Pre-lecture brain teaser

In the following languages, three are decidable and three are undecidable. Which are which?

- $A_{CFG} = \left\{ \langle G, w \rangle \;\middle|\; G \text{ is a } CFG \text{ that generates string } w \right\}.$
- $E_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a } CFG \text{ and } L(G) = \emptyset \right\}.$
- $ALL_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a } CFG \text{ and } L(G) = \Sigma^* \right\}.$
- $A_{LBA} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } LBA \text{ that accepts string } w \right\}.$
- $E_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } LBA \text{ where } L(M) = \emptyset \right\}.$
- $ALL_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } LBA \text{ where } L(M) = \Sigma^* \right\}.$

ECE-374-B: Lecture 25 - Midterm 3 Review

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A_{CFG} decidable?

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YES!

A_{CFG} decidable?

YES!

- $V = \{S\}$
- $T = \{0, 1\}$
- $P = \{S \to \epsilon \mid 0S0 \mid 1S1\}$
  (abbrev. for $S \to \epsilon, S \to 0S0, S \to 1S1$)

A_{CFG} decidable?

YES!

Lemma
A CFG in Chomsky normal form can derive a string w in at most 2^n steps!

Knowing this, we can just simulate all the possible rule combinations for 2^n steps and see if any of the resulting strings matches w.

E_{CFG} decidable?

E_{CFG} decidable?

YES!

E_{CFG} decidable?

YES!

In this case, we just need to know if we can get from the start variable to a string with only terminal symbols.

1. Mark all terminal symbols in G
2. Repeat until no new variables get marked:
   2.1 Mark any variable A where G has the rule $A \to U_1 U_2 \ldots U_k$ where U_i is a marked terminal/variable
3. If start variable is not marked, accept. Otherwise reject.

- $V = \{S\}$
- $T = \{0, 1\}$
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ALL_{CFG} decidable?

ALL_{CFG} decidable?

Nope!

ALL_{CFG} decidable?

Nope!

Proof requires computation histories which are outside the scope of this course.

A_{LBA} decidable?

A_{LBA} decidable?

YES!

A_{LBA} decidable?

YES!

Remember a LBA has a finite tape. Therefore we know:

1. A tape of length n where each cell can contain g symbols, you have g^n possible configurations.
2. The tape head can be in one of n positions and has q states yielding a tape that can be in qn configurations.
3. Therefore the machine can be in qng^n configurations.

A_{LBA} decidable?

YES!

Remember a LBA has a finite tape. Therefore we know:

1. A tape of length n where each cell can contain g symbols, you have g^n possible configurations.
2. The tape head can be in one of n positions and has q states yielding a tape that can be in qn configurations.
3. Therefore the machine can be in qng^n configurations.

Lemma
If an LBA does not accept or reject in qng^n then it is stuck in a loop forever.

A_{LBA} decidable?

Decider for A_{LBA} will do the following.

1. Simulate $\langle M \rangle$ on w for qng^n steps.
   1.1 if accepts, then accept
   1.2 if rejects, then reject
2. If neither accepts or rejects, means it's in a loop in which case, reject.

E_{LBA} decidable?

E_{LBA} decidable?

Nope!

E_{LBA} decidable?

Nope!

Proof requires computational history trick, a story for another time ...

ALL_{LBA} decidable?

ALL_{LBA} decidable?

Nope!

ALL_{LBA} decidable?

Nope!

No standard proof for this, but let's look at a pattern as follows.

Decidability across grammar complexities

	DFA	*CFG*	*PDA*	*LBA*	*TM*
A	D	D	D	D	U
E	D	D	D	U	U
ALL	D	U	U	U	U

Eventually problems get too tough ...

ALL_{LBA} decidable?

Nope!

No standard proof for this, but let's look at a pattern:

So we sort of know that ALL_{LBA} isn't decidable because we knew ALL_{CFG} wasn't (though intuition is never sufficient evidence).

Rice's theorem

Rice's theorem: Any 'non-trivial' property about the language recognized by a Turing machine is undecidable.

Un-/decidability practice problems

Available Undecidable languages

- $L_{Accept} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and accepts } w \right\}.$
- $L_{HALT} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and halts on } \varepsilon \right\}.$

Practice 1: Halt on Input

Is the following language undecidable?

$$L_{HaltOnInput} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and halts on } w \right\}.$$

Practice 2: L has an infinite fooling set

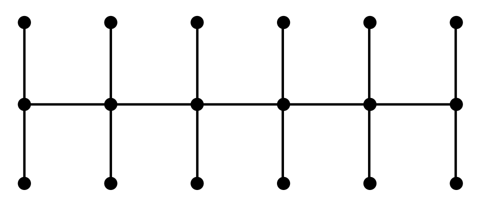
Is the following language undecidable?

$$L_{HasFooling} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and } L(M) \text{ has a fooling set } \right\}.$$

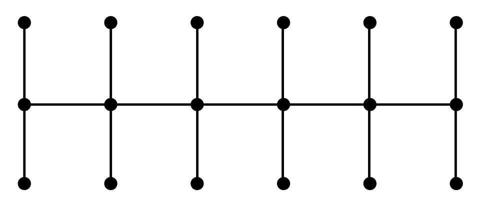
NP-Complete practice problems

Practice: NP-Complete Reduction I

A *centipede* is an undirected graph formed by a path of length k with two edges (legs) attached to each node on the path as shown in the below figure. Hence, the centipede graph has $3k$ vertices. The **CENTIPEDE** problem is the following: given an undirected graph $G = (V, E)$ and an integer k, does G contain a *centipede* of $3k$ distinct vertices as a subgraph? Prove that **CENTIPEDE** is **NP-Complete**.

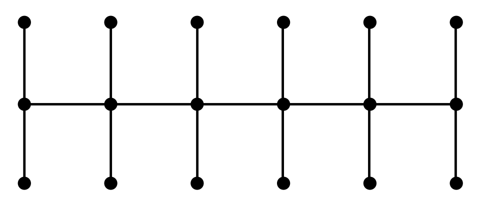


What do we need to do to prove Centipede is NP-Complete?



Practice: NP-Complete Reduction I

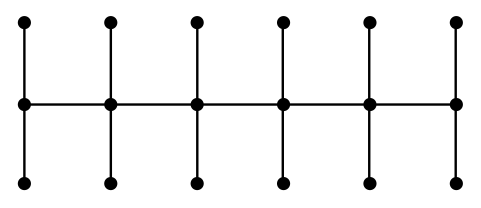
Prove Centipede is in NP:



The problem is in NP. We let the certificate be three ordered lists of length k. The first list is the main path and the other two lists form the legs. We can easily verify in polynomial time that the lists form a *centipede* by checking that in the first list any two consecutive vertices have an edge between them in G and checking that a vertex from the second or third list has an edge to a vertex in the first list of the same order (position in the list).

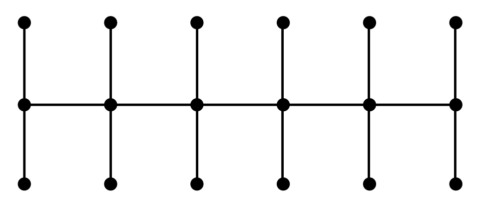
Practice: NP-Complete Reduction I

Prove Centipede is in **NP-hard**:



Practice: NP-Complete Reduction I

Prove Centipede is in NP-hard:



Hamiltonian Path (HP): Given a graph G (either directed or undirected), is there a path that visits every vertex exactly once.

Practice: NP-Complete Reduction I

$$HP \leq_P Centipede$$

The problem is NP-Hard by reduction from HAMILTONIAN-PATH to CENTIPEDE. Given an instance of HAMILTONIAN-PATH, a graph G with n vertices $v_1, v_2, \cdots, v_n$, create a new graph G′ by adding $2n$ vertices: $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_n$. Then, add an edge (u_i, v_i) and (x_i, v_i) for $1 \leq i \leq n$. The reduction is polynomial time since we only added $2n$ of vertices and edges to the graph.

Practice: NP-Complete Reduction II

A **quasi-satisfying assignment (quasiSAT)** for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

Practice: NP-Complete Reduction II

A **quasi-satisfying assignment (quasiSAT)** for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

Prove quasiSAT is in NP.

Practice: NP-Complete Reduction II

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Prove quasiSAT is NP-hard.

Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard.

Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard.

3SAT: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment.

Good luck on the exam