Pre-lecture brain teaser

We know that SAT is NP-complete which means that it is in NP-Hard. HALT is also in NP-Hard. Is SAT reducible to HALT? YES! How?

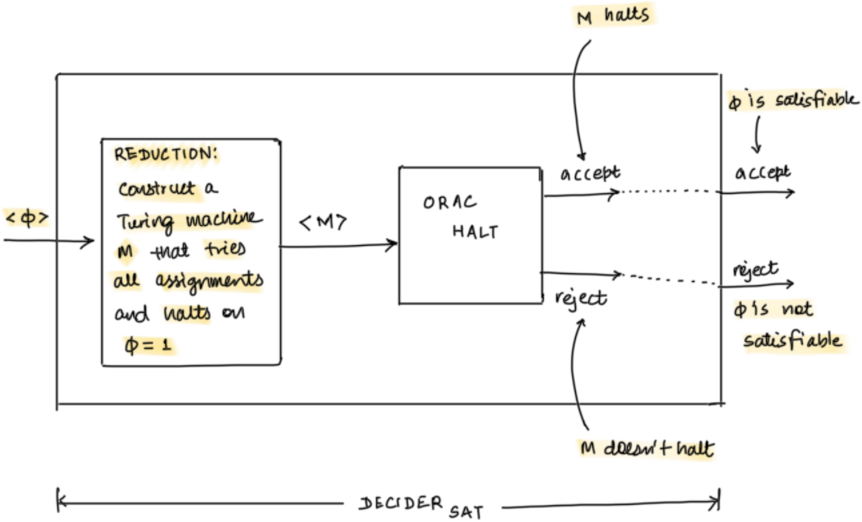
SAT $\Rightarrow$ HALT   SAT $\leq$ HALT

- Construct a Turing machine that considers all possible assignments. Using for loops.
- if satisfying assignment is solved then halt.

see next page.

Clearly oracle for HALT can find if the following Turing machine halts and therefore if the CNF is satisfiable.
Is this ok? The turing machine runs in exponential time?

SAT ⇒ HALT


M halts
φ is satisfiable
REDUCTION:
Construct a Turing machine M that tries all assignments and halts on φ = 1
<φ>
<M>
ORAC HALT
accept
accept
reject
reject
φ is not satisfiable
M doesn't halt
DECIDER SAT

ECE-374-B: Lecture 23 - Decidability II

Instructor: Abhishek Kumar Umrawal

April 18, 2024

University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

We know that SAT is NP-complete which means that it is in NP-Hard. HALT is also in NP-Hard. Is SAT reducible to HALT? How?

- Construct a Turing machine that considers all possible assignments. Using for loops.
- if satisfying assignment is solved then halt.

Clearly oracle for HALT can find if the following Turing machine halts and therefore if the CNF is satisfiable.
Is this ok? The turing machine runs in exponential time?

Reductions

Reduction

Meta definition: Problem **X** <u>reduces</u> to problem **Y**, if given a solution to **Y**, then it implies a solution for **X**. Namely, we can solve **Y** then we can solve **X**. We will done this by **X** $\implies$ **Y**.

$$X \Rightarrow Y \quad : \quad X \leq Y$$

Reduction

Meta definition: Problem $\mathbf{X}$ reduces to problem $\mathbf{Y}$, if given a solution to $\mathbf{Y}$, then it implies a solution for $\mathbf{X}$. Namely, we can solve $\mathbf{Y}$ then we can solve $\mathbf{X}$. We will done this by $\mathbf{X} \implies \mathbf{Y}$.

Definition

Oracle ORAC for language L is a function that receives as a word w, returns TRUE $\iff w \in L$.

GIVEN DECIDER

Meta definition: Problem **X** reduces to problem **Y**, if given a solution to **Y**, then it implies a solution for **X**. Namely, we can solve **Y** then we can solve **X**. We will done this by **X** $\implies$ **Y**.

Definition

Oracle ORAC for language L is a function that receives as a word w, returns TRUE $\iff w \in L$.

Lemma

A language X reduces to a language Y, if one can construct a TM decider for X using a given oracle ORAC_Y for Y.

We will denote this fact by $X \implies Y$.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of **Y**.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of **Y**.
- Assume L is decided by TM M.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of **Y**.

ORACLE for Y

- Assume L is decided by TM M.
- Create a decider for known undecidable problem **X** using M.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of **Y**.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem **X** using M.
- Result in decider for **X** (i.e., A_{TM}).

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of **Y**.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem **X** using M.
- Result in decider for **X** (i.e., A_{TM}).
- Contradiction **X** is not decidable.

Reduction proof technique

- **Y**: Problem/language for which we want to prove undecidable.
- Proof via reduction. Result in a proof by contradiction.
- L: language of **Y**.
- Assume L is decided by TM M.
- Create a decider for known undecidable problem **X** using M.
- Result in decider for **X** (i.e., A_{TM}).
- Contradiction **X** is not decidable.
- Thus, L must be not decidable.

Reduction implies decidability (RIY)

Lemma
Let X and Y be two languages, and assume that $X \implies Y$. If Y is decidable then X is decidable.

Proof.
Let $\mathtt{T}$ be a decider for Y (i.e., a program or a TM). Since X reduces to Y, it follows that there is a procedure $\mathtt{T}_{X|Y}$ (i.e., decider) for X that uses an oracle for Y as a subroutine. We replace the calls to this oracle in $\mathtt{T}_{X|Y}$ by calls to $\mathtt{T}$. The resulting program $\mathtt{T}_X$ is a decider and its language is X. Thus X is decidable (or more formally TM decidable). □

Lemma

Let X and Y be two languages, and assume that $X \implies Y$. If X is undecidable then Y is undecidable.

If $X \Rightarrow Y$ then:

- Y is decidable $\Rightarrow$ X is decidable
- X is undecidable $\Rightarrow$ Y is undeciable

Halting

The halting problem

Language of all pairs $\langle M, w \rangle$ such that M halts on w:

$$A_{\mathrm{Halt}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ stops on } w \right\}.$$

Similar to language already known to be undecidable:

$$\mathrm{A}_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and } M \text{ accepts } w \right\}.$$

Undecidable (we know!)

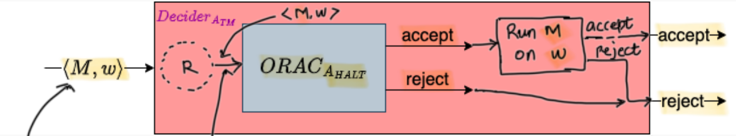
(*) $A_{TM} \Rightarrow A_{Halt}$ (we want this to prove the undecidability of A_{Halt})

One way to proving that Halting is undecidable...

$A_{TM} \Rightarrow A_{Halt}$

Lemma
The language A_{TM} *reduces to* A_{Halt}. *Namely, given an oracle for* A_{Halt} *one can build a decider (that uses this oracle) for* A_{TM}.

Lemma
The language A_{TM} *reduces to* A_{Halt}. *Namely, given an oracle for* A_{Halt} *one can build a decider (that uses this oracle) for* A_{TM}.



An instance of X : A_{TM}

Decider A_{TM}

An instance of Y: A_{Halt}

$\Rightarrow$ A decider for A_{TM} : contradiction

(Given)
ORAC A_{Halt} : accept : if M accepts w or M rejects w
reject : if M loops forever

(Needed)
DECIDER A_{TM} : accept : if M accepts w
reject : otherwise : if M rejects w or loops forever

One way to proving that Halting is undecidable...

Proof.
Let ORAC_{Halt} be the given oracle for A_{Halt}. We build the following decider for A_{TM}.

AnotherDecider-$\mathrm{A}_{TM}\Big(\langle M, w\rangle\Big)$

$\quad$ $res \leftarrow \mathsf{ORAC}_{Halt}\Big(\langle M, w\rangle\Big)$

$\quad$ `// if` M `does not halt on` w `then reject.`

$\quad$ **if** $res =$ `reject` **then**

$\quad\quad$ `halt and reject.`

$\quad$ `//` M `halts on` w `since` $res =$accept`.`

$\quad$ `// Simulating` M `on` w `terminates in finite time.`

$\quad$ $res_2 \leftarrow$`Simulate` M `on` w`.`

$\quad$ **return** res_2.

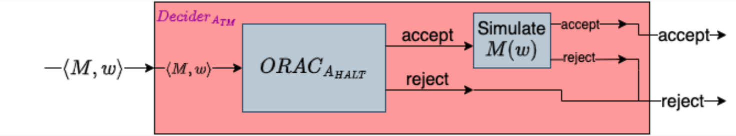
This procedure always return and as such its a decider for A_{TM}. $\square$

The Halting problem is not decidable

Theorem
The language A_{Halt} is not decidable.

Proof.
Assume, for the sake of contradiction, that A_{Halt} is decidable. As such, there is a TM, denoted by TM_{Halt}, that is a decider for A_{Halt}. We can use TM_{Halt} as an implementation of an oracle for A_{Halt}, which would imply that one can build a decider for A_{TM}. However, A_{TM} is undecidable. A contradiction. It must be that A_{Halt} is undecidable. $\square$

The same proof by figure...



... if A_{Halt} is decidable, then A_{TM} is decidable, which is impossible.

Emptiness

The language of empty languages

- $E_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) = \emptyset \right\}$.
- TM_{ETM}: Assume we are given this decider for E_{TM}.
- Need to use TM_{ETM} to build a decider for A_{TM}.
- Decider for A_{TM} is given M and w and must decide whether M accepts w.
- Restructure question to be about Turing machine having an empty language.
- Somehow make the second input (w) disappear.

We want show that E_{TM} is undecidable!

How? $A_{TM} \leq E_{TM}$

OR

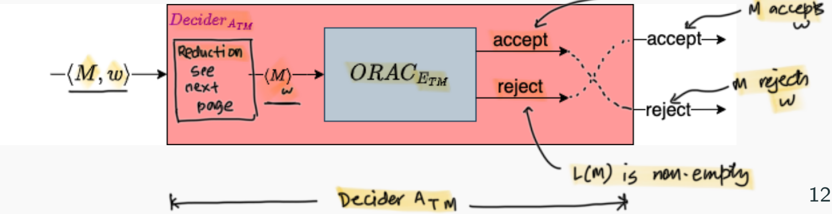
$A_{Halt} \leq E_{TM}$

The language of empty languages

- $E_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) = \emptyset \right\}$.

$A_{TM} \Rightarrow E_{TM}$

- TM_{ETM}: Assume we are given this decider for E_{TM}.
- Need to use TM_{ETM} to build a decider for A_{TM}.
- Decider for A_{TM} is given M and w and must decide whether M accepts w.
- Restructure question to be about Turing machine having an empty language.
- Somehow make the second input (w) disappear.



Reduction:

$$\langle M, w\rangle \xrightarrow{\text{Embed 'w'}} \langle M_w\rangle \; : \; \langle \underbrace{1001\ldots}_{M}, \underbrace{101\ldots}_{w}\rangle \rightarrow \langle \underbrace{1001\ldots\ 101\ldots}_{M_w}\rangle$$

E.g. $f(x)$:

$w = 2$

return w^2

$f(2) = 4$

$f(8) = 4$

$$\langle M, w\rangle \xrightarrow{\text{Reduction}} \langle M_w\rangle \begin{cases} \text{give accept } \overbrace{\text{if } M \text{ accepts } w} \Rightarrow L(M_w) = \Sigma^* : L \text{ is non-empty} \\ \text{give reject } \overbrace{\text{if } M \text{ rejects } w} \Rightarrow L(M_w) = \phi : L \text{ is empty} \end{cases}$$

The language of empty languages

- $E_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) = \emptyset \right\}$.
- TM_{ETM}: Assume we are given this decider for E_{TM}.
- Need to use TM_{ETM} to build a decider for A_{TM}.
- Decider for A_{TM} is given M and w and must decide whether M accepts w.
- Restructure question to be about Turing machine having an empty language.
- Somehow make the second input (w) disappear.
- Idea: hard-code w into M, creating a TM M_w which runs M on the fixed string w.
- TM $M_w(x)$:
    1. Input $= x$ (which will be ignored)
    2. Simulate M on w.
    3. If the simulation accepts, accept. Else, reject.

Embedding strings...

- Given program $\langle M \rangle$ and input w...
- ...can output a program $\langle M_w \rangle$.
- The program M_w simulates M on w. And accepts/rejects accordingly.
- **EmbedString**($\langle M, w \rangle$) input two strings $\langle M \rangle$ and w, and output a string encoding (TM) $\langle M_w \rangle$.

Embedding strings...

- Given program $\langle M \rangle$ and input w...
- ...can output a program $\langle M_w \rangle$.
- The program M_w simulates M on w. And accepts/rejects accordingly.
- **EmbedString**($\langle M, w \rangle$) input two strings $\langle M \rangle$ and w, and output a string encoding (TM) $\langle M_w \rangle$.
- What is $L(M_w)$?

Embedding strings...

- Given program $\langle M \rangle$ and input w...
- ...can output a program $\langle M_w \rangle$.
- The program M_w simulates M on w. And accepts/rejects accordingly.
- **EmbedString**($\langle M, w \rangle$) input two strings $\langle M \rangle$ and w, and output a string encoding (TM) $\langle M_w \rangle$.
- What is $L(M_w)$?
- Since M_w ignores input x.. language M_w is either Σ^* or $\emptyset$. It is Σ^* if M accepts w, and it is $\emptyset$ if M does not accept w.

Emptiness is undecidable

Theorem
The language E_{TM} is undecidable.

- Assume (for contradiction), that E_{TM} is decidable.
- TM_{ETM} be its decider.
- Build decider **AnotherDecider-**A_{TM} for A_{TM}:

AnotherDecider-$\mathrm{A}_{TM}(\langle M, w \rangle)$
    $\langle M_w \rangle \leftarrow$ **EmbedString** $(\langle M, w \rangle)$
    $r \leftarrow TM_{ETM}(\langle M_w \rangle)$.
    if $r =$ accept **then**
        return reject
    // $TM_{ETM}(\langle M_w \rangle)$ `rejected its input`
    return accept

Emptiness is undecidable...

Consider the possible behavior of **AnotherDecider-**A_{TM} on the input $\langle M, w\rangle$.

- If TM_{ETM} accepts $\langle M_w\rangle$, then $L(M_w)$ is empty. This implies that M does not accept w. As such, **AnotherDecider-**A_{TM} rejects its input $\langle M, w\rangle$.
- If TM_{ETM} accepts $\langle M_w\rangle$, then $L(M_w)$ is not empty. This implies that M accepts w. So **AnotherDecider-**A_{TM} accepts $\langle M, w\rangle$.

Emptiness is undecidable...

Consider the possible behavior of **AnotherDecider-**A_{TM} on the input $\langle M, w \rangle$.

- If TM_{ETM} accepts $\langle M_w \rangle$, then $L(M_w)$ is empty. This implies that M does not accept w. As such, **AnotherDecider-**A_{TM} rejects its input $\langle M, w \rangle$.
- If TM_{ETM} accepts $\langle M_w \rangle$, then $L(M_w)$ is not empty. This implies that M accepts w. So **AnotherDecider-**A_{TM} accepts $\langle M, w \rangle$.

$\implies$ **AnotherDecider-**A_{TM} is decider for A_{TM}.

But A_{TM} is undecidable...

Emptiness is undecidable...

Consider the possible behavior of **AnotherDecider-**A_{TM} on the input $\langle M, w \rangle$.

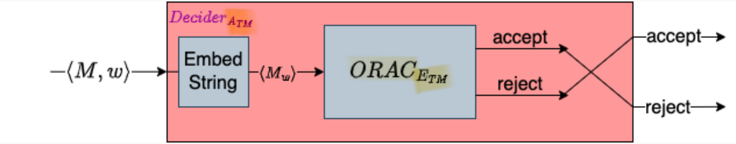
- If TM_{ETM} accepts $\langle M_w \rangle$, then $L(M_w)$ is empty. This implies that M does not accept w. As such, **AnotherDecider-**A_{TM} rejects its input $\langle M, w \rangle$.
- If TM_{ETM} accepts $\langle M_w \rangle$, then $L(M_w)$ is not empty. This implies that M accepts w. So **AnotherDecider-**A_{TM} accepts $\langle M, w \rangle$.

$\Longrightarrow$ **AnotherDecider-**A_{TM} is decider for A_{TM}.

But A_{TM} is undecidable...

...must be assumption that E_{TM} is decidable is false.

Emptiness is undecidable via diagram



AnotherDecider-A_{TM} never actually runs the code for M_w. It hands the code to a function TM_{ETM} which analyzes what the code would do if run it. So it does not matter that M_w might go into an infinite loop.

Equality

Equality is undecidable

$$EQ_{TM} = \left\{ \langle M, N \rangle \;\middle|\; M \text{ and } N \text{ are TM's and } L(M) = L(N) \right\}.$$

Lemma
The language EQ_{TM} is undecidable.

Equality is undecidable

$$EQ_{TM} = \left\{ \langle M, N \rangle \;\middle|\; M \text{ and } N \text{ are TM's and } L(M) = L(N) \right\}.$$

Lemma
The language EQ_{TM} is undecidable.

Let's try something different. We know E_{TM} is undecidable. Let's use that:

Equality is undecidable

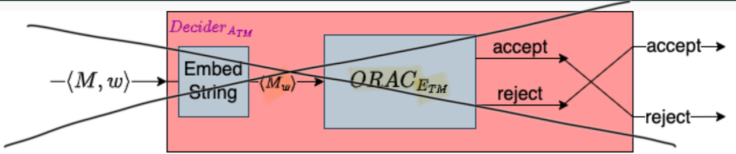
$$EQ_{TM} = \left\{ \langle M, N \rangle \;\middle|\; M \text{ and } N \text{ are TM's and } L(M) = L(N) \right\}.$$

Lemma
The language EQ_{TM} is undecidable.

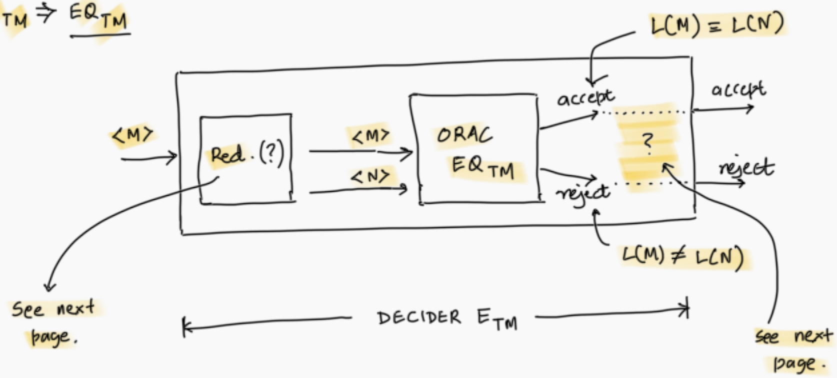
Let's try something different. We know E_{TM} is undecidable. Let's use that:

$$E_{TM} \implies EQ_{TM}$$

Equality diagram



$E_{TM} \not\Rightarrow EQ_{TM}$



Reduction:

$\langle M \rangle \longrightarrow \langle M \rangle, \langle N \rangle$

Idea: Take N to be a TM such that $L(N) = \phi$!

$$\langle M \rangle \longrightarrow \left.\begin{array}{c} \langle M \rangle \\ \langle N \rangle \\ (L(N) = \phi) \end{array}\right\} \longrightarrow \text{ORAC } EQ_{TM}$$

$\rightarrow$ accept if $L(M) = L(N) = \phi$

$\rightarrow$ reject if $L(M) \neq L(N) = \phi$

Proof

Proof.
Suppose that we had a decider **DeciderEqual** for EQ_{TM}. Then we can build a decider for E_{TM} as follows:

TM R:

1. Input $= \langle M \rangle$
2. Include the (constant) code for a TM T that rejects all its input. We denote the string encoding T by $\langle T \rangle$.
3. Run **DeciderEqual** on $\langle M, T \rangle$.
4. If **DeciderEqual** accepts, then accept.
5. If **DeciderEqual** rejects, then reject.

$\square$

DFAs

DFAs are empty?

$$E_{DFA} = \left\{ \langle A \rangle \;\middle|\; A \text{ is a DFA and } L(A) = \emptyset \right\}.$$

What does the above language describe?

All the DFA encodings that edescribe empty languages.

DFAs are empty?

$$E_{DFA} = \left\{ \langle A \rangle \;\middle|\; A \text{ is a DFA and } L(A) = \emptyset \right\}.$$

Is the language above decidable? Yes ofcourse. It's a simple DFA.

DFAs are empty?

$$E_{DFA} = \left\{ \langle A \rangle \;\middle|\; A \text{ is a DFA and } L(A) = \emptyset \right\}.$$

Is the language above decidable? Yes ofcourse. It's a simple DFA.

Lemma
The language E_{DFA} is decidable:

We can make a decider for E_{DFA}. The decider is a TM that simulates the working of A and checks if $L(A) = \phi$. simple DFA computation.

See page 22 for the formal description of the decider.

Scratch

Proof.
Unlike in the previous cases, we can directly build a decider (**DeciderEmptyDFA**) for E_{DFA}

TM R:

1. Input $= \langle A \rangle$
2. Mark start state of A as visited.
3. Repeat until no new states get marked:
   - Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, then accept.
5. Otherwise, then reject.

□

Equal DFAs

DFAs are equal?

$$EQ_{DFA} = \left\{ \langle A, B \rangle \;\middle|\; A \text{ and } B \text{ are DFAs and } L(A) = L(B) \right\}.$$

What does the above language describe?

All the DFA string pairs that represent equivalent languages

DFAs are equal?

$$EQ_{DFA} = \left\{ \langle A, B \rangle \;\middle|\; A \text{ and } B \text{ are DFAs and } L(A) = L(B) \right\}.$$

Is the language above decidable? Yes of course. Typically when we're dealing with simple machines, they're fairly decidable

DFAs are equal?

$$EQ_{DFA} = \left\{ \langle A, b \rangle \;\middle|\; A \text{ and } B \text{ are DFAs and } L(A) = L(B) \right\}.$$

Is the language above decidable? Yes of course. Typically when we're dealing with simple machines, they're fairly decidable

Lemma
The language E_{DFA} is decidable.

DFAs are equal?

$$EQ_{DFA} = \left\{ \langle A, B \rangle \;\middle|\; A \text{ and } B \text{ are DFAs and } L(A) = L(B) \right\}.$$

Is the language above decidable? Yes of course. Typically when we're dealing with simple machines, they're fairly decidable

A, B: DFA

Lemma
The language $\overset{EQ_{DFA}}{\cancel{E_{DFA}}}$ *is decidable.*

$\underbrace{L(A) = L(B)?}$ $(L(A) = \phi\,?)$

Can we show this using reductions? $\boxed{EQ_{DFA} \implies E_{DFA}}$

We want to show that this is decidable.

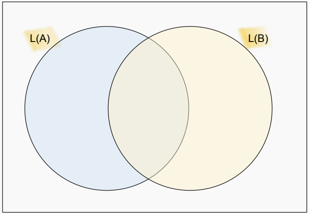
Decidable! (Just saw.)

Using an ORACLE for E_{DFA}, construct a decider for EQ_{DFA}. $\leftarrow$ How?

↳ next page.

Equal DFA trick I

Need a way to determine if there any strings in one language and not the other....



Equal DFA trick I

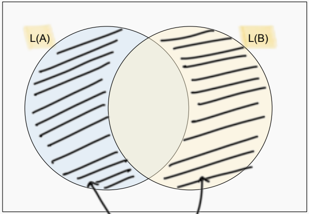
Need a way to determine if there any strings in one language and not the other....

$L(C)$: Equality, Empty

if $L(A) = L(B) \Rightarrow L(C) = \phi$

if $L(A) \neq L(B) \Rightarrow L(C) \neq \phi$

See next page!



L(C)

(denoted as Δ)

DFA	Language
A:	L(A)
B:	L(B)

Consider:

$\Delta(L(A), L(B))$ 'C'

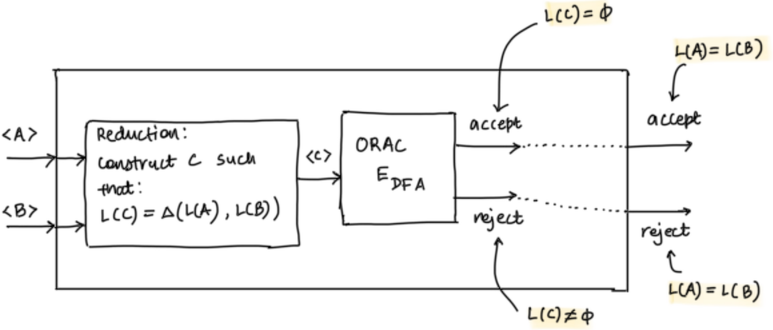
construct a DFA for $\Delta(L(A), L(B))$! We can due to the closure properties of regular languages.

This is known as the symmetric difference. Can create a new DFA (C) which represents the symmetric difference of L_A and L_B.

C is the DFA constructed.

$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right) \quad (1)$$

$EQ_{DFA} \Rightarrow E_{DFA}$



Equal DFA trick II

Notice with $L(C)$:

- If $L(A) = L(B)$ then $L(C) = \emptyset$
- If $L(A) \neq L(B)$ then $L(C)$ is not empty

Good time to use E_{DFA} proof from before.....How do we show EQ_{DFA} is decidable using a reduction?

Equal DFA trick II

Notice with $L(C)$:

- If $L(A) = L(B)$ then $L(C) = \emptyset$
- If $L(A) \neq L(B)$ then $L(C)$ is not empty

Good time to use E_{DFA} proof from before.....How do we show EQ_{DFA} is decidable using a reduction?

Want to show $EQ_{DFA} \implies E_{DFA}$

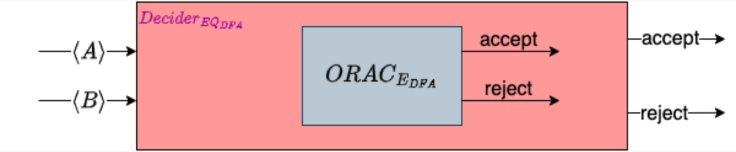
Equal DFA trick II

Notice with $L(C)$:

- If $L(A) = L(B)$ then $L(C) = \emptyset$
- If $L(A) \neq L(B)$ then $L(C)$ is not empty

Good time to use E_{DFA} proof from before.....How do we show EQ_{DFA} is decidable using a reduction?

Want to show $EQ_{DFA} \implies E_{DFA}$



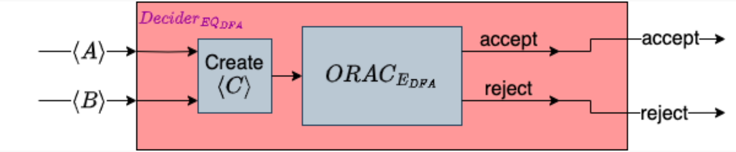
Equal DFA trick II

Notice with $L(C)$:

- If $L(A) = L(B)$ then $L(C) = \emptyset$
- If $L(A) \neq L(B)$ then $L(C)$ is not empty

Good time to use E_{DFA} proof from before.....How do we show EQ_{DFA} is decidable using a reduction?

Want to show $EQ_{DFA} \implies E_{DFA}$



Equal DFA decider

TM F:

1. Input $= \langle A, B \rangle$ where A and B are DFAs
2. Construct DFA C as described before
3. Run **DeciderEmptyDFA** from previous slide on C
4. If accepts, then accept.
5. If rejects, then reject.

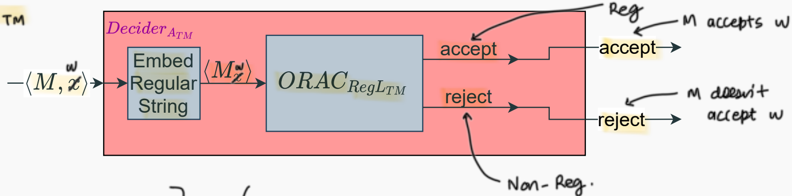
Regularity (R1Y)

Many undecidable languages

- Almost any property defining a TM language induces a language which is undecidable.
- proofs all have the same basic pattern.
- Regularity language:
  $\mathrm{Regular}_{TM} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a TM and } L(M) \text{ is regular} \right\}.$
- **DeciderRegL**: Assume TM decider for $\mathrm{Regular}_{TM}$.
- Reduction from halting requires to turn problem about deciding whether a TM M accepts w (i.e., is $w \in \mathrm{A}_{TM}$) into a problem about whether some TM accepts a regular set of strings.

Outline of IsRegular? reductionr

$A_{TM} \Rightarrow RegL_{TM}$



$M_w(x)$:

if x is of the form $a^n b^n$

  accept

otherwise

  Simulate M on w

  if accept

    accept

  if reject

    reject

$\Rightarrow$

M accepts w then any x is accepted by M

$\Rightarrow L(M) = \Sigma^*$: reg.

M doesn't accept w then only $x \in \{a^n b^n : n \geq 0\}$ is accepted by M.

$\Rightarrow$ L(M) has an infinite fooling set.

$\therefore$ Non-Reg.

Proof continued...

- Given M and w, consider the following TM M'_w:
  TM M'_w:
  (i) Input $= x$
  (ii) If x has the form $a^n b^n$, halt and accept.
  (iii) Otherwise, simulate M on w.
  (iv) If the simulation accepts, then accept.
  (v) If the simulation rejects, then reject.
- **not** executing M'_w!
- feed string $\langle M'_w \rangle$ into **DeciderRegL**
- **EmbedRegularString**: program with input $\langle M \rangle$ and w, and outputs $\langle M'_w \rangle$, encoding the program M'_w.
- If M accepts w, then any x accepted by M'_w: $L(M'_w) = \Sigma^*$.
- If M does not accept w, then $L(M'_w) = \{a^n b^n \mid n \geq 0\}$.

Proof continued...

- $a^n b^n$ is not regular...
- Use **DeciderRegL** on M'_w to distinguish these two cases.
- Note - cooked M'_w to the decider at hand.
- A decider for A_{TM} as follows.

AnotherDecider-A_{TM} $(\langle M, w\rangle)$
$\langle M'_w\rangle \leftarrow$ **EmbedRegularString** $(\langle M, w\rangle)$
$r \leftarrow$ **DeciderRegL**$(\langle M'_w\rangle)$.
return r

- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*)

Proof continued...

- $a^n b^n$ is not regular...
- Use **DeciderRegL** on M'_w to distinguish these two cases.
- Note - cooked M'_w to the decider at hand.
- A decider for A_{TM} as follows.

$$\boxed{\begin{array}{l} \textbf{AnotherDecider-}\mathrm{A}_{TM}(\langle M, w\rangle) \\ \qquad \langle M'_w\rangle \leftarrow \textbf{EmbedRegularString}\,(\langle M, w\rangle) \\ \qquad r \leftarrow \textbf{DeciderRegL}(\langle M'_w\rangle). \\ \qquad \textbf{return } r \end{array}}$$

- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*) $\implies M$ accepts w. So **AnotherDecider-**A_{TM} should accept $\langle M, w\rangle$.

Proof continued...

- $a^n b^n$ is not regular...
- Use **DeciderRegL** on M'_w to distinguish these two cases.
- Note - cooked M'_w to the decider at hand.
- A decider for A_{TM} as follows.

> **AnotherDecider-**$\mathrm{A}_{TM}$$(\langle M, w\rangle)$
>
> $\langle M'_w\rangle \leftarrow$ **EmbedRegularString** $(\langle M, w\rangle)$
>
> $r \leftarrow$ **DeciderRegL**$(\langle M'_w\rangle)$.
>
> **return** r

- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*) $\implies M$ accepts w. So **AnotherDecider-**A_{TM} should accept $\langle M, w\rangle$.
- If **DeciderRegL** rejects $\implies L(M'_w)$ is not regular $\implies$ $L(M'_w) = a^n b^n$

Proof continued...

- $a^n b^n$ is not regular...
- Use **DeciderRegL** on M'_w to distinguish these two cases.
- Note - cooked M'_w to the decider at hand.
- A decider for A_{TM} as follows.

> **AnotherDecider-**$\mathrm{A}_{TM}(\langle M, w\rangle)$
>
> $\langle M'_w\rangle \leftarrow$ **EmbedRegularString** $(\langle M, w\rangle)$
>
> $r \leftarrow$ **DeciderRegL**$(\langle M'_w\rangle)$.
>
> **return** r

- If **DeciderRegL** accepts $\implies L(M'_w)$ regular (its Σ^*) $\implies M$ accepts w. So **AnotherDecider-**A_{TM} should accept $\langle M, w\rangle$.
- If **DeciderRegL** rejects $\implies L(M'_w)$ is not regular $\implies L(M'_w) = a^n b^n \implies M$ does not accept $w \implies$ **AnotherDecider-**A_{TM} should reject $\langle M, w\rangle$.

Rice's theorem

The above proofs were somewhat repetitious...

...they imply a more general result.

Theorem (Rice's Theorem.)
Suppose that L *is a language of Turing machines; that is, each word in* L *encodes a TM. Furthermore, assume that the following two properties hold.*

(a) *Membership in* L *depends only on the Turing machine's language, i.e. if* $L(M) = L(N)$ *then* $\langle M \rangle \in \mathsf{L} \Leftrightarrow \langle N \rangle \in \mathsf{L}$.

(b) *The set* L *is "non-trivial," i.e.* $\mathsf{L} \neq \emptyset$ *and* L *does not contain all Turing machines.*

Then L *is a undecidable.*

Philosophical Question:

What can we do to decide about languages accepted by Turing machines / programs?

"NOTHING" except "TRIVIAL" things!