many undecidable languages

- Almost any property defining a TM language induces a language which is undecidable.
- proofs all have the same basic pattern.
- Regularity language:
  $\mathrm{Regular}_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) \text{ is regular} \right\}.$
- **DeciderRegL**: Assume TM decider for $\mathrm{Regular}_{TM}$.
- Reduction from halting requires to turn problem about deciding whether a TM M accepts w (i.e., is $w \in \mathrm{A}_{TM}$) into a problem about whether some TM accepts a regular set of strings.

Outline of IsRegular? reductionr



Proof continued...

- Given M and w, consider the following TM M'_w:
  TM M'_w:
  (i) Input $= x$
  (ii) If x has the form $a^n b^n$, halt and accept.
  (iii) Otherwise, simulate M on w.
  (iv) If the simulation accepts, then accept.
  (v) If the simulation rejects, then reject.
- **not** executing M'_w!
- feed string $\langle M'_w \rangle$ into **DeciderRegL**
- **EmbedRegularString**: program with input $\langle M \rangle$ and w, and outputs $\langle M'_w \rangle$, encoding the program M'_w.
- If M accepts w, then any x accepted by M'_w: $L(M'_w) = \Sigma^*$.
- If M does not accept w, then $L(M'_w) = \{a^n b^n \mid n \geq 0\}$.

Proof continued...

- $a^n b^n$ is not regular...
- Use **DeciderRegL** on M'_w to distinguish these two cases.
- Note - cooked M'_w to the decider at hand.
- A decider for A_{TM} as follows.

AnotherDecider-$\mathrm{A}_{TM}$$(\langle M, w\rangle)$

  $\langle M'_w\rangle \leftarrow$ **EmbedRegularString** $(\langle M, w\rangle)$

  $r \leftarrow$ **DeciderRegL**$(\langle M'_w\rangle)$.

  return r

- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*)

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- If **DeciderRegL** rejects $\implies L(M'_w)$ is not regular $\implies L(M'_w) = a^n b^n$

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- If **DeciderRegL** rejects $\implies$ $L(M'_w)$ is not regular $\implies$ $L(M'_w) = a^n b^n$ $\implies$ M does not accept w $\implies$ **AnotherDecider-**A_{TM} should reject $\langle M, w\rangle$.

Rice's theorem

The above proofs were somewhat repetitious...

...they imply a more general result.

Theorem (Rice's Theorem.)
Suppose that L *is a language of Turing machines; that is, each word in* L *encodes a* *TM*. *Furthermore, assume that the following two properties hold.*

(a) *Membership in* L *depends only on the Turing machine's language, i.e. if* $L(M) = L(N)$ *then* $\langle M \rangle \in \mathsf{L} \Leftrightarrow \langle N \rangle \in \mathsf{L}$.

(b) *The set* L *is "non-trivial," i.e.* $\mathsf{L} \neq \emptyset$ *and* L *does not contain all Turing machines.*

Then L *is a undecidable.*